ANDRZEJ GRANAS
JAMES DUGUNDJI

Fixed Point Theory

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Andrzej Granas James Dugundji

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&

TO THE MEMORY OF

Jim Dugundji

Preface

The aim of this monograph is to give a unified account of the classical topics in fixed point theory that lie on the border-line of topology and non-linear functional analysis, emphasizing developments related to the Leray–Schauder theory. Using for the most part geometric methods, our study centers around formulating those general principles of the theory that provide the foundation for many of the modern results in diverse areas of mathematics.

The main text is self-contained for readers with a modest knowledge of topology and functional analysis; the necessary background material is collected in an appendix, or developed as needed. Only the last chapter presupposes some familiarity with more advanced parts of algebraic topology.

The "Miscellaneous Results and Examples", given in the form of exercises, form an integral part of the book and describe further applications and extensions of the theory. Most of these additional results can be established by the methods developed in the book, and no proof in the main text relies on any of them; more demanding problems are marked by an asterisk. The "Notes and Comments" at the end of paragraphs contain references to the literature and give some further information about the results in the text.

This monograph evolved from *Fixed Point Theory*, *Vol. I*, published in the Monografie Matematyczne series in 1982. An outline of the entire treatise was conceived by the authors in 1978; in spite of its appearance many years later than expected, the content follows the original plan.

The following is a brief note about the life and work of my close friend Jim Dugundji (1919–1985). Jim Dugundji received his B.A. degree from New York University in 1940 and, for the next two years, studied at the University of North Carolina. After serving in the US Air Force from 1942 to 1946 he enrolled at the Massachussetts Institute of Technology, where he earned his Ph.D. in 1948 under Witold Hurewicz. Since 1948, Jim Dugundji taught at the University of Southern California in Los Angeles, where he became a

full professor in 1958. For many years he served as one of the editors of the *Pacific Journal of Mathematics* and of *Topology and its Applications*.

While Dugundji's mathematical work lay mainly in the field of topology, he also contributed to dynamical systems and functional analysis, and to problems in applied mathematics (electrical engineering, geology, and theoretical chemistry). Among his books are *Topology* (Allyn and Bacon, 1965) and *Perspectives in Theoretical Stereochemistry* (Springer, 1984), the latter written with I. Ugi, R. Kopp and D. Marquarding.

Jim Dugundji's mathematical publications are marked by their lucidity and frequently by the decisiveness of his results. His work was, in fact, in many ways an expression of his character. Although he was self-effacing and lacking in any wish for self-advancement, he was totally independent and would not tolerate anything which he considered second best. He spent his life for science's sake, aware of the sacrifice and dedication this requires, and what he asked from himself—which was quite a lot—he expected from others. Man of high integrity and moral strength, he had a great sensitivity, and all who were close to him could testify to his caring concern.

I wish to express my gratitude to our many friends and collaborators who so generously assisted us during the long years of preparation of this book. Most important, without the help and encouragement of Merope Dugundji and my wife Monique, it would not have been possible for us to conceive this project nor for me to bring it to its completion. First and foremost my thanks go to Cezary Bowszyc, who read and commented upon the entire manuscript; his detailed and constructive criticism has led to many improvements and has been of a very great help. Alberto Abbondandolo, Robert Burckel, Haïm Brézis, Ed Fadell, Marlène Frigon, Kazimierz Gęba, Tadeusz Iwaniec, Marc Lassonde, Isaac Namioka, and Gencho Skordev offered valuable suggestions in various stages of the writing, all of which are sincerely appreciated. Special thanks go to Jerzy Trzeciak for his excellent editorial job; to Anna Rudnik for her considerable help with the typesetting; and to the staff of Springer-Verlag for their most efficient handling of publication matters.

I thank also the Killam Foundation and the National Research Council of Canada for providing support for my research projects on various topics, which are now summarized in this book.

Finally, I would like to express my gratitude to Albrecht Dold, Ky Fan and Louis Nirenberg for their encouragement and inspiration over the years.

Montreal and Olsztyn, September 2002 *Andrzej Granas*

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§0. Introduction

In this introduction we take a brief general look at the subject and discuss some simple notions and techniques of fixed point theory. It is hoped that this discussion will help the reader to grasp some of the ideas and results of the theory, before entering the detailed and systematic study needed for a deeper understanding.

Throughout the book, by *space* we understand a (Hausdorff) topological space. Unless specifically stated otherwise, a *map* is a continuous transformation; however, we do occasionally add the word "continuous" for emphasis.

1. Fixed Point Spaces

We begin with a basic

(1.1) Definition. Let X be any space and f a map of X, or of a subset of X, into X. A point $x \in X$ is called a *fixed point* for f if $x = f(x)$. The set of all fixed points of f is denoted by $\mathrm{Fix}(f)$.

In this definition one recognizes the form typical of existence theorems in analysis. For example, finding a solution of the equation $P(z) = 0$, where P is a complex polynomial, is equivalent to searching for a fixed point of the self-map $z \mapsto z - P(z)$ of $\boldsymbol{C}$. More generally, if D is any operator acting on a subset of a linear space, then showing that the equation $Du = 0$ (respectively $u - \lambda Du = 0$) has a solution is equivalent to showing that the map $u \mapsto u - Du$ (respectively $u \mapsto \lambda Du$) has a fixed point. Thus, conditions on an operator, or on its domain of definition, that guarantee the existence of a fixed point can be reinterpreted as existence theorems in analysis and therefore have considerable interest.

For a given space X and map $f : X \to X$, the existence of a fixed point for f may be due entirely to the nature of the space X itself, rather than to

any special feature that the map f has. This is formalized in

(1.2) DEFINITION. A space X is called a *fixed point space* provided every map $f : X \to X$ has a fixed point.

EXAMPLES. (i) Any bounded closed interval $J = [a, b] \subset \boldsymbol{R}$ is a fixed point space. Indeed, given $f : J \to J$, we have $a - f(a) \leq 0$ and $b - f(b) \geq 0$; the intermediate value theorem ensures that the equation $x - f(x) = 0$ has a solution in J, and therefore f has a fixed point.

(ii) The real line $\boldsymbol{R}$ is not a fixed point space, since the translation $x \mapsto x + 1$ has no fixed point.

In general, it is difficult to decide whether or not a given space is a fixed point space; such results usually have many interesting topological consequences. An example is the Brouwer fixed point theorem, which asserts that: *Every compact convex set in $\boldsymbol{R}^n$ is a fixed point space.*

The property of being a fixed point space is topologically invariant: for if X is a fixed point space and $h : X \to Y$ a homeomorphism, then for any $g : Y \to Y$ the map $h^{-1} \circ g \circ h : X \to X$ has a fixed point x_0, so $g \circ h(x_0) = h(x_0)$ and $h(x_0)$ is a fixed point for g.

EXAMPLE. The graph of any continuous $f : [a, b] \to \boldsymbol{R}$, for example, the graph of

$$f(x) = \begin{cases} x \sin(1/x), & 0 < x \leq 1, \\ 0, & x = 0, \end{cases}$$

being homeomorphic to $[a, b]$, is a fixed point space.

If X is not a fixed point space, it may still be true that every map having some well-defined general property will have a fixed point. To formalize this notion, we enlarge the scope of Definition (1.2):

(1.3) DEFINITION. Let X be a space and $\mathcal{M}$ a class of maps $f : X \to X$. If each $f \in \mathcal{M}$ has a fixed point, then X is called a *fixed point space relative to* $\mathcal{M}$.

For example, the Banach contraction principle asserts that: *Every complete metric space is a fixed point space for contractive maps.*

The notion introduced is particularly important when $\mathcal{M}$ is taken as the class of compact maps, i.e., those maps $f : X \to X$ such that the closure $\overline{f(X)}$ of $f(X)$ is compact; maps of this type arise naturally in many problems of nonlinear analysis.

EXAMPLES. (i) We have seen that $\boldsymbol{R}$ is not a fixed point space. However, $\boldsymbol{R}$ is, in fact, a fixed point space relative to the class of compact maps. For let $f : \boldsymbol{R} \to \boldsymbol{R}$ be compact; then $f(\boldsymbol{R})$ is contained in some finite interval $[a, b]$; in particular, f maps $[a, b]$ into itself, so has a fixed point.

(ii) The Schauder fixed point theorem, which has numerous applications in analysis, asserts that: *Every convex set in a normed linear space is a fixed point space for compact maps.*

Because the continuous image of a compact set is compact, the same argument as before shows that being a fixed point space relative to the class of compact maps is a topologically invariant property. Thus, for example, any open $(a, b) \subset \boldsymbol{R}$, as well as the graph of $\sin(1/x)$, $0 < x \leq 1$, is a fixed point space for compact maps.

2. Forming New Fixed Point Spaces From Old

In general, a subspace of a fixed point space need not be a fixed point space: for example, $\{a, b\} \subset [a, b]$ does not have the fixed point property. However, certain subspaces do, in fact, always inherit this property. To describe these subspaces, we need

(2.1) DEFINITION. A subset $A \subset X$ is called a *retract* of X if there is a continuous $r : X \to A$ such that $r(a) = a$ for each $a \in A$; the map r is then called a *retraction* of X onto A.

We note that a retract of a Hausdorff space is necessarily closed, since $A = \{x \mid r(x) = \mathrm{id}(x)\}$.

For example, if E is a normed space and $K_\varrho = \{x \in E \mid \|x\| \leq \varrho\}$ is the closed ball in E with center 0 and radius ϱ, then $r : E \to K_\varrho$ given by

$$(*) \qquad r(y) = \begin{cases} y & \text{for } \|y\| \leq \varrho, \\ \varrho y/\|y\| & \text{for } \|y\| > \varrho, \end{cases}$$

defines a retraction (called the *standard retraction*) of E onto K_ϱ.

The importance of this concept in fixed point theory stems from

(2.2) THEOREM. *If X is a fixed point space (respectively a fixed point space for compact maps), so also is every retract of X.*

PROOF. Let $r : X \to A$ be the retraction, and $i : A \to X$ be the inclusion; we have $r \circ i = \mathrm{id}_A$. Consider any $f : A \to A$; then $i \circ f \circ r : X \to X$ has a fixed point, x_0. From $x_0 = i \circ f \circ r(x_0)$ follows $r(x_0) = r \circ i \circ f \circ r(x_0) = f[r(x_0)]$ and $r(x_0)$ is a fixed point for f. The proof of the second part of the theorem is similar. □

On the other hand, if X has a retract that is a fixed point space, it does not follow that X itself is one, since any one-point subspace $\{a\}$ is a retract of any space.

We further illustrate the retraction technique by deriving from the Schauder fixed point theorem two basic results that have many consequences and applications.

(2.3) THEOREM (Nonlinear alternative). *Let E be a normed linear space and K_ϱ the closed ball in E with center 0 and radius ϱ. Then each compact map $F : K_\varrho \to E$ has at least one of the following two properties:*
(a) *F has a fixed point,*
(b) *there exist $x \in \partial K_\varrho$ and $\lambda \in (0, 1)$ such that $x = \lambda F(x)$.*

PROOF. Let $r : E \to K_\varrho$ be the standard retraction. By the Schauder theorem the compact composite $r \circ F : K_\varrho \to K_\varrho$ has a fixed point $x = rF(x)$. If $F(x)$ is in K_ϱ, then $x = rF(x) = F(x)$, so F has a fixed point; if $F(x)$ does not belong to K_ϱ, then by $(*)$, $x = rF(x) = \varrho F(x)/\|F(x)\|$, so $x \in \partial K_\varrho$, and taking $\lambda = \varrho/\|F(x)\| < 1$ completes the proof. □

As an obvious immediate consequence of (2.3) we obtain:

(2.4) THEOREM (Leray–Schauder alternative). *Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F) = \{x \in E \mid x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point. □

To conclude we briefly mention one other way of forming new fixed point spaces. For Cartesian products, the result depends on the number of factors. The Cartesian product of two compact fixed point spaces need not be a fixed point space; however, an infinite Cartesian product of compact fixed point spaces will be a fixed point space if and only if every finite product of those spaces is a fixed point space. Thus, by Brouwer's theorem, the Hilbert cube I^∞, and in fact any Tychonoff cube $I^\aleph$, are fixed point spaces.

3. Topological Transversality

Sometimes, a map f of Y, or of a subset X of Y, into Y will have a fixed point because of its behavior on some particular subset. Indeed, many existence theorems of analysis are of this type. For example, consider an open set U in a normed linear space E and an operator $D : \overline{U} \to E$ such that $Du \neq 0$ on ∂U; does the equation $Du = 0$ have a solution in U? In terms of maps, we are given a map $f(u) = u - Du$ that is fixed point free on ∂U, and seek to determine, from its behavior on ∂U, whether or not it has a fixed point in U.

Putting this now in a more abstract setting, we shall confine our attention to compact maps. Let Y be an arbitrary space; by a *pair* (X, A) *in* Y is meant a closed subset X of Y and an $A \subset X$ closed in X. By $\mathscr{K}_A(X, Y)$

we denote the set of all compact maps $f : X \to Y$ such that the restriction $f|A : A \to Y$ is fixed point free.

(3.1) DEFINITION. Let (X, A) be a pair in Y. A map $g \in \mathscr{K}_A(X, Y)$ is called *essential* if every $g^* \in \mathscr{K}_A(X, Y)$ such that $g^*|A = g|A$ has a fixed point. A map that is not essential is called *inessential*.

In geometric terms, a compact map $g : X \to Y$ is essential if the graph of $g|A$ does not meet the diagonal $\Delta \subset X \times Y$, but the graph of every compact $g^* : X \to Y$ that coincides with g on A must cross (i.e. traverse) the diagonal.

EXAMPLES. (i) Let $X = [a, b] \subset \boldsymbol{R} = Y$ and $A = [a, b]$. A map $g : X \to \boldsymbol{R}$ is essential if and only if $[a - g(a)][b - g(b)] < 0$.

(ii) Let Y be a fixed point space for compact maps, U an open subset of Y, and $(\overline{U}, \partial U)$ the pair consisting of the closure of U in Y and the boundary of U in Y. Then for any $u_0 \in U$, the constant map $g|\overline{U} = u_0$ is essential. Indeed, let $g^* : \overline{U} \to Y$ be a compact extension of the map $g|\partial U$ over $\overline{U}$. To show that g^* has a fixed point, we extend g^* to a compact map $\overline{g} : Y \to Y$ by setting $\overline{g}|(Y - U) = u_0$. By assumption, $\overline{g}$ must have a fixed point; and since no point in $Y - U$ is fixed, that point $x = \overline{g}(x)$ must be in U, and hence $x = g^*(x)$.

The concept of an essential map is closely related to the notion of a continuous deformation. To formulate this in a precise way, we need the notion of homotopy.

Let (X, A) be a pair in Y. By a *homotopy* is meant a parametrized family $\{h_t : X \to Y\}$ of maps indexed by $t \in \boldsymbol{I} = [0, 1]$ such that the map $h : X \times \boldsymbol{I} \to Y$ given by $h(x, t) = h_t(x)$ is continuous. We say that a homotopy $h_t : X \to Y$ is *compact* if the map $h : X \times \boldsymbol{I} \to Y$ is compact; a compact homotopy $h_t : X \to Y$ is said to be *admissible* (for the pair (X, A)) if it is fixed point free on $A \subset X$, i.e., if for each $t \in \boldsymbol{I}$, the map $h_t|A : A \to Y$ has no fixed point.

(3.2) DEFINITION. Two maps $f, g \in \mathscr{K}_A(X, Y)$ are called *admissibly homotopic*, written $f \simeq g$ in $\mathscr{K}_A(X, Y)$, if there is an admissible compact homotopy $h_t : X \to Y$ $(0 \le t \le 1)$ such that $h_0 = f$ and $h_1 = g$. Clearly, the relation "$\simeq$" of admissible homotopy divides the set $\mathscr{K}_A(X, Y)$ into disjoint equivalence classes.

(3.3) THEOREM. *Let Y be a completely regular space, (X, A) a pair in Y, and let $f, g \in \mathscr{K}_A(X, Y)$ be admissibly homotopic. If the map g is essential, then* $\operatorname{Fix}(f) \neq \emptyset$.

PROOF. Suppose $\mathrm{Fix}(f) = \emptyset$. Let $h : X \times I \to Y$ be an admissible compact homotopy such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in X$.

Let $B = \{x \in X \mid x = h(x, t) \text{ for some } t \in I\}$. There is no loss of generality in assuming that B is nonempty; then B is a closed subset of the compact set $\overline{h(X \times I)}$, so it is compact in X. Clearly, $A \cap B = \emptyset$, because h is admissible; consequently, because X is completely regular, there is an Urysohn function $\lambda : X \to I$ with $\lambda|A = 0$ and $\lambda|B = 1$.

Define $g^*(x) = h(x, 1 - \lambda(x))$ for $x \in X$. Clearly, $g^*|A = g|A$ and g^* is compact; since g is essential, g^* has a fixed point; but if $g^*(x) = h(x, 1 - \lambda(x)) = x$, then $x \in B$ so $\lambda(x) = 1$ and $x = h(x, 0) = f(x)$, which contradicts the assumption that $\mathrm{Fix}(f) = \emptyset$. □

We can now formulate a general result of the Leray-Schauder type:

(3.4) THEOREM (Leray–Schauder principle). *Let Y be a completely regular fixed point space for compact maps, U open in Y, and $(\overline{U}, \partial U)$ the pair consisting of the closure of U in Y and the boundary of U in Y. Let $\{h_t : \overline{U} \to Y\}$ be an admissible compact homotopy such that $h_0 = f$ and $h_1 = g$, where g is the constant map sending $\overline{U}$ to a point $u_0 \in U$. Then f has a fixed point in $\overline{U}$*

PROOF. Because $\{h_t\}$ is admissible, f has a fixed point by Theorem (3.3) and Example (ii). □

We remark that Theorem (2.3) follows at once from (3.4).

4. Factorization Technique

In this section we describe the factorization technique, which can be used to establish that a given space is a fixed point space. First we indicate some special properties of individual maps that will ensure the existence of fixed points.

The following simple general result is of importance:

(4.1) LEMMA. *Let $\alpha : K \to X$ and $\beta : X \to K$ be two maps. Then $f = \beta\alpha : K \to K$ has a fixed point if and only if $F = \alpha\beta : X \to X$ has a fixed point. In other words, given the commutative diagrams*

$$\begin{tikzcd} & K \arrow[dl, "\alpha"'] \arrow[d, "f"] \\ X \arrow[r, "\beta"] & K \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & X \arrow[dl, "\beta"'] \arrow[d, "F"] \\ K \arrow[r, "\alpha"] & X \end{tikzcd}$$

we have: $\mathrm{Fix}(f) \neq \emptyset \Leftrightarrow \mathrm{Fix}(F) \neq \emptyset$.

PROOF. From $x_0 = \beta\alpha(x_0)$ it follows that $\alpha(x_0) = \alpha\beta[\alpha(x_0)]$. □

For a subset A of a fixed point space K that is not a retract of that space, this leads to the conclusion that a map $f : A \to A$ will have a fixed point simply if it is extendable to a map $f^* : K \to A$. We recall that given $A \subset X$, a map $f : A \to Y$ is *extendable* over X if there is an $f^* : X \to Y$ with $f^*|A = f$.

(4.2) PROPOSITION. *Let K be a fixed point space and $A \subset K$. If $f : A \to A$ is extendable to $f^* : K \to A$, then f has a fixed point.*

PROOF. Letting $i : A \hookrightarrow X$ be the inclusion, we have $f = f^*i$; since $if^* = f^*$ has a fixed point, so also does f. □

More generally, if K is not a fixed point space, the existence of a fixed point for $f : K \to K$ can sometimes be determined by factorization of the map:

(4.3) LEMMA. *Suppose that a map $f : K \to K$ factors through a fixed point space X, i.e., there are maps α and β making the diagram*

$$\begin{tikzcd} & K \arrow[dl, "\alpha"'] \arrow[d, "f"] \\ X \arrow[r, "\beta"] & K \end{tikzcd}$$

commutative. Then f has a fixed point.

PROOF. This is an obvious consequence of (4.1). □

(4.4) THEOREM. *Let (K, d) be a compact metric space. Assume that for each $n = 1, 2, \ldots$, there is a fixed point space Z_n and maps $\alpha_n : K \to Z_n$ and $\beta_n : Z_n \to K$ such that $d(x, \beta_n\alpha_n(x)) \le 1/n$ for all x. Then K is a fixed point space.*

PROOF. Let $f : K \to K$ be given. For each n, there is a $z_n \in Z_n$ with $\alpha_n f \beta_n(z_n) = z_n$; letting $x_n = \beta_n z_n$ we find that $\beta_n\alpha_n f(x) = x_n$ and, by our hypothesis, $d(f(x_n), x_n) \le 1/n$. We can assume $x_n \to x_0$; then $fx_n \to x_0$; but by continuity of f also $fx_n \to fx_0$, so that $fx_0 = x_0$ and f has a fixed point. □

EXAMPLES. (i) Consider the space $Z = \{(x, y) \mid y = \sin(1/x),\ 0 < x \le 1\} \cup \{(0, y) \mid -1 \le y \le 1\}$. By projecting Z onto the portion of the sine curve on the intervals $1/n \le x \le 1$, application of (4.4) shows that Z is a fixed point space.

(ii) Consider the Hibert cube

$$I^\infty = \{x = (x_1, x_2, \ldots) \mid |x_i| \le 1/2^i \text{ for all } i\}$$

and its subspaces I^n, where

$$I^n = \{x = \{x_i\} \in I^\infty \mid x_i = 0 \text{ for all } i \geq n+1\}.$$

Projecting I^∞ onto the subspaces I^n and using the Brouwer fixed point theorem, one finds that I^∞ is a fixed point space. A similar technique can be used for the proof of the Schauder fixed point theorem.

The factorization method is very effective whenever X has a sufficient structure. To illustrate this, recall that X is an *absolute retract*, written X is an AR, if X is metrizable and given any metric space Y and $B = \overline{B} \subset Y$, any $f_0 : B \to X$ extends over Y to a map $f : Y \to X$.

(4.5) THEOREM. *Let K be a compact metric space, and let $f : K \to K$ be a map. Assume that f can be factored as $K \xrightarrow{\alpha} X \xrightarrow{\beta} K$, where X is an AR. Then f has a fixed point.*

PROOF. Consider the diagram

$$\begin{tikzcd}
I^\infty \arrow[dd, dashed, "\varphi"'] & & \widehat{K} \arrow[ll, hook', "j"'] \arrow[dl, shift left, "s^{-1}"] \\
& K \arrow[ur, shift left, "s"] \arrow[dl, "\alpha"'] \arrow[d, "f"] & \\
X \arrow[r, "\beta"] & K &
\end{tikzcd}$$

in which I^∞ is the Hilbert cube and $s : K \to \widehat{K}$ is a homeomorphism of K onto $\widehat{K} \subset I^\infty$ with inverse $s^{-1} : \widehat{K} \to K$. Since X is an AR, there is an extension $\varphi : I^\infty \to X$ of the map $\alpha s^{-1} : \widehat{K} \to X$ over I^∞, i.e., $\varphi j = \alpha s^{-1}$, where $j : \widehat{K} \hookrightarrow I^\infty$ is the inclusion. Consider now the composite

$$K \xrightarrow{js} I^\infty \xrightarrow{\beta\varphi} K.$$

We have $(\beta\varphi)(js) = \beta[\varphi j]s = \beta[\alpha s^{-1}]s = \beta\alpha = f$ Thus, f factors through the Hilbert cube I^∞ and consequently has a fixed point. □

As an immediate consequence, we have

(4.6) THEOREM. *Let X be an AR, and let $f : X \to X$ be a compact map. Then f has a fixed point.* □

Because every convex set in a normed linear space is an AR, the last result is a generalization of the Schauder fixed point theorem.

I.

Elementary Fixed Point Theorems

In this chapter we provide an introduction to those topics of fixed point theory that for the most part involve only the notions of *completeness*, *order*, and *convexity*. In spite of their elementary character, the results given here have a number of significant applications. Some of these are presented at the end of the chapter.

§1. Results Based on Completeness

The fixed point theorems presented in this paragraph are all related to the Banach contraction principle, which asserts that every complete metric space is a fixed point space for the class of contractive mappings.

1. Banach Contraction Principle

The Banach contraction principle is the simplest and one of the most versatile elementary results in fixed point theory. Being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive map: it produces approximations of any required accuracy, and moreover, even the number of iterations needed to get a specified accuracy can be determined.

A map $F : (X, d) \to (Y, \varrho)$ of metric spaces that satisfies $\varrho(F(x), F(z)) \leq Md(x, z)$ for some fixed constant M and all $x, z \in X$ is called *Lipschitzian*; the smallest such M is called the *Lipschitz constant* $L(F)$ of F. If $L(F) < 1$, the map F is called *contractive* with *contraction constant* $L(F)$; if $L(F) \leq 1$, the map F is said to be *nonexpansive*. Note that a Lipschitzian map is necessarily continuous.

Let Y be any set and $F : Y \to Y$ a map of Y into itself. For any given $y \in Y$, define $F^n(y)$ inductively by $F^0(y) = y$ and $F^{n+1}(y) = F(F^n(y))$; we

call $F^n(y)$ the *nth iterate* of y under F, and the set $\{F^n(y) \mid n = 0, 1, \ldots\}$ is the *orbit* of y under F.

(1.1) THEOREM (Banach contraction principle). *Let (Y, d) be a complete metric space and $F : Y \to Y$ be contractive. Then F has a unique fixed point u, and $F^n(y) \to u$ for each $y \in Y$.*

PROOF. Let $\alpha < 1$ be the contraction constant for F. There is at most one fixed point: for if $F(x_0) = x_0$ and $F(y_0) = y_0$, then $x_0 \neq y_0$ gives the contradiction

$$d(x_0, y_0) = d(F(x_0), F(y_0)) \leq \alpha d(x_0, y_0) < d(x_0, y_0).$$

To prove existence, we shall show that for any given $y \in Y$, the sequence $\{F^n y\}$ of iterates converges to a fixed point. For this purpose, observe first that $d(Fy, F^2 y) \leq \alpha d(y, Fy)$ and, by induction, that $d(F^n y, F^{n+1} y) \leq \alpha^n d(y, Fy)$. Thus, for any n and any $p > 0$, we have

$$\begin{aligned} d(F^n y, F^{n+p} y) &\leq \sum_{i=n}^{n+p-1} d(F^i y, F^{i+1} y) \\ &\leq (\alpha^n + \cdots + \alpha^{n+p-1}) d(y, Fy) \leq \frac{\alpha^n}{1-\alpha} d(y, Fy); \end{aligned}$$

since $\alpha < 1$, so that $\alpha^n \to 0$, this shows that $\{F^n y\}$ is a Cauchy sequence and, because d is complete, that $F^n y \to u$ for some $u \in Y$ By continuity of F, we must have $F(F^n y) \to Fu$; but $\{F^{n+1} y\}$ is a subsequence of $\{F^n y\}$, so $Fu = u$ and u is a fixed point for F We have therefore shown that for each $y \in Y$, the limit of the sequence $\{F^n y\}$ exists and is a fixed point; since F has at most one fixed point, every sequence $\{F^n y\}$ converges to the same point. □

Observe that from

$$d(F^n y, F^{n+p} y) \leq \frac{\alpha^n}{1-\alpha} d(y, Fy) \quad \text{for every } p > 0,$$

we find

$$d(F^n y, u) = \lim_{p \to \infty} d(F^n y, F^{n+p} y) \leq \frac{\alpha^n}{1-\alpha} d(y, Fy);$$

the error of the nth iteration when starting from a given $y \in Y$ is therefore completely determined by the contraction constant α and the initial displacement $d(y, Fy)$.

The Banach principle has a useful local version that involves an open ball B in a complete metric space Y and a contractive map of B into Y which does not displace the center of the ball too far:

(1.2) COROLLARY. *Let (Y,d) be complete and $B = B(y_0,r) = \{y \mid d(y,y_0) < r\}$. Let $F : B \to Y$ be a contractive map with constant $\alpha < 1$. If $d(F(y_0),y_0) < (1-\alpha)r$, then F has a fixed point.*

PROOF. Choose $\varepsilon < r$ so that $d(Fy_0,y_0) \le (1-\alpha)\varepsilon < (1-\alpha)r$. We show that F maps the closed ball $K = \{y \mid d(y,y_0) \le \varepsilon\}$ into itself: for if $y \in K$, then

$$d(Fy,y_0) \le d(Fy,Fy_0) + d(Fy_0,y_0) \le \alpha d(y,y_0) + (1-\alpha)\varepsilon \le \varepsilon.$$

Since K is complete, the conclusion follows from Banach's principle. □

2. Elementary Domain Invariance

In most applications, the complete metric space Y will be a Banach space; because of this richer structure, the Banach theorem leads to a result especially useful in applications.

Let X be a subset of a Banach space E. Given a map $F : X \to E$, the map $x \mapsto x - F(x)$ of X into E is called the *field associated with* F, and is denoted by the corresponding lowercase letter: $f(x) = x - F(x)$. The field $f : X \to E$ determined by a contractive $F : X \to E$ is called a *contractive field.*

(2.1) THEOREM (Invariance of domain for contractive fields). *Let E be a Banach space, $U \subset E$ open, and $F : U \to E$ contractive with contraction constant $\alpha < 1$. Let $f : U \to E$ be the associated field, $f(x) = x - F(x)$. Then:*
(a) *$f : U \to E$ is an open mapping; in particular, $f(U)$ is open in E,*
(b) *$f : U \to f(U)$ is a homeomorphism.*

PROOF. To show that f is an open mapping, it is enough to establish that for any $u \in U$, if $B(u,r) \subset U$, then $B[f(u),(1-\alpha)r] \subset f[B(u,r)]$. For this purpose, choose any $y_0 \in B[f(u),(1-\alpha)r]$ and define $G : B(u,r) \to E$ by $G(y) = y_0 + F(y)$; then G is contractive with constant $\alpha = L(F) < 1$ and

$$\|G(u) - u\| = \|y_0 + F(u) - u\| = \|y_0 - f(u)\| < (1-\alpha)r,$$

so by (1.2), there is a $u_0 \in B(u,r)$ with $u_0 = y_0 + F(u_0)$, that is, $f(u_0) = y_0$. Thus, $B[f(u),(1-\alpha)r] \subset f[B(u,r)]$ so $f : U \to E$ is an open mapping, and in particular, $f(U)$ is open in E. To prove (b), we observe that if $u,v \in U$, then

$$\|f(u) - f(v)\| \ge \|u - v\| - \|F(u) - F(v)\| \ge (1-\alpha)\|u - v\|,$$

so that f is injective; since $f : U \to f(U)$ is a continuous open bijection, it is therefore a homeomorphism. □

(2.2) COROLLARY. *Let E be a Banach space and $F : E \to E$ be contractive. Then the corresponding field $f = I - F$ is a homeomorphism of E onto itself.*

PROOF. By (2.1) we need to show only that $f(E) = E$. Given $y_0 \in E$, define $G : E \to E$ by $x \mapsto y_0 + F(x)$; G is contractive, so it has a fixed point $x_0 = y_0 + F(x_0)$, that is, $y_0 = f(x_0)$. □

3. Continuation Method for Contractive Maps

Let (Y, d) be a complete metric space, and X a closed subset in Y with nonempty interior U and boundary $A = \partial X$. By $\mathscr{C}(X, Y)$ we denote the set of all contractive maps from X to Y

For a contractive map F from X to Y, we are concerned with the existence of solutions of the equation $x = F(x)$. One method of determining whether or not such an equation has a solution starts by embedding F in a parametrized family $\{H_\lambda\}$ of maps "joining" F to a simpler map G and then attempts to reduce the problem to that of the equation $x = G(x)$. In geometric terms, one "deforms" the graph of F to that of G and seeks to conclude from the nature of the deformation that if the graph of G intersects the diagonal $\Delta \subset X \times Y \subset Y \times Y$, then the graph of F must also do so.

Our main result of this section gives conditions under which such a conclusion is valid. Let (Λ, ϱ) be a "parameter" space with a metric ϱ. In our discussion, some special families $\{H_\lambda \mid \lambda \in \Lambda\}$ of maps in $\mathscr{C}(X, Y)$, depending on a parameter $\lambda \in \Lambda$, will be needed:

(3.1) DEFINITION. A family $\{H_\lambda \mid \lambda \in \Lambda\}$ of maps in $\mathscr{C}(X, Y)$ is called *α-contractive*, where $0 \leq \alpha < 1$, provided for some $M > 0$ and some $0 < \varkappa \leq 1$, we have

$$(*) \qquad d[H_\lambda(x_1), H_\lambda(x_2)] \leq \alpha d(x_1, x_2) \text{ for all } \lambda \in \Lambda \text{ and } x_1, x_2 \in X,$$

$$(**) \qquad d[H_\lambda(x), H_\mu(x)] \leq M[\varrho(\lambda, \mu)]^\varkappa \text{ for all } x \in X \text{ and } \lambda, \mu \in \Lambda.$$

Observe that:

(i) if $\{H_\lambda\}$ is α-contractive, then the map $H : \Lambda \times X \to Y$ given by $(\lambda, x) \mapsto H(\lambda, x) = H_\lambda(x)$ is continuous;

(ii) the map H determines the family $\{H_\lambda\}$ and vice versa;

(iii) for any parameter $\lambda \in \Lambda$, the fixed point set $\mathrm{Fix}(H_\lambda)$ is either empty or consists of exactly one fixed point denoted by x_λ;

(iv) given $x_\lambda = H_\lambda(x_\lambda)$ and $x_\mu = H_\mu(x_\mu)$, and using $(*)$ and $(**)$, we get

$$\begin{aligned} d(x_\lambda, x_\mu) &\leq d[H_\lambda(x_\lambda), H_\mu(x_\lambda)] + d[H_\mu(x_\lambda), H_\mu(x_\mu)] \\ &\leq M[\varrho(\lambda, \mu)]^\varkappa + \alpha d(x_\lambda, x_\mu) \end{aligned}$$

and therefore

$$(***)\qquad d(x_\lambda, x_\mu) \le \frac{M}{1-\alpha}[\varrho(\lambda,\mu)]^{\varkappa}$$

Let $\mathscr{C}_A(X,Y)$ be the set of all maps F in $\mathscr{C}(X,Y)$ such that the restriction $F|A : A \to Y$ is fixed point free on the boundary A of X. We are now ready to formulate the main result:

(3.2) THEOREM (Elementary implicit function theorem). *Let Λ be connected, and let $\{H_\lambda \mid \lambda \in \Lambda\}$ be an α-contractive family in $\mathscr{C}_A(X,Y)$. Then:*

(i) *if the equation $H_\lambda(x) = x$ has a solution for some $\lambda \in \Lambda$, then it has a unique solution x_λ for each $\lambda \in \Lambda$,*

(ii) *if $x_\lambda = H_\lambda(x_\lambda)$ for $\lambda \in \Lambda$, then the map $\lambda \mapsto x_\lambda$ from Λ to U is Hölder continuous.*

PROOF. (i) Consider the nonempty (by assumption) set

$$Q = \{\lambda \in \Lambda \mid x_\lambda = H_\lambda(x_\lambda) \text{ for some } x_\lambda \in U\}$$

and observe that:

(a) *Q is closed in Λ*: Indeed, let $\{\lambda_n\}$ be a sequence in Q such that $\lambda_n \to \lambda_0$; for $x_{\lambda_n} = H_{\lambda_n}(x_{\lambda_n})$ and $x_{\lambda_m} = H_{\lambda_m}(x_{\lambda_m})$, using (***), we get

$$d(x_{\lambda_n}, x_{\lambda_m}) \le \frac{M}{1-\alpha}[\varrho(\lambda_n,\lambda_m)]^{\varkappa},$$

showing that $\{x_{\lambda_n}\}$ is a Cauchy sequence. By completeness of d, $x_{\lambda_n} \to x_0$ for some x_0 in X, and hence, by continuity of H, $x_{\lambda_n} = H_{\lambda_n}(x_{\lambda_n}) \to H_{\lambda_0}(x_0)$; this gives $x_0 = H_{\lambda_0}(x_0)$, and thus we conclude that λ_0 is in Q.

(b) *Q is open in Λ*: Letting λ_0 be in Q with $x_{\lambda_0} = H_{\lambda_0}(x_{\lambda_0})$, we fix an open ball $B(x_{\lambda_0}, r) = \{x \in X \mid d(x, x_{\lambda_0}) < r\} \subseteq U$, and choose $\varepsilon > 0$ so that $\varepsilon^{\varkappa} \le (1-\alpha)r/M$, where the constants M and $\varkappa$ come from (**). Now, if λ is any point of the open ball $B(\lambda_0, \varepsilon) = \{\lambda \in \Lambda \mid \varrho(\lambda_0, \lambda) < \varepsilon\}$, then

$$d[H_\lambda(x_{\lambda_0}), x_{\lambda_0}] = d[H_\lambda(x_{\lambda_0}), H_{\lambda_0}(x_{\lambda_0})] \le M[\varrho(\lambda,\lambda_0)]^{\varkappa} < (1-\alpha)r.$$

Because, by (1.2), any such H_λ has a fixed point, we infer that $B(\lambda_0, \varepsilon) \subset Q$ and, consequently, $\lambda_0 \in \operatorname{Int}(Q)$.

Because Q is nonempty, the connectedness of the parameter space Λ, in view of (a) and (b), implies that $Q = \Lambda$; thus the proof of (i) is complete. As for (ii), it follows at once from (i) and (***). □

4. Nonlinear Alternative for Contractive Maps

For applications, we now assume that our metric space Y is a closed convex subset C of a Banach space E and our parameter space Λ is $[0,1]$. Because of this richer structure we are now able to derive the desired result:

(4.1) THEOREM (Nonlinear alternative). *Let U be a (relatively) open subset of C with $0 \in U$ Then any bounded contractive map $F : \overline{U} \to C$ has at least one of the following properties:*
(i) *F has a unique fixed point,*
(ii) *there exist $y_0 \in \partial U$ and $\lambda \in (0,1)$ such that $y_0 = \lambda F(y_0)$.*

PROOF. For $(\lambda, x) \in [0,1] \times \overline{U}$ we let $H_\lambda(x) = \lambda F(x)$. It is easily seen that $\{H_\lambda \mid \lambda \in [0,1]\}$ is an α-contractive family in $\mathscr{C}(\overline{U}, C)$ with $\varkappa = 1$. Assume first that $\{H_\lambda\}$ is fixed point free on the boundary ∂U. In this case, since $H_0(0) = 0$, we conclude, by Theorem (3.2), that $H_1 = F$ also has a fixed point in U. If $\{H_\lambda\}$ is not in $\mathscr{C}_{\partial U}(\overline{U}, C)$, then λF must have a fixed point on the boundary ∂U for some $\lambda \in [0,1]$; clearly $\lambda \neq 0$ (because $0 \in U$) and therefore: either F has a fixed point on ∂U, or property (ii) holds. □

Several fixed point theorems for contractive maps are obtained from (4.1) by imposing conditions that prevent occurrence of the second possibility:

(4.2) COROLLARY. *Let U be a (relatively) open subset of C with $0 \in U$, and let $|\ |$ be any norm in the Banach space E in which the convex set C is contained. Assume $F : \overline{U} \to C$ is a bounded contractive map such that for all $x \in \partial U$ one of the following conditions is satisfied:*
(i) $|F(x)| \leq |x|$,
(ii) $|F(x)| \leq |x - F(x)|$,
(iii) $|F(x)|^2 \leq |x|^2 + |x - F(x)|^2$,
(iv) $\langle F(x), x\rangle \leq \langle x, x\rangle$, *where $\langle\ ,\ \rangle$ is a scalar product in E.*
Then F has a unique fixed point.

PROOF. As an illustration, we prove the assertion under the hypothesis (iii). If F had no fixed point, there would be a $z \in \partial U$ with $z = \lambda F(z)$ for some $0 < \lambda < 1$; in particular $F(z) \neq 0$. Because of (iii), we would have

$$|F(z)|^2 \leq |\lambda F(z)|^2 + |\lambda F(z) - F(z)|^2$$

and therefore

$$1 \leq \lambda^2 + (\lambda - 1)^2;$$

but this is a contradiction, because $\lambda^2 + (1-\lambda)^2 < \lambda + (1-\lambda) = 1$ for every $0 < \lambda < 1$. □

The next result represents an elementary version of the antipodal theorem of Borsuk.

(4.3) COROLLARY (Antipodal theorem). *Let U be an open subset of a Banach space $(E, \|\ \|)$, symmetric with respect to the origin and with $0 \in U$, and let $F : \overline{U} \to E$ be a contractive bounded map such that $F(x) = -F(-x)$ for all $x \in \partial U$. Then F has a unique fixed point.*

PROOF. Because U is symmetric with respect to 0 and F is contractive and odd on ∂U, it follows at once that $\|F(x)\| \leq \|x\|$ for any $x \in \partial U$. Thus our assertion follows from (4.2)(i). □

5. Extensions of the Banach Theorem

There are various generalizations of the Banach theorem in arbitrary complete metric spaces where the contractive nature of the map is weakened. Many such results rely on a general principle involving the images of balls when their centers are not moved too much:

(5.1) THEOREM. *Let (X, d) be a complete metric space and $F : X \to X$ a map, not necessarily continuous. Assume*

(*) *for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $d(x, Fx) < \delta$, then $F[B(x, \varepsilon)] \subset B(x, \varepsilon)$.*

Then, if $d(F^n u, F^{n+1}u) \to 0$ for some $u \in X$, the sequence $\{F^n u\}$ converges to a fixed point for F.

PROOF. Write $F^n u = u_n$; we first show $\{u_n\}$ is a Cauchy sequence. Given $\varepsilon > 0$, choose N so large that $d(u_n, u_{n+1}) < \delta(\varepsilon)$ for all $n \geq N$; since $d(u_N, Fu_N) < \delta$, we have $F[B(u_N, \varepsilon)] \subset B(u_N, \varepsilon)$ so $Fu_N = u_{N+1} \in B(u_N, \varepsilon)$ and, by induction, $F^k u_N = u_{N+k} \in B(u_N, \varepsilon)$ for all $k \geq 0$. Thus, $d(u_k, u_s) < 2\varepsilon$ for all $s, k \geq N$ and $\{u_n\}$ is a Cauchy sequence, therefore it converges to some $z \in X$. To show z is a fixed point for F, we argue by contradiction: if $d(z, Fz) = a > 0$, we could choose a $u_n \in B(z, a/3)$ such that $d(u_n, u_{n+1}) < \delta(a/3)$; we would then have $F[B(u_n, a/3)] \subset B(u_n, a/3)$ by hypothesis, so $Fz \in B(u_n, a/3)$; but this is impossible because $d(Fz, u_n) \geq d(Fz, z) - d(u_n, z) \geq 2a/3$ so $Fz \notin B(u_n, a/3)$. Thus, $d(z, Fz) = 0$. □

To illustrate some of the techniques used in applying (5.1), we derive two generalizations of the Banach principle.

(5.2) THEOREM. *Let (X, d) be complete, and let $F : X \to X$ be a map satisfying*

$$d(Fx, Fy) \leq \varphi[d(x, y)],$$

where $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ is any nondecreasing (not necessarily continuous) function such that $\varphi^n(t) \to 0$ for each fixed $t > 0$. Then F has a unique fixed point u, and $F^n x \to u$ for each $x \in X$.

PROOF. Observe that $\varphi(t) < t$ for each $t > 0$; for if $t \leq \varphi(t)$ for some $t > 0$, then monotonicity gives $\varphi(t) \leq \varphi[\varphi(t)]$ and, by induction, $t \leq \varphi^n(t)$ for all $n > 0$. With this observation, we begin the proof. The hypothesis shows $d(F^n x, F^{n+1}x) \leq \varphi^n[d(x, Fx)]$, so $d(F^n x, F^{n+1}x) \to 0$ for each $x \in X$. Now

let $\varepsilon > 0$ be given, and choose $\delta(\varepsilon) = \varepsilon - \varphi(\varepsilon)$; if $d(x, Fx) < \delta(\varepsilon)$, then for any $z \in B(x, \varepsilon)$ we have

$$d(Fz, x) \leq d(Fz, Fx) + d(Fx, x) < \varphi[d(z, x)] + \delta \leq \varphi(\varepsilon) + \varepsilon - \varphi(\varepsilon) = \varepsilon$$

so $Fz \in B(x, \varepsilon)$. Thus (5.1) ensures that F has a fixed point, and the remainder of the proof is obvious. □

Situations arise in approximation theory where a map is known to be contractive, with contraction constant depending on the distance between the points considered. This prompts the following weakened version of the Banach theorem.

(5.3) THEOREM. *Let (X, d) be complete and $F : X \to X$ be a map satisfying*

$$d(Fx, Fy) \leq \alpha(x, y) d(x, y),$$

where $\alpha : X \times X \to \mathbb{R}^+$ has the property: for any closed interval $[a, b] \subset \mathbb{R}^+ - \{0\}$,

$$\sup\{\alpha(x, y) \mid a \leq d(x, y) \leq b\} = \lambda(a, b) < 1.$$

Then F has a unique fixed point u, and $F^n x \to u$ for each $x \in X$.

PROOF. The case $\lambda(a, a) \to 1$ as $a \to 0$ in this version does not follow from (5.2), so we give a direct proof of this theorem. For each $x \in X$, the sequence $\{d(F^n x, F^{n+1} x)\}$ is nonincreasing, therefore convergent to some $a \geq 0$. We must have $a = 0$: otherwise, $d(F^n x, F^{n+1} x) \in [a, a+1]$ for all large n; we could then choose such an n and use $c = \lambda(a, a+1)$ to get, by induction,

$$a \leq d(F^{n+k} x, F^{n+1+k} x) \leq c^k d(F^n x, F^{n+1} x) \leq c^k (a+1)$$

for all $k > 0$, which, because $c < 1$, is a contradiction.

Now, let $\varepsilon > 0$, let $\lambda = \lambda(\varepsilon/2, \varepsilon)$ and choose $\delta = \min[\varepsilon/2, \varepsilon(1 - \lambda)]$. Let $d(x, Fx) < \delta$ and $z \in B(x, \varepsilon)$; then $d(Fz, x) \leq d(Fz, Fx) + d(Fx, x)$ and we consider two cases:

(a) $d(z, x) < \varepsilon/2$: then

$$d(Fz, x) \leq d(z, x) + d(Fx, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon;$$

(b) $\varepsilon/2 \leq d(z, x) < \varepsilon$: then

$$d(Fz, x) \leq \alpha(z, x) d(z, x) + d(Fx, x) < \lambda\varepsilon + (1 - \lambda)\varepsilon = \varepsilon.$$

Thus, $F[B(x, \varepsilon)] \subset B(x, \varepsilon)$, and (5.1) ensures that F has a fixed point. The remainder of the proof is left for the reader. □

There is another way of extending the Banach theorem that does not rely on comparing $d(Fx, Fy)$ with $d(x, y)$, but concentrates instead on the

behavior of $d(x, Fx)$. Several such generalizations are based on the following general principle involving minimizing sequences for suitable real-valued functions:

(5.4) THEOREM. *Let (X, d) be complete and $\varphi : X \to R^+$ an arbitrary (not necessarily continuous) nonnegative function. Assume that*

$$(**) \qquad \inf\{\varphi(x) + \varphi(y) \mid d(x, y) \geq a\} = \mu(a) > 0 \quad \text{for all } a > 0.$$

Then each sequence $\{x_n\}$ in X for which $\varphi(x_n) \to 0$ converges to one and the same point $u \in X$.

PROOF. Let $A_n = \{x \mid \varphi(x) \leq \varphi(x_n)\}$; these sets are nonempty, and any finite family has a nonempty intersection. We show $\delta(A_n) \to 0$: given any $\varepsilon > 0$, choose N so large that $\varphi(x_n) < \frac{1}{2}\mu(\varepsilon)$ for all $n \geq N$; then for any $n \geq N$ and $x, y \in A_n$ we have $\varphi(x) + \varphi(y) < \mu(\varepsilon)$; therefore, $(**)$ gives $d(x, y) < \varepsilon$, so $\delta(A_n) \leq \varepsilon$. Thus, $\delta(A_n) \to 0$; because $\delta(\overline{A}_n) = \delta(A_n) \to 0$, we conclude from Cantor's theorem that there is a unique $u \in \bigcap_n \overline{A}_n$ and, since $x_n \in \overline{A}_n$ for each n, that $x_n \to u$. For any other sequence $\{y_n\}$ satisfying $\varphi(y_n) \to 0$ we get $\varphi(x_n) + \varphi(y_n) \to 0$, so, from $(**)$ as before, $d(x_n, y_n) \to 0$ and therefore $y_n \to u$ also. □

The following fixed point theorem is an obvious consequence:

(5.5) THEOREM. *Let (X, d) be complete and $F : X \to X$ continuous. Assume that the function $\varphi(x) = d(x, Fx)$ has the property $(**)$ and $\inf_{x \in X} d(x, Fx) = 0$. Then F has a unique fixed point.*

Observe that the Banach theorem follows from (5.5); for if $d(Fx, Fy) \leq \alpha d(x, y)$ with constant $\alpha < 1$, then the condition $(**)$ is satisfied with $\varphi(x) = d(x, Fx)$ because

$$(1 - \alpha)d(x, y) \leq d(x, y) - d(Fx, Fy) \leq d(x, Fx) + d(y, Fy)$$

and $\inf_{x \in X} d(x, Fx) = 0$, since we have seen that

$$d(F^n x, F^{n+1} x) \to 0 \quad \text{for each } x \in X.$$

6. Miscellaneous Results and Examples

A. Fixed point theorems in complete metric spaces

(A.1) Let (X, d) be complete and $F : X \to X$ a map such that $F^N : X \to X$ is contractive for some N (easy examples show that F itself need not be continuous). Prove: F has a unique fixed point u, and the sequence of iterates $F^n x \to u$ for each $x \in X$.

[Given any set X and $F : X \to X$, if F^N has a unique fixed point, then so does F.]

(A.2) Let X be a complete metric space, and $F_n : X \to X$ a sequence of continuous maps. Assume that each F_n has a fixed point x_n.

(a) Let $F_n \to F$ uniformly on X. Show: (i) If $x_n \to x_0$, or if $F(x_n) \to x_0$, then x_0 is a fixed point for F; (ii) if F is contractive then x_n converges to the unique fixed point of F.

(b) Let $F_n \to F$ pointwise, with each F_n Lipschitzian, $L(F_n) \leq M < \infty$ for all n. Prove: (i) F is Lipschitzian with $L(F) \leq M$; (ii) if $x_n \to x_0$, then x_0 is a fixed point for F; (iii) if $M < 1$, then $\{x_n\}$ converges to the unique fixed point of F.

(c) The condition $L(F_n) \leq M < 1$ in (b)(iii) cannot be relaxed to $L(F_n) < 1$ even if $L(F) < 1$. Define $F_n : l^2 \to l^2$ by

$$F_n(x_1, \ldots, x_n, \ldots) = (0, \ldots, (1 - 1/n)x_n + 1/n, 0, \ldots).$$

Then $L(F_n) < 1$ for each n and $\|x_n\| = \|(0, \ldots, 1, 0, \ldots)\| = 1$, but F_n converges pointwise to the function $F \equiv 0$.

(A.3) Let (X, d) be a locally compact complete metric space and $F : X \to X$ be contractive. Assume $F_n : X \to X$ is a sequence of contractive maps converging pointwise to F. Let x_n (respectively $\widehat{x}$) be the fixed point of F_n (respectively of F). Show: x_n converges to $\widehat{x}$ (Nadler [1968]).

(A.4) (*Parametrized version of the Banach theorem*) Let (X, d) be complete, (Λ, ϱ) be metric and let $\{H_\lambda \mid \lambda \in \Lambda\}$ be a family of contractive maps of X into itself. Assume $H : X \times \Lambda \to X$ is continuous in the second variable and for each $\lambda \in \Lambda$, let x_λ be the unique fixed point of H_λ. Prove:

(a) If all H_λ are α-contractive with $0 < \alpha < 1$, then $\lambda \mapsto x_\lambda$ is continuous.

(b) If X is locally compact, then $\lambda \mapsto x_\lambda$ is continuous.

[For (b), use (A.3).]

(A.5) Let (X, d) be complete and $\{\alpha_n\}$ a sequence of nonnegative numbers with $\sum_{n=1}^{\infty} \alpha_n < \infty$. Let $F : X \to X$ be such that $d(F^n x, F^n y) \leq \alpha_n d(x, y)$ for all $x, y \in X$. Prove: F has a unique fixed point u and $F^n x \to u$ for each $x \in X$ (Weissinger [1952]).

(A.6) Let (X, d) be complete and $F : X \to X$ be such that for any closed $A \subset X$ with $\delta(A) \neq 0$, we have $\delta(F(A)) \leq \alpha\delta(A)$, where $0 \leq \alpha < 1$. Prove: F has a fixed point (H. Amann).

(A.7) Let (X, d) be complete and $F : X \to X$ be a map satisfying $d(Fx, Fy) < d(x, y)$ for $x \neq y$.

(a) Prove: If for some $x_0 \in X$, the sequence $\{F^n x_0\}$ has a convergent subsequence, then F has a unique fixed point.

(b) Prove: If $\overline{F(X)}$ is compact (i.e., F is a compact map), then F has a unique fixed point u and $F^n x \to u$ for each $x \in X$.

(c) Construct a map $F : X \to X$ satisfying the inequality above without fixed points and such that for some $x_0, y_0 \in X$, the sequence $d(F^n x_0, F^n y_0)$ does not converge to 0.

[Consider the map $x \mapsto \ln(1 + e^x)$ of $\boldsymbol{R}$ into itself.]

(A.8) Let (Y, d) be complete. A map $F : Y \to Y$ is *expanding* if $d(Fx, Fy) \geq \beta d(x, y)$ for some $\beta > 1$ and all $x, y \in Y$. Let $F : Y \to Y$ be surjective and expanding. Prove:

(a) F is bijective.

(b) F has a unique fixed point u and $F^{-n} y \to u$ for each $y \in Y$.

(A.9) Let E be a Banach space and $F : E \to E$ a linear operator such that $(I - F)^{-1}$ exists.

(a) Let $G : E \to E$ be Lipschitzian with $\|(I - F)^{-1}\| L(G) < 1$. Show: The map $x \mapsto Fx + Gx$, $x \in E$, has a unique fixed point.

(b) Let r, λ be positive numbers with $\lambda < 1$, and let $K = K(0, r)$. Assume G $K(0, r) \to E$ is a Lipschitzian map satisfying $\|G(0)\| \leq (1 - \lambda)r / \|(I - F)^{-1}\|$. Show: If $\|(I - F)^{-1}\| L(G) < \lambda$, then the map $x \mapsto Fx + Gx$, $x \in K$, has a unique fixed point.

(A.10) Let (X, d) be complete metric and $F : X \to X$ be α-contractive with $x_0 = Fx_0$. Show: For each $\epsilon > 0$, there exists a $\beta > 0$ with $\alpha + \beta < 1$ such that for any $(\alpha + \beta)$-contractive map $G : X \to X$, the condition $d(Fx, Gx) < \beta$ for all $x \in X$ implies $y_0 = Gx_0 \in B(x_0, \epsilon)$.

(A.11) Let U be an open bounded set in a Banach space E with $0 \in U$ and $F, G : \overline{U} \to E$ be two contractive maps such that $F|\partial U = G|\partial U$. Show: $\text{Fix}(F) \neq \emptyset \Leftrightarrow \text{Fix}(G) \neq \emptyset$.

(A.12) Let U be an open bounded set in a Banach space E such that $U = -U$, and $V \subset U$ be an open nbd of the origin with $\overline{V} \subset U$. Let $F : \overline{U} \to E$ be a contractive map such that:

(i) $Fx = -F(-x)$ for all $x \in \partial U$,

(ii) there exists x_0 such that $x \neq Fx + \lambda x_0$ for all $x \in \partial V$ and $\lambda \geq 0$.

Show: F has a unique fixed point $u \in \overline{U} - V$

(A.13) Let $E = A \oplus B$ be a Banach space represented as a direct sum of two closed linear subspaces A and B with linear projections $P_A : E \to A$ and $P_B : E \to B$. Let $F : A \to E$ and $G : B \to E$ be two Lipschitzian maps, and let $f : A \to E$ and $g : B \to E$ be given by $a \mapsto a - F(a)$ and $b \mapsto b - G(b)$ respectively. Show: If

$$\|P_A\| L(F) + \|P_B\| L(G) < 1,$$

then the intersection $f(A) \cap g(B)$ consists of exactly one point.

[Prove that if $H : E \to E$ is contractive, then $x \mapsto x + H(x)$ is a homeomorphism of E onto itself.]

(A.14) (*Discrete Banach theorem*) Let Y be a set, and $\{R_n \mid n = 0, 1, \ldots\} \subset Y \times Y$ a sequence of equivalence relations such that (a) $Y \times Y = R_0 \supset R_1 \supset \cdots$, (b) $\bigcap_{n=0}^{\infty} R_n =$ the diagonal in $Y \times Y$, and (c) if $\{y_n\}$ is any sequence in Y such that $(y_n, y_{n+1}) \in R_n$ for each n, then there is a $y \in Y$ such that $(y_n, y) \in R_n$ for each n. Let $F : Y \to Y$ be a map such that whenever $(x, y) \in R_n$, then $(Fx, Fy) \in R_{n+1}$. Prove: F has a unique fixed point u, and $(F^n y, u) \in R_n$ for each n and each $y \in Y$ (S. Eilenberg).

B. Extensions of the Banach theorem

(B.1) A metric space (X, d) is *ϵ-chainable* if for each pair $x, y \in X$ there are finitely many points $x = x_0, x_1, \ldots, x_n, x_{n+1} = y$ such that $d(x_i, x_{i+1}) < \epsilon$ for all $0 \leq i \leq n$. Let (X, d) be complete, and let $F : X \to X$ be a map. Assume that there is an $\epsilon > 0$ and a $0 \leq k < 1$ such that $d(Fx, Fy) \leq k d(x, y)$ whenever $d(x, y) < \epsilon$. Prove: If (X, d) is ϵ-chainable, then F has a unique fixed point (M. Edelstein).

(B.2) Let (X, d) be complete and $F : X \to X$ a map satisfying $d(Fx, Fy) \leq \varphi[d(x, y)]$ for all $x, y \in X$, where $\varphi : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is any function such that (i) φ is nondecreasing, (ii) $\varphi(t) < t$ for each $t > 0$, (iii) φ is right continuous. Prove: F has a unique fixed point u and $F^n x \to u$ for each $x \in X$ (Browder [1968]).

[Apply (5.2) by showing that $\varphi^n(t) \to 0$ for each $t > 0$.]

(B.3) Let (X, d) be complete and $F : X \to X$ continuous. Assume that

$$d(Fx, Fy) \leq k \max[d(x,y), d(x,Fx), d(y,Fy), d(x,Fy), d(y,Fx)]$$

for some $k \in [0,1)$ and all $x, y \in X$. Prove: F has a unique fixed point u and $F^n x \to u$ for each $x \in X$ (L. B. Ćirić).

[Use diam(orbit $F^n x) \leq k$ diam(orbit $F^{n-1}x$).]

(B.4) Let (X, d) be complete and $F \; X \to X$ continuous. Assume that for each $\varepsilon > 0$ and each pair $x, y \in X$, there is an $n = n(x, y, \varepsilon)$ such that $d(F^n x, F^n y) < \varepsilon$. Prove: If the function $\varphi(x) = d(x, Fx)$ has the property (**) of (5.4), then F has a fixed point (D. F. Bailey).

(B.5) Let (X, d) be complete and $F : X \to X$ continuous. Assume that there exists an integer n and $0 \leq k < 1$ such that

$$d(Fx, Fy) \leq k[d(x, F^n z) + d(y, F^n z)]$$

for all $x, y, z \in X$. Prove: F has a unique fixed point (F. Pittnauer).

(B.6) Let (X, d) be an arbitrary metric space, and let $A \subset X$ be compact. Let $\varphi : X \to \boldsymbol{R}^+$ be an arbitrary (not necessarily continuous) nonnegative function such that

$$\inf\{\varphi(x) \mid d(x, A) \geq a\} > 0$$

for each $a > 0$. Prove: Each sequence $\{x_n\}$ in X for which $\varphi(x_n) \to 0$ contains a subsequence converging to some point of A.

(B.7) Let (X, d) be an arbitrary metric space and $F : X \to X$ a map satisfying $d(Fx, Fy) < d(x, y)$ whenever $x \neq y$. Assume that for some $z \in X$, the sequence $\{F^n z\}$ has a subsequence converging to a point u. Prove: u is a fixed point for F.

[Use (B.6) with $\varphi(x) = d(x, Fx) - d(Fx, F^2 x) + d(x, u)$.]

(B.8) If (X, d) is a metric space and $F : X \to X$ is a map, we denote the diameter of the orbit $\{F^n x \mid n = 0, 1, \ldots\}$ of $x \in X$ by $\delta(x)$. The map F is said to have *shrinking orbits* if for each x with $\delta(x) > 0$, there is an n with $\delta(F^n x) < \delta(x)$.

Let (X, d) be a bounded metric space, and $F : X \to X$ a map satisfying $d(Fx, Fy) \leq d(x, y)$ for all $x, y \in X$. Assume that for some $z \in X$, the sequence $\{F^n z\}$ has a subsequence converging to a point u. Prove: If F has shrinking orbits, then u is a fixed point for F.

[Show that $x \mapsto \delta(x)$ is continuous on X, then apply (B.6) using $\varphi(x) = \delta(x) - \delta(F^s x) + d(x, u)$ for each $s \geq 1$.]

(B.9) Let (X, d) be a metric space and $F : X \to X$ continuous. Assume that for some $z \in X$, the orbit $\{F^n z\}$ contains a convergent subsequence $\{F^{n_i} z\}$. Prove: If $d(F^{n_i} z, F^{1+n_i} z) \to 0$, then F has a fixed point.

[Use (B.6) with $\varphi(x) = d(x, u) + d(Fx, u)$, where $u = \lim F^{n_i}(x)$.]

(B.10) Let (X, d) be a metric space and $F : X \to X$ continuous. Assume that there is a continuous nonnegative real-valued function $V : X \times X \to \boldsymbol{R}^+$ with $V^{-1}(0)$ contained in the graph of F, and such that $\inf\{V(x, x) \mid x \in X\} = 0$. Show:

(a) If the function $\varphi(x) = V(x, x)$ has the property in (B.6) relative to some compact $A \subset X$, then F has a fixed point.

(b) If (X, d) is complete and $\varphi(x) = V(x, x)$ has the property (**) in (5.4), then F has a fixed point.

C. Some applications of the Banach theorem

(C.1) Let $E = (BC^1(\boldsymbol{R}^n, \boldsymbol{R}^n), \| \ \|_1)$ be the Banach space of C^1-bounded functions from $\boldsymbol{R}^n$ into itself equipped with the C^1 norm. Let $f \in E$ have a fixed point x_0 and assume that $Df_{x_0} \in \mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^n)$ does not have eigenvalue one. Show: For each $\varepsilon > 0$ there exists $\delta > 0$ such that every function $g \in E$ with $\|f - g\|_1 \le \delta$ has a unique fixed point $\widehat{x}_0$ satisfying $\|x_0 - \widehat{x}_0\| \le \varepsilon$.

[Use the inverse function theorem in $\boldsymbol{R}^n$ and the contraction principle.]

(C.2) (*Square roots in Banach algebras*) (a) Let E be a Banach algebra, and let $z \in E$ with $\|z\| < 1$. Show: There exists a unique $x \in E$ such that $\|x\| < 1$ and $x^2 - 2x + z = 0$.

[Letting $E(z)$ be the subalgebra of E generated by z, d any number satisfying $\|z\| < d < 1$, and $\widehat{K} = K(0, d) \cap E(z)$, prove that the map $F : \widehat{K} \to \widehat{K}$ given by

$$Fx = \tfrac{1}{2}(x^2 + z), \qquad x \in \widehat{K},$$

is contractive and apply the Banach theorem; use the fact that $E(z)$ is commutative.]

(b) Assume E has a unit e (i.e., $ex = xe = x$ for all $x \in E$). Show: If $\|e - z\| < 1$, then z has a "square root", i.e., there exists a unique element $y = e - x$ with $x \in E(z)$, $\|x\| < 1$ and $y^2 = z$.

(c) Let X be an arbitrary set and $B(X)$ be an algebra of bounded real functions on X equipped with the sup norm (the product being pointwise multiplication). Show: If $B(X)$ is complete and contains the unit function, then: (i) any nonnegative function $f \in B(X)$ has a square root $f^{1/2} \in B(X)$, (ii) if $f \in B(X)$, then $|f| = (f^2)^{1/2} \in B(X)$, (iii) if $f, g \in B(X)$, then $\max(f, g)$ and $\min(f, g)$ also belong to $B(X)$.

(d) A subset S of $B(X)$ *separates points* of X if for every pair (x, y) of distinct points of X there is a function $f \in S$ such that $f(x) \neq f(y)$. Show: If S is a subalgebra of $B(X)$ that separates points of X and contains the constant functions, then for any numbers $\alpha, \beta \in \boldsymbol{R}$ and any distinct $x, y \in X$, there is a $g \in S$ such that $g(x) = \alpha$ and $g(y) = \beta$.

(The above results are due to Bonsall–Stirling [1972] and J. Zemánek [1978].)

(C.3) (*Weierstrass–Stone theorem*) Let X be a compact space, $C(X)$ be the Banach algebra of all continuous real-valued functions on X, and E be any subalgebra of $C(X)$ containing the unit function and separating points of X. Show: E is dense in $C(X)$.

[Fix $f \in C(X)$ and $\varepsilon > 0$ and proceed (using (C.2)(c), (d) with $S = E$, $B(X) = \overline{E}$) in three steps:

1. For each $x, y \in X$, find $g_{x,y} \in \overline{E}$ such that $g_{x,y}(x) = f(x)$ and $g_{x,y}(y) = f(y)$.

2. For each $y \in X$ choose a nbd U_y of y such that $g_{x,y}(z) < f(z) + \varepsilon$ for all $z \in U_y$. Supposing $\{U_{y_j} \mid j = 1, \ldots, t\}$ cover X, define $h_x \in \overline{E}$ as the minimum of the corresponding $\{g_{x,y_j} \mid j = 1, \ldots, t\}$ and note that $h_x(x) = f(x)$ for all $x \in X$ and

$$h_x(z) < f(z) + \varepsilon \qquad \text{for all } z \in X.$$

3. For each $x \in X$ choose a nbd V_x of x on which $h_x(z) > f(z) - \varepsilon$; letting $\{V_{x_i} \mid i = 1, \ldots, s\}$ cover X, define $h \in \overline{E}$ as the maximum of the corresponding $\{h_{x_i} \mid i = 1, \ldots, s\}$ and show $\|f - h\| < \varepsilon$.]

(C.4) (*Geometry of fractals*) Let (X, d) be a metric space and $(\mathcal{CB}(X), D)$ the space of nonempty closed bounded subsets of X with the Hausdorff metric

$$D(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

We denote by $\mathcal{K}(X)$ the subspace of $\mathcal{CB}(X)$ consisting of the compact subsets of X.

By an *iterated function system* (or briefly an *IF-system*) we understand a system

$$(*) \qquad ((X,d); f_1, \dots, f_k)$$

consisting of a complete metric space (X,d) together with α_i-contractions $f_i : X \to X$, $i = 1, \dots, k$. Given such a system we define $F : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$F(A) = \bigcup_{i=1}^{k} f_i(A) \qquad \text{for } A \in \mathcal{K}(X).$$

Show:

(a) F is an α-contraction with $\alpha = \max\{\alpha_1, \dots, \alpha_k\}$.

(b) There exists a unique $B \in \mathcal{K}(X)$ (called the *attractor* for the IF-system $(*)$) such that $B = \bigcup_{i=1}^{k} f_i(B)$.

(c) For each $x \in X$, the iterates $F^k(x)$ converge to B in the Hausdorff metric on $\mathcal{K}(X)$.

[Use the fact that $(\mathcal{K}(X), D)$ is complete; cf. Kuratowski [1966].]

(The above results are due to Hutchinson [1981].)

(C.5) (*Stability of attractors*) Let (Λ, ϱ) be a metric space and

$$(*_\lambda) \qquad ((X,d); f_{1,\lambda}, \dots, f_{k,\lambda})$$

be a family of IF-systems depending on a parameter $\lambda \in \Lambda$. Assume furthermore that for any $i = 1, \dots, k$ we have:

(a) $f_{i,\lambda} : X \to X$ is α_i-contractive for all $\lambda \in \Lambda$.

(b) $\lambda \mapsto f_{i,\lambda}(x)$ is continuous for each $x \in X$.

Show: The attractor B_λ of the IF-system $(*_\lambda)$ depends continuously on λ (cf. Jachymski [1996]).

(C.6) (*Dynamic programming*) Let X be an arbitrary set and $B(X)$ the Banach space of all bounded real-valued functions on X equipped with the sup norm and the natural partial order relation $f \le g$. Let E be a linear subspace of $B(X)$ containing the constant functions and $F : E \to E$ be a map (not necessarily continuous) satisfying:

(a) $f \le g \Rightarrow F(f) \le F(g)$ for all $f, g \in E$,

(b) there exists a positive constant $\alpha < 1$ such that for any constant function $x \mapsto c$ on X we have

$$F(f + c) \le F(f) + \alpha c \qquad \text{for all } f \in E.$$

Prove: F has a unique fixed point (Blackwell [1965]).

(C.7) (*Shadowing property*) Let (X,d) be a metric space and $f : X \to X$. A sequence $\{x_n\}_{n \in Z}$ is called an *orbit* (respectively a *δ-orbit*, with $\delta > 0$) for f provided $f(x_n) = x_{n+1}$ (respectively $d(f(x_n), x_{n+1}) \le \delta$) for all $n \in Z$. We say that f has the *shadowing property* if for each $\varepsilon > 0$ we can find a $\delta > 0$ such that given any δ-orbit $\{x_n\}$, there exists an orbit $\{y_n\}$ such that $d(x_n, y_n) \le \varepsilon$ for all $n \in Z$.

(a) Let (X,d) be complete and $f : X \to X$ be α-contractive. Show: f has the shadowing property.

[Given $\varepsilon > 0$, let $\{x_n\}$ be a δ-orbit for f with $\delta = \varepsilon(1-\alpha)$. Define (M, ϱ) by $M = \{y = \{y_n\}_{n \in Z} \mid d(x_n, y_n) \le \varepsilon \text{ for all } n \in Z\}$, $\varrho(y,z) = \sup\{d(y_n, z_n) \mid n \in Z\}$ and apply the Banach principle to the map $F : M \to M$ given by $y \mapsto \{f(y_{n-1})\}_{n \in Z}$.]

(b) Let E be a Banach space and $L \in \mathrm{GL}(E)$ a linear isomorphism. We say that L is *hyperbolic* provided $E = E_1 \oplus E_2$ and $L = L_1 \oplus L_2$, where $L_i \in \mathrm{GL}(E_i)$, $i = 1, 2$, with

$\|L_1\| < 1$ and $\|L_2^{-1}\| < 1$. Show: If $L \in \mathrm{GL}(E)$ is hyperbolic, then L has the shadowing property.

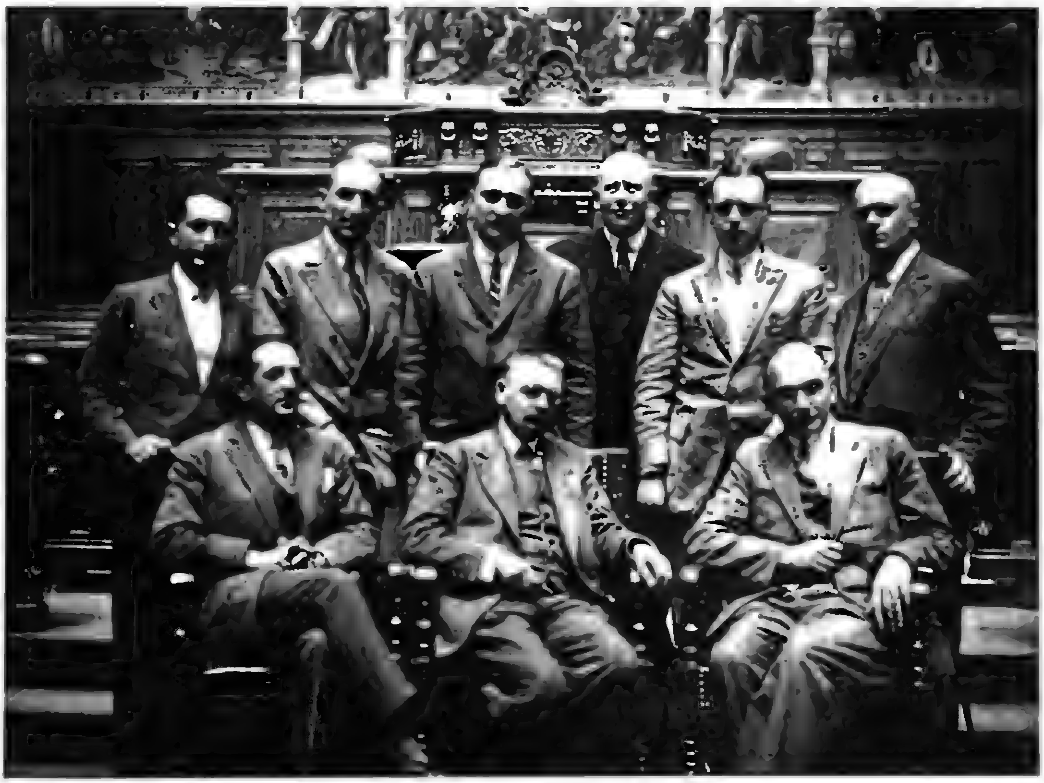
(The above results are due to Ombach [1993].)

7. Notes and Comments

Contraction principle and elementary domain invariance

Theorem (1.1) is due to Banach [1922]; it is an abstraction of the classical method of successive approximations, introduced by Liouville [1837] and developed systematically for the first time by Picard [1890].

The arguments based on domain invariance and their applications to analysis were introduced by Schauder. Under some additional (unnecessary) assumptions Theorem (2.1) appeared in Schauder [1933], as a special case of domain invariance for maps of the form $f(x) = x - [F(x) + G(x)]$, where F is a compact map and G is contractive.



Lwów, 1929. *Top row*: K. Kuratowski, B. Knaster, S. Banach, W. Stożek, E. Żyliński, S. Ruziewicz. *Bottom row*: H. Steinhaus, E. Zermelo, S. Mazurkiewicz

Continuation method for contractive maps

The presentation of this method follows Granas [1994]. The elementary implicit function theorem (3.2) appears here for the first time; its "continuous"

and "differentiable" versions can be found in (4.6.E.1). When the space Λ of parameters is $[0,1]$ and (X,d) is a Banach space, Theorem (3.2) yields the nonlinear alternative and Leray–Schauder type results for contractive maps. For some applications of these results to differential equations in Hilbert and Banach spaces, see Frigon–Granas [1994] and Lee -O'Regan [1995].

More general and related results

For (5.2) see Matkowski [1975]; for generalized contractions of (5.3) see the book by Krasnosel'skiĭ et al. [1969] and Dugundji–Granas [1978b]. Eilenberg (see (A.9)) formulated a discrete analogue of the Banach theorem, which has applications in automata theory. For various other extensions of the Banach theorem see Edelstein [1961], [1962], Dugundji [1976], the survey by Browder [1976], and "Miscellaneous Results and Examples". Some noteworthy extensions and applications of the Banach theorem are also given in Section I(a) of "Additional References"

Concerning extensions of (2.1), we remark that for the generalized contractions in (5.2) and (5.3) (or, more generally, for a continuous map having: (i) the property $(*)$ of (5.1) and (ii) a unique fixed point u to which $F^n x$ converges for each $x \in X$) the corresponding field in a Banach space will have the domain invariance property; this can be seen directly, by examining the proof of (2.1). Even more important, however, is that if a continuous map F of a complete metric space (X,d) into itself satisfies (i) and (ii), then X has an equivalent complete metric under which F becomes contractive; this follows from the result of Meyers quoted below.

Concerning some extensions of the continuation method for contractive maps, see Frigon–Granas [1994], [1998] and Frigon [1996].

Converse of the Banach theorem

Bessaga [1959] obtained the following result: *Let $F : X \to X$ be a map of an abstract set X into itself such that each iterate F^n $(n = 1, 2, \dots)$ has a unique fixed point. Let α be any number with $0 < \alpha < 1$. Then there is a complete metric d for X such that F is contractive with contraction constant α.* In another direction goes the following theorem of Meyers [1967]: *Let X be a complete metric space and $F : X \to X$ be a map satisfying* (i) $F(x_0) = x_0$ *for some* $x_0 \in X$, (ii) $\lim_{n\to\infty} F^n(x) = x_0$ *for all* $x \in X$, (iii) *there is a nbd U of x_0 such that for any nbd V of x_0 there is n_V such that $F^n(V) \subset U$ for all $n \geq n_V$. Then for each $\alpha \in (0,1)$ there is an equivalent complete metric on X such that F is contractive with contraction constant α.* For more details on the converse of the Banach theorem the reader is referred to the survey by Opoĭtsev [1976].

§2. Order-Theoretic Results

The Banach theorem and its generalizations are founded on the notion of completeness. We now present some fixed point theorems based on the considerations of order; these results have proved of importance in algebra, the theory of automata, mathematical linguistics, linear functional analysis, approximation theory and the theory of critical points.

1. The Knaster–Tarski Theorem

Let $(P, \preceq)$ be a partially ordered set and $M \subset P$ a nonempty subset. Recall that an *upper* (respectively *lower*) *bound* for M is an element $p \in P$ with $m \preceq p$ (respectively $m \succeq p$) for each $m \in M$; the *supremum* of M, if it exists, is an upper bound for M which is a lower bound for the set of all upper bounds of M. Recall also that a linearly ordered subset in P is called a *chain*. A map $F : P \to P$ is *isotone* if $F(x) \preceq F(y)$ whenever $x \preceq y$.

(1.1) THEOREM (Knaster–Tarski). *Let $(P, \preceq)$ be a partially ordered set and $F : P \to P$ isotone. Assume that there is a $b \in P$ such that both*
(1) *$b \preceq F(b)$ and*
(2) *every chain in $\{x \in P \mid x \succeq b\}$ has a supremum.*
Then the set of fixed points of F is not empty, and among them there is a fixed point λ maximal in P (i.e. $F(\lambda) = \lambda$ and there is no fixed point a with $a \succ \lambda$).

PROOF. Consider the partially ordered set

$$Q = \{x \mid x \preceq F(x)\} \cap \{x \mid x \succeq b\};$$

this is not empty, because $b \in Q$. Every chain C in Q has an upper bound: for if $u = \sup C$, then $c \preceq u$ for each $c \in C$, so by the isotone property, $c \preceq F(c) \preceq F(u)$ for each $c \in C$, showing that $F(u)$ is an upper bound for C, therefore that $u \preceq F(u)$ and thus $u \in Q$. By the Kuratowski–Zorn lemma, there is a maximal element λ in Q; since $\lambda \preceq F(\lambda)$, we have $F(\lambda) \preceq F[F(\lambda)]$, so $F(\lambda) \in Q$, and if $\lambda \neq F(\lambda)$, this would contradict the maximality of λ. Thus, λ is a fixed point, and it is clearly maximal in P □

In a partially ordered set P, it has been fruitful to regard a countable chain as being a "sequence" and the supremum of that chain, if it exists, as being the limit of that "sequence". Guided by this, we define a map $F : P \to P$ to be *continuous* if for each countable chain $\{c_i\}$ having a supremum, $F(\sup\{c_i\}) = \sup[\{F(c_i)\}]$. Observe that a continuous map $F : P \to P$ is necessarily isotone; for if $x \preceq y$, then $y = \sup\{x, y\}$, so by continuity, we must have $F(y) = \sup\{F(x), F(y)\}$ therefore $F(x) \preceq F(y)$. For continuous

maps $F : P \to P$, the conditions on P in (1.1) can be relaxed, and a fixed point can be found by the method of successive approximations. This is given in

(1.2) THEOREM (Tarski–Kantorovitch). *Let $(P, \preceq)$ be a partially ordered set and $F : P \to P$ continuous. Assume that there is a $b \in P$ such that both*

(1) $b \preceq F(b)$ and

(2) every countable chain in $\{x \mid x \succeq b\}$ has a supremum.

Then F has a fixed point $\mu = \sup_n F^n(b)$, and μ is the infimum of the set of fixed points of F in $\{x \mid x \succeq b\}$.

PROOF. Because $b \preceq F(b)$ and F is isotone, we find $F(b) \preceq F^2(b)$ and, inductively, that $F^n(b) \preceq F^{n+1}(b)$ for each $n \geq 1$. Thus, $\{F^n(b) \mid n \geq 1\}$ is a chain in $\{x \mid x \succeq b\}$ so $\mu = \sup F^n(b)$ exists. Since F is continuous,

$$F(\mu) = \sup F\{F^n(b)\} = \sup F^{n+1}(b) = \mu,$$

and μ is a fixed point. To complete the proof, we must show that if $\widehat{\mu}$ is any other fixed point in $\{x \mid x \succeq b\}$, then $\mu \preceq \widehat{\mu}$. To establish this, observe that since $b \preceq \widehat{\mu}$, we have $F(b) \preceq F(\widehat{\mu}) = \widehat{\mu}$ and, by induction, that $F^n(b) \preceq \widehat{\mu}$ for every $n \geq 1$; thus $\widehat{\mu}$ is an upper bound for $\{F^n(b) \mid n \geq 1\}$, so $\mu \preceq \widehat{\mu}$ and the proof is complete. □

The constructive character of the formula for the minimal fixed point μ in (1.2) is quite important for applications.

2. Order and Completeness. Theorem of Bishop–Phelps

The notion of order and the notion of completeness have each led to a fixed point theorem. We now obtain some results based on an interplay of these two notions.

Let $\varphi : X \to \boldsymbol{R}$ be any real-valued function on a metric space (X, d) and λ a positive number. Following Bishop–Phelps, define a relation $\preceq_{\varphi,\lambda}$ on X by

$$x \preceq_{\varphi,\lambda} y \quad \text{if and only if} \quad \lambda d(x, y) \leq \varphi(x) - \varphi(y).$$

It is easy to verify that $\preceq_{\varphi,\lambda}$ is a partial ordering in X; the transitivity of $\preceq_{\varphi,\lambda}$ follows from the triangle inequality. The space X, together with this partial ordering, is denoted by $X_{\varphi,\lambda}$; we let $X_\varphi = X_{\varphi,1}$ and write $\preceq_\varphi$ or simply $\preceq$ for $\preceq_{\varphi,1}$.

Observe that if $x, y \in X_{\varphi,\lambda}$ are known to be related, then the condition $\varphi(y) \leq \varphi(x)$ alone ensures that $\lambda d(x, y) \leq \varphi(x) - \varphi(y)$, i.e., $x \preceq_{\varphi,\lambda} y$.

Recall that a function $\varphi : X \to \boldsymbol{R}$ is called *lower semicontinuous* (l.s.c.) whenever $\{x \in X \mid \varphi(x) \leq a\}$ is closed for each $a \in \boldsymbol{R}$, and *upper semicontinuous* (u.s.c.) if $-\varphi$ is lower semicontinuous.

(2.1) THEOREM (Bishop–Phelps). *Let (X, d) be complete and $\varphi : X \to \boldsymbol{R}$ be a lower semicontinuous function with a finite lower bound. Then for any $x_0 \in X_{\varphi,\lambda}$, there is a maximal element $x^* \in X_{\varphi,\lambda}$ with $x_0 \preceq_{\varphi,\lambda} x^*$ Precisely: for any $x_0 \in X$ there is an $x^* \in X$ such that*

$$\varphi(x^*) + \lambda d(x_0, x^*) \leq \varphi(x_0)$$

and

$$\varphi(x^*) < \varphi(x) + \lambda d(x, x^*) \qquad \textit{for any } x \neq x^*$$

PROOF. Clearly (for the proof) we may suppose that $\lambda = 1$ and consider $X_\varphi = X_{\varphi,1}$. For any $z \in X_\varphi$, denote the terminal tail $\{y \mid y \succeq z\}$ by $T(z)$. We observe that since $T(z) = \{y \mid \varphi(y) + d(z, y) \leq \varphi(z)\}$ and the map $y \mapsto \varphi(y) + d(z, y)$ is lower semicontinuous, each $T(z)$ is closed in X.

Now let $x_0 \in X_\varphi$ be given. We construct an ascending sequence $x_0 \preceq x_1 \preceq x_2 \preceq \cdots$ inductively, first choosing $x_1 \in T(x_0)$ so that $\varphi(x_1) \leq 1 + \inf[\varphi | T(x_0)]$ and when $x_0, \ldots, x_{n-1}$ have been selected, choosing $x_n \in T(x_{n-1})$ so that

$$\varphi(x_n) \leq 1/n + \inf[\varphi | T(x_{n-1})].$$

The sequence $T(x_0) \supset T(x_1) \supset \cdots$ of closed sets is clearly descending. We estimate the diameter of each $T(x_n)$ whenever $n \geq 1$: given $\xi \in T(x_n) \subset T(x_{n-1})$, we have $\varphi(\xi) \geq \inf[\varphi | T(x_{n-1})] \geq \varphi(x_n) - 1/n$, so since $x_n \preceq \xi$, we find $d(x_n, \xi) \leq \varphi(x_n) - \varphi(\xi) \leq 1/n$. This implies that $\operatorname{diam} T(x_n) \leq 2/n$ for each $n \geq 1$, so by Cantor's theorem, there is a unique $x^* \in \bigcap_{n=0}^{\infty} T(x_n)$. Since $x^* \in T(x_0)$, we have $x^* \succeq x_0$; moreover, x^* is maximal in X_φ: for if $z \succeq x^*$, then $z \succeq x^* \succeq x_n$ for all $n \geq 0$, so $z \in \bigcap_{n=0}^{\infty} T(x_n)$ and therefore $z = x^*$ □

This leads immediately to

(2.2) THEOREM (Caristi). *Let (X, d) be complete and $\varphi : X \to \boldsymbol{R}$ a lower semicontinuous function with a finite lower bound. Let $F : X \to X$ be any (not necessarily continuous) function such that $d(x, Fx) \leq \varphi(x) - \varphi(Fx)$ for each $x \in X$. Then F has a fixed point.*

PROOF. Consider the partially ordered set X_φ, and let x_0 be a maximal element. Since $d(x_0, Fx_0) \leq \varphi(x_0) - \varphi(Fx_0)$, we have $x_0 \preceq Fx_0$ in X_φ, and since x_0 is maximal, it follows that $x_0 = Fx_0$. □

We remark that the existence of a fixed point for a contractive map F in a complete metric space (X, d) is a consequence of (2.2); for if we have

$d(Fx, Fy) \le \alpha d(x, y)$ with $\alpha < 1$, then $d(Fx, F^2x) \le \alpha d(x, Fx)$; therefore

$$d(x, Fx) - \alpha d(x, Fx) \le d(x, Fx) - d(Fx, F^2x),$$

so with the nonnegative function $\varphi(x) = (1-\alpha)^{-1} d(x, Fx)$, the conditions of (2.2) are satisfied.

3. Fixed Points for Set-Valued Contractive Maps

Let (X, d) and (Y, ϱ) be metric spaces and $(\mathcal{CB}(Y), D)$ be the space of nonempty closed bounded subsets of Y with the Hausdorff metric

$$D(A, B) = \max\{\sup_{a \in A} \varrho(a, B), \sup_{b \in B} \varrho(b, A)\}.$$

A set-valued map $\mathcal{F} : X \to \mathcal{CB}(Y)$ is called *α-contractive*, where $0 \le \alpha < 1$, if

$$D(\mathcal{F}x, \mathcal{F}y) \le \alpha d(x, y) \quad \text{for all } x, y \in X.$$

The following result extends the Banach principle to set-valued contractive maps.

(3.1) THEOREM (Nadler). *Let (X, d) be a complete metric space and $\mathcal{F} : X \to \mathcal{CB}(X)$ an α-contractive map. Then $\mathcal{F}$ has a fixed point.*

PROOF. First, notice that for any $x \in X$ and any $y \in \mathcal{F}x$, we have

$$d(y, \mathcal{F}y) \le D(\mathcal{F}x, \mathcal{F}y) \le \alpha d(x, y).$$

Now fix $\varepsilon > 0$. For any $x \in X$ there exists $y_\varepsilon(x) \in \mathcal{F}x$ such that

$$d(x, y_\varepsilon(x)) \le (1+\varepsilon) d(x, \mathcal{F}x).$$

From this we get

$$\begin{aligned}\left[\left(\frac{1}{1+\varepsilon}\right) - \alpha\right] d(x, y_\varepsilon(x)) &\le d(x, \mathcal{F}x) - \alpha d(x, y_\varepsilon(x)) \\ &\le d(x, \mathcal{F}x) - d(y_\varepsilon(x), \mathcal{F}y_\varepsilon(x)).\end{aligned}$$

Let

$$\varphi_\varepsilon(x) = \left[\left(\frac{1}{1+\varepsilon}\right) - \alpha\right]^{-1} d(x, \mathcal{F}x).$$

If ε is chosen such that $(1+\varepsilon)^{-1} > \alpha$, then φ_ε is continuous and bounded below. Furthermore, we have just shown that for any $x \in X$,

$$x \preceq_{\varphi_\varepsilon} y_\varepsilon(x) \quad \text{and} \quad y_\varepsilon(x) \in \mathcal{F}x.$$

Let x^* be a maximal element for the partial order $\preceq_{\varphi_\varepsilon}$ in X_{φ_ε}. From $x^* \preceq_{\varphi_\varepsilon} y_\varepsilon(x^*)$ we get $x^* = y_\varepsilon(x^*)$, and therefore $x^* \in \mathcal{F}x^*$. □

4. Applications to Geometry of Banach Spaces

Let $K = K(z, r)$ be a closed ball in a Banach space. For any $x \notin K$, the convex hull of x and K is called a *drop* and is denoted by $\mathcal{D}(x, K)$; it is clear that if $y \in \mathcal{D}(x, K)$, then $\mathcal{D}(y, K) \subset \mathcal{D}(x, K)$, and if $z = 0$, that $\|y\| \leq \|x\|$.

(4.1) THEOREM (Daneš). *Let A be a closed subset of a Banach space E, let $z \in E - A$, and let $K = K(z, r)$ be a closed ball of radius $r < d(z, A) = R$. Let $F : A \to A$ be any map such that $F(a) \in A \cap \mathcal{D}(a, K)$ for each $a \in A$. Then for each $x \in A$, the map F has a fixed point in $A \cap \mathcal{D}(x, K)$.*

PROOF. We can assume $z = 0$. Let $\|x\| = \varrho \geq R$ and let $X = A \cap \mathcal{D}(x, K)$; clearly, F maps X into itself; we shall estimate $\|y - F(y)\|$ on X.

Given $y \in X$, there is a $b \in K$ with $F(y) = tb + (1 - t)y$; since $\|F(y)\| \leq t\|b\| + (1 - t)\|y\|$, we have

$$t(\|y\| - \|b\|) \leq \|y\| - \|F(y)\|,$$

so because $\|y\| - \|b\| \geq R - r$, we find

$$t \leq \frac{\|y\| - \|Fy\|}{R - r}.$$

Thus,

$$\begin{aligned} \|y - F(y)\| &\leq t\|y - b\| \leq t(\|y\| + \|b\|) \leq t(\varrho + r) \\ &\leq \frac{\varrho + r}{R - r}(\|y\| - \|F(y)\|). \end{aligned}$$

Therefore, applying the theorem of Caristi with

$$\varphi(x) = \frac{\varrho + r}{R - r}\|x\|$$

yields the result. □

As a consequence we obtain

(4.2) THEOREM (Supporting drops theorem). *Let A be a closed set in a Banach space E, and $z \in E - A$ a point with $d(z, A) = R > 0$. Then for any $r < R < \varrho$ there is an $x_0 \in \partial A$ with*

$$\|z - x_0\| \leq \varrho \quad \textit{and} \quad A \cap \mathcal{D}(x_0, K(z, r)) = \{x_0\}.$$

PROOF. Let $\widetilde{A} = A \cap K(z, \varrho)$. It is a nonempty closed subset of E. Let $K = K(z, r)$. For each $x \in \widetilde{A}$ choose $F(x) \in \widetilde{A} \cap \mathcal{D}(x, K)$ such that $F(x) \neq x$ if $A \cap \mathcal{D}(x, K) \neq \{x\}$. One can easily see that fixed points x_0 of F can occur only at points of ∂A and that $\widetilde{A} \cap \mathcal{D}(x_0, K) = A \cap \mathcal{D}(x_0, K)$. □

5. Applications to the Theory of Critical Points

Let $\varphi : X \to \boldsymbol{R}$ be a real-valued function on a metric space X with finite $\eta = \inf\{\varphi(x) \mid x \in X\}$. Recall that a *minimizer* (respectively a *strict minimizer*) of φ is an element $x_0 \in X$ with $\varphi(x_0) = \eta$ (respectively such that the relation $\varphi(z) \leq \varphi(x_0)$ implies $z = x_0$). Recall also that a sequence $\{x_n\}$ in X for which $\varphi(x_n) \to \eta$ is called a *minimizing sequence* for φ.

(5.1) THEOREM (Ekeland). *Let (X, d) be complete, and let $\varphi : X \to \boldsymbol{R}$ be a lower semicontinuous function with finite lower bound η. Let $\{x_n\}$ be a minimizing sequence for φ and $\lambda_n = (\varphi(x_n) - \eta)^{1/2} > 0$. Then there exists a minimizing sequence $\{y_n\}$ for φ such that for any natural n we have:*

(i) $\varphi(y_n) \leq \varphi(x_n)$ *and* $d(x_n, y_n) \leq \lambda_n$,

(ii) y_n *is a strict minimizer of the function* $\varphi_n : X \to \boldsymbol{R}$ *given by*

$$\varphi_n(z) = \varphi(z) + \lambda_n d(z, y_n) \quad \textit{for } z \in X,$$

(iii) $\varphi(y_n) = \varphi_n(y_n) \leq \varphi(z) + \lambda_n d(z, y_n)$ *for* $z \in X$.

PROOF. We first describe the construction of $\{y_n\}$. For a given natural n, consider the space X_{φ,λ_n}, where $\lambda_n = (\varphi(x_n) - \eta)^{1/2}$. By the Bishop–Phelps theorem applied in X_{φ,λ_n}, for the point x_n there exists an element y_n in X_{φ,λ_n} such that (a) $x_n \preceq_{\varphi,\lambda_n} y_n$ and (b) y_n is maximal in X_{φ,λ_n}. We now show that y_n and the function φ_n defined in (ii) have the properties (i)–(iii).

Indeed, the relation $x_n \preceq_{\varphi,\lambda_n} y_n$ in X_{φ,λ_n} translates into the estimate

$$\lambda_n d(x_n, y_n) \leq \varphi(x_n) - \varphi(y_n),$$

and gives

$$d(x_n, y_n) \leq \frac{1}{\lambda_n}(\varphi(x_n) - \varphi(y_n)) \leq \frac{1}{\lambda_n}(\eta + \lambda_n^2 - \eta) = \lambda_n;$$

thus (i) is satisfied.

To establish (ii), suppose that $\varphi_n(z) \leq \varphi_n(y_n)$ for some z in X; then

$$\varphi_n(z) = \varphi(z) + \lambda_n d(z, y_n) \leq \varphi(y_n) = \varphi_n(y_n),$$

which (by the definition of the order in X_{φ,λ_n}) gives $y_n \preceq_{\varphi,\lambda_n} z$. Since y_n is maximal in X_{φ,λ_n}, the last relation implies $y_n = z$, showing that y_n is a strict minimizer of φ_n, as asserted.

(iii) is an obvious consequence of (ii).

Thus we constructed a minimizing sequence $\{y_n\}$ satisfying (i)–(iii). $\square$

(5.2) COROLLARY. *Let E be a Banach space, $\varphi : E \to \boldsymbol{R}$ be a differentiable function on E with finite lower bound η, and $\{x_n\}$ a minimizing sequence for φ. Then there exists a minimizing sequence $\{y_n\}$ in E for φ such that $\varphi(y_n) \leq \varphi(x_n)$ for each n and $D\varphi(y_n) \to 0$ in E^*.*

PROOF. By (5.1), there exists a minimizing sequence $\{y_n\}$ in E for φ such that for all n, $\varphi(y_n) \leq \varphi(x_n)$, and with $\lambda_n = (\varphi(x_n) - \eta)^{1/2}$,

$$(*) \qquad \varphi(y_n) \leq \varphi(z) + \lambda_n \|z - y_n\| \quad \text{for all } z \in E.$$

For a given n, letting $z = y_n + v$ we obtain from $(*)$ the estimate

$$\begin{aligned} \varphi(y_n) &\leq \varphi(y_n + v) + \lambda_n \|(y_n + v) - y_n\| \\ &= \varphi(y_n + v) + \lambda_n \|v\| \quad \text{for all } v \in E, \end{aligned}$$

and consequently,

$$\|D\varphi(y_n)\|_{E^*} = \lim_{\varrho \to 0} \sup_{\substack{\|v\| \leq \varrho \\ v \neq 0}} \frac{\varphi(y_n) - \varphi(y_n + v)}{\|v\|} \leq \lambda_n.$$

Thus, $\|D\varphi(y_n)\|_{E^*} \leq \lambda_n$ for all n, and as $\lambda_n \to 0$, our assertion follows. □

6. Miscellaneous Results and Examples

A. Fixed points in partially ordered sets

(A.1) Let X, Y be two sets and $f : X \to Y$, $g : Y \to X$ two maps. Show that X and Y can be written as disjoint unions, $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$, where $f(X_1) = Y_1$ and $g(Y_2) = X_2$. Derive the Bernstein–Schroeder theorem: If both f and g are injective, then there exists a bijective $F : X \to Y$

[Consider the map $\varphi : 2^X \to 2^X$ given by $A \mapsto X - g[Y - f(A)]$ and use (1.1).]

(A.2) A partially ordered set is called a *complete semilattice* if every nonempty subset has a supremum. In a complete semilattice L, show:

(a) Every nonempty set having a lower bound has an infimum.

(b) An isotone map $f : L \to L$ such that $b \preceq f(b)$ for some $b \in L$, has a maximal fixed point and a minimal fixed point in $\{x \mid b \preceq x\}$.

[For (a), consider the sup of the set of lower bounds.]

(A.3) A partially ordered set is a *Dedekind-complete semilattice* if every nonempty subset having an upper bound has a supremum. Show that (1.1) need not be true for Dedekind-complete semilattices.

[Consider the map $x \mapsto x + 1$ of $\boldsymbol{R}$ into itself.]

(A.4) Let f, g be isotone maps of a complete semilattice L into itself such that $fg(x) = gf(x)$ for each $x \in L$. Prove: If gf has a fixed point, then f and g have a common fixed point.

[By (1.1), gf has a maximal fixed point.]

(A.5) Let f, g be isotone maps of a complete semilattice into itself such that (a) if $x \preceq f(x)$, then $f(x) \preceq g(x)$, and (b) if $f(x) \preceq x$, then $g(x) \preceq f(x)$. Show: If either f or g has a fixed point, then f and g have a common fixed point.

(A.6) Let P be a partially ordered set and $f\ \ P \to P$ be isotone. Assume $f(u) \preceq u$ for some $u \in P$ and that every chain in $\{x \mid x \preceq u\}$ has an infimum. Show: f has a fixed point minimal in $\{x \mid x \preceq u\}$.

[Consider the set P with the partial order $x < y$ if and only if $y \preceq x$ and use (1.1).]

(A.7) Let P be a partially ordered set and $f : P \to P$ a map such that $f(\inf C) = \inf f(C)$ for every countable chain. Assume $f(u) \preceq u$ for some $u \in P$ and that every countable chain in $\{x \mid x \preceq u\}$ has an infimum. Show: f has a fixed point maximal in $\{x \mid x \preceq u\}$.

(A.8) Let P be a partially ordered set and $\mathscr{F}$ a nonempty commutative family of isotone maps (i.e., $f \circ g = g \circ f$ for every $f, g \in \mathscr{F}$). Assume that $f(u) \preceq u$ for some $u \in P$ and all $f \in \mathscr{F}$, and that every chain in $\{x \mid x \preceq u\}$ has an infimum. Show: There is a minimal common fixed point for $\mathscr{F}$ (i.e., $f(a) = a$ for all $f \in \mathscr{F}$ and no $a_0 < a$ has this property) (DeMarr [1964]).

(A.9) Let E be a topological space and $C \subset E$ a nonempty compact subset. Let $P = \{A \subset E \mid A \cap C$ is closed and nonempty$\}$, partially ordered by inclusion. Let $\mathscr{F}$ be any nonempty commutative family of isotone maps of P into itself. Show: There exists a minimal nonempty $A \subset E$ with $f(A) = A$ for all $f \in \mathscr{F}$

(A.10) Let E be a connected compact space, and $\mathscr{F}$ a commutative family of continuous maps $f : E \to E$. Show: There is a nonempty connected closed $A \subset E$ with $f(A) = A$ for all $f \in \mathscr{F}$ and such that no nonempty connected closed $B \subset E$ which is a proper subset of A has this property.

[Partially order the nonempty connected closed sets by inclusion and use (A.8).]

B. Results related to the Bishop–Phelps theorem

(B.1) (*Expanding maps*) Let $X = (X, \preceq)$ be a partially ordered set; for every $z \in X$, denote the terminal tail $\{y \mid z \preceq y\}$ by $T(z)$.

(a) A map $F : X \to X$ is said to be *expanding* if $x \preceq F(x)$ for each $x \in X$. Show: If $F : X \to X$ is expanding, then: (i) every tail in X is invariant under F (i.e., $F(T(z)) \subset T(z)$), (ii) each maximal element of X is a fixed point of F.

(b) A set-valued map $\mathcal{F} : X \to 2^X$ is *expanding* if it admits a single-valued expanding selector $F\ \ X \to X$. Prove: If $\mathcal{F} : X \to 2^X$ is expanding then: (i) for every tail $T(z)$ in X, $x \mapsto \mathcal{F}(x) \cap T(z)$ defines a set-valued expanding map from $T(z)$ to $T(z)$, (ii) each maximal x_0 in X is a fixed point of $\mathcal{F}$, i.e., $x_0 \in \mathcal{F}(x_0)$.

(B.2) Let $X = (X, d, \preceq)$ be a metric space with a partial order. We say that X admits *arbitrarily small tails* if given any tail $T(z)$, for each $\varepsilon > 0$ there exists $T(y) \subset T(z)$ with $\operatorname{diam} T(y) \leq \varepsilon$. Assume that X is complete and admits arbitrarily small tails. Show:

(a) For any $x_0 \in X$ there exists an ascending sequence $x_0 \preceq x_1 \preceq \cdots$ in X such that $\lim x_n = \widehat{x} \in \bigcap \overline{T(x_n)}$.

(b) If $f\ \ X \to X$ is continuous and expanding, then there is an $\widehat{x} \in X$ such that $\widehat{x} = f(\widehat{x}) \in \overline{T(x_0)}$.

(B.3) (*Cantor spaces*) Let $X = (X; d, \preceq)$ be a metric space in which a partial order $\preceq$ is defined. We say that X is a *partially ordered Cantor space* or simply a *Cantor space* if:

(i) d is complete, (ii) each tail $T(z)$ is closed in X, and (iii) X admits arbitrarily small tails. Show: If X is a Cantor space, then:

(i) every tail $T(z)$ in X is a Cantor space,
(ii) X has a maximal element x^*,
(iii) every tail $T(z)$ contains at least one maximal element $z^* \in T(z)$,
(iv) if $\mathcal{F} : X \to 2^X$ is expanding, then every tail $T(z)$ in X contains at least one fixed point of $\mathcal{F}$

[For (ii), use the Cantor theorem.]

(B.4) Show that the statement (ii) of (B.3) implies the Cantor theorem: If (X, d) is complete and $\{A_n\}$ a descending sequence of closed sets in X with $\delta(A_n) \to 0$, then $\bigcap A_n$ contains exactly one point.

(B.5) Let (X, d) be a complete metric space, $\varphi : X \to \boldsymbol{R}$ an arbitrary function and $\lambda > 0$. Recall that $X_{\varphi,\lambda}$ is the space appearing in (2.1).

(a) Assume φ has a finite lower bound. Show: $X_{\varphi,\lambda}$ admits arbitrarily small tails.
(b) Assume φ is l.s.c. and has a finite lower bound. Prove: $X_{\varphi,\lambda}$ is a Cantor space.

(B.6) Let (X, d) be complete and φ $X \to \boldsymbol{R}$ an arbitrary function with a finite lower bound. Suppose $F : X \to X$ is a continuous function such that $d(x, Fx) \le \varphi(x) - \varphi(Fx)$ for each $x \in X$. Show: F has a fixed point (Brøndsted [1974]).

(B.7) Let (X, d) be a complete metric space, $\varphi : X \to \boldsymbol{R}$ a l.s.c. function with a finite lower bound, and $\lambda > 0$. Show: If $F : X \to X$ is any map (not necessarily continuous) such that $d(x, Fx) \le \varphi(x) - \varphi(Fx)$ for all $x \in X$, then for each $z \in X$ there exists a fixed point z^* of F such that $d(z, z^*) \le \varphi(z) - \varphi(z^*)$.

(B.8) Let (X, d) be a complete metric space and $\varphi : X \to \boldsymbol{R}$ a l.s.c. function bounded below. Let $\mathcal{F} : X \to 2^X$ be a set-valued map such that for each $x \in X$ there is a $y \in \mathcal{F}(x)$ satisfying $d(x, y) \le \varphi(x) - \varphi(y)$. Show: $\mathcal{F}$ has a fixed point (W. Takahashi).

[Consider the Cantor space X_φ and observe that $\mathcal{F}$ is expanding.]

(B.9) Let (X, d) be a metric space. We say that X is *metrically convex* if for any distinct points $x, y \in X$ the *metric interval* $(x, y)_X = \{z \in X \mid x \ne z \ne y,\ d(x, y) = d(x, z) + d(z, y)\}$ is nonempty. Let (X, d) be complete and metrically convex and $F : X \to X$ be a continuous map such that for each $x \ne F(x)$ there exists a point $y \in (x, Fx)_X$ satisfying $d(Fx, Fy) \le \alpha d(x, y)$ for some $0 \le \alpha < 1$. Show: F has a fixed point (F. Clarke).

[Consider the space X_φ with $\varphi(x) = \frac{2}{1-\alpha} d(x, Fx)$ and show that the assumption $x^* \ne F(x^*)$, where x^* is a maximal element in X_φ, leads to a contradiction.]

(B.10) Let X be a space with a partial order $\preceq$ such that: (a) each tail $T(x) = \{y \mid y \succeq x\}$ is closed, and (b) each nondecreasing sequence $x_1 \preceq x_2 \preceq \cdots$ converges. Assume that for each chain C, there is some map $\varphi_C : C \to \boldsymbol{R}$ such that $\varphi_C(x) > \varphi_C(y)$ whenever $x \prec y$. Prove: Given any $x_0 \in X$ there is a maximal element $x^* \in X$ such that $x_0 \preceq x^*$

[Given any chain C, choose $c_n \in C$ so that $\varphi_C(c_n) \downarrow \inf \varphi_C$ and show that $S = \{c_n\}$ is cofinal in C; the limit of the sequence $\{c_n\}$ is an upper bound for C; now use the Kuratowski–Zorn lemma.]

C. Continuation method and fixed points for set-valued contractive maps

Let (X, d) and (Y, ρ) be metric spaces, and let $(\mathcal{CB}(Y), D)$ be the space of nonempty closed bounded subsets of Y with the Hausdorff metric. A family $\{H_t : X \to \mathcal{CB}(Y)\}$ of set-valued maps, depending on $t \in [0, 1]$ is called *α-contractive* provided:

(i) $D(H_t(x_1), H_t(x_2)) \le \alpha d(x_1, x_2)$ for all $t \in [0,1]$ and $x_1, x_2 \in X$,
(ii) there is a continuous and strictly increasing function $\varphi : [0,1] \to \boldsymbol{R}$ such that

$$D(H_{t_1}(x), H_{t_2}(x)) \le |\varphi(t_1) - \varphi(t_2)| \quad \text{for all } x \in X \text{ and } t_1, t_2 \in [0,1].$$

(C.1) Let (X,d) be complete, $x_0 \in X$ and $r > 0$. Let $T : K(x_0, r) \to CB(X)$ be an α-contractive map $(0 \le \alpha < 1)$ such that $d(Tx_0, x_0) < (1-\alpha)r$. Prove: T has a fixed point.

[Define by induction a sequence $\{y_n\}$ in $K(x_0, r)$ such that $y_{n+1} \in T(y_n)$ and $d(y_{n+1}, y_n) \le \alpha^n(1-\alpha)r$. Then, using the estimate

$$d(y_{n+p}, y_n) \le [1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}]\alpha^n(1-\alpha)r,$$

show that $\{y_n\}$ is a Cauchy sequence and hence converges to a certain $y_0 \in K(x_0, r)$. From $D(Ty_0, Ty_n) \le \alpha d(y_0, y_n)$ conclude that $y_0 \in Ty_0$.]

(C.2) Let (X,d) be complete, $U \subset X$ open, and $\{H_t \;\; \overline{U} \to CB(X)\}$ an α-contractive family of maps which are fixed point free on the boundary ∂U.

(a) Suppose $Q = \{(t, x) \in [0,1] \times U \mid x \in H_t(x)\}$ is nonempty and partially ordered by

$$(t', x') \preceq (t'', x'') \Leftrightarrow t' \le t'' \text{ and } d(x', x'') \le \frac{2(\varphi(t'') - \varphi(t'))}{1-\alpha}.$$

Establish the existence of a maximal element (t_0, x_0) in Q.

[Let P be a chain in Q and $t^* = \sup\{t \in [0,1] \mid (t, x) \in P\}$; take a sequence $\{(t_n, x_{t_n})\}$ in P with $(t_n, x_{t_n}) \preceq (t_{n+1}, x_{t_{n+1}})$ and $t_n \to t^*$ Show that $x_{t_n} \to x^* \in U$ and (t^*, x^*) is an upper bound for P. Apply the Kuratowski–Zorn lemma.].

(b) Show: If H_0 has a fixed point, then so does H_1 (Frigon–Granas [1994]).

[Letting (t_0, x_0) be maximal in Q, suppose to the contrary that $t_0 < 1$ and choose $t \in (t_0, 1]$ with

$$r = \frac{2(\varphi(t) - \varphi(t_0))}{1-\alpha} \le \frac{1}{2} d(x_0, \partial U).$$

Then, using (C.1), find a fixed point $x \in H_t(x)$ in $K(x_0, r)$, so that $(t_0, x_0) \prec (t, x)$, a contradiction.]

(C.3) (*Nonlinear alternative*) Let E be a Banach space, U a domain in E containing 0 and $T : \overline{U} \to CB(E)$ an α-contractive map with $T(\overline{U})$ bounded. Prove: Either (i) T has a fixed point, or (ii) there exist $y_0 \in \partial U$ and $\lambda \in (0,1)$ such that $y_0 \in \lambda T y_0$.

7. Notes and Comments

Order-theoretic fixed point results

The first fixed point result based on considerations of order was established by Knaster and Tarski in 1927 (Knaster [1927]). Some refinements of the initial result were established by Tarski in 1939; see Tarski [1955], where further bibliographical comments can be found, and also Kantorovitch [1939].

The following theorem is due to Tarski [1955]: *Let $(L, \preceq)$ be a complete lattice. (i) The fixed point set of every isotone map $\varphi : L \to L$ is nonempty and forms a complete lattice under the inherited order.* (ii) *If L is in addition*

a Boolean algebra, then given two isotone maps $\varphi, \psi : L \to L$ *and* $a, b \in L$ *there are* $c, d \in L$ *with* $\varphi(a - c) = d$ *and* $\psi(b - d) = c$. Several theorems on equality of cardinalities (for example, the Cantor–Bernstein theorem) reduce directly to (ii); (i) and (ii) are applicable to topological functions (closure, derivative); for example (i) gives a familiar result: every closed set has a largest perfect subset. Tarski's theorem can also be applied to partial differential equations (Simon et al. [1992], Simon [2000]), and to variational inequalities (Jerome [1984]). For applications of the Knaster–Tarski theorem to differential equations, see Volkmann [1995] in "Additional References" IVb.

In Amann's survey [1976], the following strengthening of the Knaster–Tarski theorem is proved: *The fixed point set* $\mathrm{Fix}(F|Q)$, *where* $Q = \{x \mid x \preceq F(x)\} \cap \{x \mid x \succeq b\}$, *contains a fixed point minimal in the set* $\mathrm{Fix}(F|Q)$; Amann found a number of applications of this extended version to fixed point problems of functional analysis.

We mention two other results: (i) Fuchssteiner [1977] obtained a theorem that contains Tarski's theorem as a special case and implies several fixed point results of functional-analytic type. (ii) Baclawski and Björner [1979] established a (discrete) Hopf–Lefschetz theorem for isotone maps on finite partially ordered sets. Acyclic sets (i.e., those that are like a point homologically) are shown to have the fixed point property for isotone maps; various sufficient conditions for acyclicity are developed.

Order-theoretic fixed point theorems also have applications in automata theory (Scott [1971], [1975]). For a number of order-theoretic fixed point results in functional analysis, see the surveys by E. Bohl [1974], Amann [1976] and books by Krasnosel'skiĭ [1964a] and Zeidler [1986].

Bishop–Phelps theorem and its applications

The formulation and proof of (2.1) are taken from Dugundji–Granas [1982]. The original formulation appeared as a remark (in a somewhat different but equivalent form) in a survey, written in 1971, by Phelps [1974]. Examining the proof of (2.1) leads to the "Cantor order-theoretic theorem", which provides a unified approach to a long list of results related to the Bishop–Phelps theorem (see Granas–Horvath [1999] and "Miscellaneous Results and Examples"). For some extensions of (2.1) (outside the scope of complete metric spaces) the reader is referred to Brøndsted [1974], Brézis–Browder [1976], and Altman [1982], among others.

In Sections 2 through 5, we give only a few applications of the Bishop–Phelps technique. For (2.2) see Caristi [1976], for (4.1), (4.2) see Daneš [1972], and for (5.1), (5.2) see Ekeland [1972]. The proof of (3.1) (Nadler [1969]) follows that given by Takahashi [1990].

For certain interrelations of these results the reader is referred to Daneš [1985] and Penot [1986]. Further applications of the Bishop–Phelps technique

can be found in Ekeland [1979] (variational problems) and the lecture notes by Phelps [1993] (convex analysis).

The Bishop-Phelps technique presented in Sections 2 through 5 originated in and evolved from the work of the above authors in the theory of support functionals in Banach spaces. Let E be a Banach space and $X \subset E$. A point $x_0 \in X$ is a *support point* of X if for some $f \in E^*$, called a *support functional* of X, we have $f(x_0) = \sup\{f(x) \mid x \in X\}$. The following theorem was established by Bishop–Phelps [1963]: *Let C be a closed convex subset of E. Then* (a) *the support points of C are dense in the boundary ∂C of C, and* (b) *the support functionals of C are norm dense in the set* $\{f \in E^* \mid \sup_C f < \infty\}$.

In connection with this result, we make the following comments:

(i) If $\text{Int}(C) \neq \emptyset$ then every $x \in C$ is a support point of C; this follows at once from the Mazur separation theorem.

(ii) If C is the closed unit ball in E, then the set $\{f \in E^* \mid f(x) = \|f\|$ for some $x \in \partial C\}$ is norm dense in E^*; this is a special case of the Bishop–Phelps theorem.

(iii) If C is the closed unit ball in E, then [each $f \in E^*$ is a support functional of C] $\Leftrightarrow$ [the space E is reflexive] (theorem of James [1972]).

(iv) Let $\varphi : E \to \boldsymbol{R}$ be convex and lower semicontinuous. We define the *subdifferential* of φ at $x \in E$ by

$$\partial\varphi(x) = \{f \in E^* \mid f(y - x) \leq \varphi(y) - \varphi(x) \text{ for all } y \in E\}.$$

Because the elements of $\partial\varphi(x)$ can be identified with support functionals of the closed convex epigraph $\text{epi}(\varphi) \subset E \times \boldsymbol{R}$ of φ at $(x, \varphi(x))$, the Bishop–Phelps theorem leads to the following result: *The set* $\{x \in E \mid \partial\varphi(x) \neq \emptyset\}$ *is dense in* E. This frequently used result (and, in fact, its "extended" version valid for functions φ possibly equal to ∞) is due to Brøndsted Rockafellar [1965].

Fixed points for set-valued contractive maps

An extension of the Banach theorem to set-valued contractive maps is due to Nadler [1969]. The continuation method for such maps can also be established (see Frigon–Granas [1994] and "Miscellaneous Results and Examples").

In the following remarks, X and Y are Banach spaces, $W \subset X$ an open set, and $f : W \to Y$ a continuous map. Let M be a subset of X and $x_0 \in M$. A vector $x \in X$ is *tangent* to M at x_0 if there is $\tau : (-\varepsilon, \varepsilon) \to X$ with $x_0 + tx + \tau(t) \in M$ for $t \in (-\varepsilon, \varepsilon)$ and $\|\tau(t)\|/t \to 0$ as $t \to 0$. We let $T_{x_0}M$ be the set of all tangent vectors to M at x_0. We say that $f : W \to Y$ is *strictly differentiable* at $x_0 \in W$ if it has a *derivative* $L = Df(x_0) \in \mathscr{L}(X, Y)$ at x_0 with the property: for each $\varepsilon > 0$ there is a nbd $V \subset W$ of x_0 such that

$$\|f(x) - f(x') - L(x - x')\| \leq \varepsilon\|x - x'\| \qquad \text{for all } x, x' \in V.$$

The following result (which is of importance in optimal control theory) was established by Lusternik [1934]: *Let $f : W \to Y$ be strictly differentiable at $x_0 \in W$ with $Df(x_0)$ surjective. Let $M = \{x \in X \mid f(x) = f(x_0)\}$. Then* (A) *there exists a nbd $U \subset W$ of x_0 and a map $\vartheta : U \to X$ such that, for all $x \in U$, we have $x + \vartheta(x) \in M$ and $\|\vartheta(x)\| \leq K\|f(x) - f(x_0)\|$, where $K > 0$, and* (B) $T_{x_0}M = \text{Ker}\, Df(x_0)$. For a proof of the above theorem that uses the fixed point theorem for (generalized) set-valued contractive maps in conjunction with the open mapping principle, see Ioffe–Tikhomirov [1978].

§3. Results Based on Convexity

In this paragraph we present an introduction to the geometric theory of KKM-maps. We first establish a geometric analogue of the Knaster–Kuratowski–Mazurkiewicz theorem and then derive from it, among other results, the theorems of von Neumann and Markoff–Kakutani. Topological aspects of the theory, based on the Brouwer theorem, will be given in the next chapter. The paragraph ends with the Kakutani theorem in normed linear spaces.

1. KKM-Maps and the Geometric KKM-Principle

The general principles of the theory of KKM-maps use simple notions related to set-valued maps. For the convenience of the reader, and to establish the terminology, we recall the relevant definitions.

Let X and Y be two sets. The set of all subsets of X is denoted by 2^X A map $S : X \to 2^Y$ is called a *set-valued map*; the sets Sx are the *values* of S and the sets

$$G_S = \{(x, y) \in X \times Y \mid y \in Sx\} \quad \text{and} \quad S(X) = \bigcup_{x \in X} Sx$$

are the *graph* and *image* of S, respectively.

The *inverse* $S^{-1} : Y \to 2^X$ and the *dual* $S^* : Y \to 2^X$ of S are the maps $y \mapsto S^{-1}y = \{x \in X \mid y \in Sx\}$ and $y \mapsto S^*y = X - S^{-1}y$. The values of S^{-1} (respectively of S^*) are called the *fibers* (respectively *cofibers*) of S.

Note that S is *surjective* (i.e., $S(X) = Y$) if and only if its fibers $S^{-1}y$ are all nonempty. By a *fixed point* of a set-valued map $S : X \to 2^X$ is meant a point $x_0 \in X$ for which $x_0 \in Sx_0$. Clearly, if S has a fixed point, then so does S^{-1}.

It will also be convenient to have the following notation at our disposal.

Given a vector space E (always over $\boldsymbol{R}$) and $A \subset E$, we shall frequently denote by $[A]$ the convex hull, $\operatorname{conv} A$, of A. For any integer $n \in \boldsymbol{N}$ we shall write

$$[n] = \{i \in \boldsymbol{N} \mid 1 \le i \le n\}.$$

We are now in a position to introduce the basic class of set-valued maps that will enter our discussion, as follows:

(1.1) DEFINITION. Let E be a vector space and $X \subset E$ an arbitrary subset. A map $G : X \to 2^E$ is called a *Knaster–Kuratowski–Mazurkiewicz map* (or simply a *KKM-map*) provided

$$[A] = \operatorname{conv}\{x_1, \ldots, x_s\} \subset G(A) = \bigcup_{i=1}^{s} Gx_i$$

for each finite subset $A = \{x_1, \ldots, x_s\}$ of X. We say that G is a *strongly KKM-map* if (i) $x \in Gx$ for each $x \in X$, and (ii) the cofibers G^*y of G are convex.

We first record two simple properties of KKM-maps for future reference.

(1.2) LEMMA. *Let E be a vector space, $C \subset E$ convex, and $G : C \to 2^E$ strongly KKM. Then G is a KKM-map.*

PROOF. Let $A = \{x_1, \ldots, x_s\} \subset X$, and let $y_0 \in [A]$. We have to show that $y_0 \in \bigcup_{i=1}^{s} Gx_i$. Since $y_0 \in Gy_0$, we have $y_0 \notin G^*y_0$, and therefore $[A]$ is not contained in G^*y_0. Since the set G^*y_0 is convex, at least one point x_i of A does not belong to G^*y_0, which means that $y_0 \in Gx_i$. □

(1.3) LEMMA. *Let E be a vector space and $C \subset E$ a nonempty convex set. Let $G : C \to 2^C$ be a set-valued map such that $G^* : C \to 2^C$ is not a KKM-map. Then:*
(i) *there exists a point $w \in C$ with $w \in \operatorname{conv}(Gw)$,*
(ii) *if G has convex values, then G has a fixed point.*

PROOF. Because (i)⇒(ii), we need only establish (i). As, by assumption, G^* is not a KKM-map, there is $w \in \operatorname{conv}\{x_1, \ldots, x_n\}$ for some $x_1, \ldots, x_n \in C$ such that

$$w \in C - \bigcup_{i=1}^{n} G^*x_i = C - \bigcup_{i=1}^{n}(C - G^{-1}x_i) = \bigcap_{i=1}^{n} G^{-1}x_i.$$

Consequently, $x_i \in Gw$ for each $i \in [n]$, and so $w \in \operatorname{conv}(Gw)$. □

Before we give a few examples of KKM-maps, we recall some definitions.

Let E be a vector space and $C \subset E$ a convex subset. A function $\varphi : C \to \boldsymbol{R}$ is said to be *convex* if $\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y)$ for all $t \in [0,1]$ and $x, y \in C$. The sum and the maximum of two convex functions are convex. A function $\psi : C \to \boldsymbol{R}$ is *concave* if $-\psi$ is convex.

A function $\varphi : C \to \boldsymbol{R}$ is *quasi-convex* if $\{y \in C \mid \varphi(y) < \lambda\}$ is convex for each $\lambda \in \boldsymbol{R}$. A function $\psi : C \to \boldsymbol{R}$ is *quasi-concave* if $-\psi$ is quasi-convex. Clearly, every convex function is quasi-convex.

EXAMPLES. In the following two examples of KKM-maps the domain of a map is convex, so it is enough to show that the map is strongly KKM. An example of a KKM-map that is not strongly KKM will appear in (4.4.9).

(i) Let X, Y be convex subsets of vector spaces E_X and E_Y, and let $f : X \times Y \to \boldsymbol{R}$ be a *concave-convex function* (i.e., $x \mapsto f(x,y)$ is concave for each $y \in Y$ and $y \mapsto f(x,y)$ is convex for each $x \in X$). Then the map $G : X \times Y \to 2^{X \times Y}$ defined by

$$G(x,y) = \{(x',y') \in E_X \times E_Y \mid f(x,y') - f(x',y) \le 0\}$$

is strongly KKM. Indeed, $(x, y) \in G(x, y)$ for each $(x, y) \in X \times Y$, and because $(x, y) \mapsto f(x, y') - f(x', y)$ is concave, the cofibers

$$G^*(x', y') = \{(x, y) \in X \times Y \mid f(x, y') - f(x', y) > 0\}$$

of G are convex.

(ii) Let C be a convex subset of a vector space E and $g : C \times C \to \boldsymbol{R}$ be a function such that:

(a) $g(x, x) \leq 0$ for each $x \in C$,

(b) $x \mapsto g(x, y)$ is quasi-concave on C for each $y \in C$.

Then the map $G : C \to 2^C$ given by $x \mapsto Gx = \{y \in C \mid g(x, y) \leq 0\}$ is strongly KKM. Indeed, it follows from (a) that $x \in Gx$ for each $x \in C$, and (b) implies that the cofibers $G^*y = \{x \in C \mid g(x, y) > 0\}$ of G are convex.

Let E be a vector space. A subset $A \subset E$ is called *finitely closed* if its intersection with each finite-dimensional flat $L \subset E$ is closed in the Euclidean topology of L.

Recall that a family $\{A_\lambda \mid \lambda \in \Lambda\}$ of subsets of some set is said to have the *finite intersection property* if the intersection of each finite subfamily is nonempty.

The basic geometric property of KKM-maps is given in

(1.4) THEOREM. *Let $X \subset E$, and let $G : X \to 2^E$ be a KKM-map with finitely closed convex values. Then the family $\{Gx\}_{x \in X}$ has the finite intersection property.*

PROOF. Let $A = \{x_1, \ldots, x_n\}$ be a finite subset of X. We are going to show that

$$[A] \cap \bigcap_{i=1}^{n} Gx_i \neq \emptyset. \tag{1}$$

The proof is by induction. For A consisting of a single element x of X our statement holds, because $x \in Gx$ for any $x \in X$. Assume the statement is true for any set containing $n - 1$ elements. For each $i \in [n]$, choose an element y_i in the set

$$\bigcap_{j \neq i} Gx_j \cap [A \setminus \{x_i\}],$$

which is nonempty by the inductive hypothesis, and consider the compact convex set

$$Y = [y_1, \ldots, y_n] \subset [A].$$

To establish (1) it is clearly enough to show that $\bigcap_{i=1}^{n} Gx_i \cap Y \neq \emptyset$. Suppose that, on the contrary,

$$\bigcap_{i=1}^{n} Gx_i \cap Y = \emptyset.$$

Working in the finite-dimensional space L spanned by A, let d be the Euclidean metric in L; note that because $Gx_i \cap L$ is closed in L, we have $d(x, Gx_i \cap Y) = 0$ if and only if $x \in Gx_i \cap Y$.

For each $j \in [n]$, define $\varphi_j : Y \to \boldsymbol{R}$ by $\varphi_j(y) = d(y, Gx_j \cap Y)$; note that because each φ_j is convex and continuous, so also is the function $\varphi : Y \to \boldsymbol{R}$ given by $\varphi(y) = \max\{\varphi_1(y), \ldots, \varphi_n(y)\}$ for $y \in Y$. Let $\widehat{y} \in Y$ be a point at which φ attains its minimum. Since by assumption $\bigcap Gx_i \cap Y = \emptyset$, we must have $\varphi(\widehat{y}) > 0$. Because G is a KKM-map, we have $Y \subset [A] \subset \bigcup_{i=1}^{n} Gx_i$, and hence the point $\widehat{y}$ belongs to one of the sets Gx_i, say Gx_n.

To get a contradiction, we shall evaluate the functions φ_i at points $z_t = t\widehat{y} + (1-t)y_n$ of the interval $[\widehat{y}, y_n] \subset Y$. First, for $i = n$, because $\varphi_n(\widehat{y}) = 0$, we have

$$\varphi_n(z_t) \le t\varphi_n(\widehat{y}) + (1-t)\varphi_n(y_n) \le (1-t)\varphi_n(y_n).$$

This implies $\varphi_n(z_t) \to 0$ as $t \to 1$, and hence, for some t_0 sufficiently close to 1, we get

$$\varphi_n(z_{t_0}) < \varphi(\widehat{y}). \tag{2}$$

Furthermore, for any $i \in [n-1]$, because $\varphi_i(y_n) = 0$, we have

$$\varphi_i(z_{t_0}) \le t_0\varphi_i(\widehat{y}) + (1-t_0)\varphi_i(y_n) < \varphi(\widehat{y}).$$

This, in view of (2), implies that $\varphi(z_{t_0}) = \max\{\varphi_i(z_{t_0}) \mid i \in [n]\} < \varphi(\widehat{y})$, and, with this contradiction, the proof is complete. □

Theorem (1.4) implies at once the main result of this section:

(1.5) THEOREM (Geometric KKM-principle). *Let E be an arbitrary linear topological space, $X \subset E$, and $G : X \to 2^E$ a KKM-map with closed convex values such that Gx_0 is compact for some $x_0 \in X$. Then the intersection $\bigcap\{Gx \mid x \in X\}$ is not empty.* □

In the next two sections we illustrate the technique of KKM-maps by deriving from Theorem (1.5) some further results, including the theorems of von Neumann and Markoff Kakutani.

2. Theorem of von Neumann and Systems of Inequalities

We give an immediate application of the geometric KKM-principle by deriving a classical result in game theory.

(2.1) THEOREM (von Neumann). *Let X and Y be two nonempty compact convex subsets of two linear topological spaces E_X and E_Y. Let $f : X \times Y \to \boldsymbol{R}$ be a real-valued function satisfying:*

(i) *$x \mapsto f(x,y)$ is concave and u.s.c. for each $y \in Y$,*

(ii) *$y \mapsto f(x,y)$ is convex and l.s.c. for each $x \in X$.*

Then:

(A) *there is a point* $(x_0, y_0) \in X \times Y$ *(called a saddle point for* f*) such that*

$$f(x, y_0) \leq f(x_0, y) \quad \textit{for all } (x, y) \in X \times Y,$$

(B) $\max\limits_{x \in X} \min\limits_{y \in Y} f(x, y) = f(x_0, y_0) = \min\limits_{y \in Y} \max\limits_{x \in X} f(x, y).$

PROOF. Because (B) follows at once from (A), we need only establish (A). To this end, define a map $G : X \times Y \to 2^{X \times Y}$ by putting

$$G(x, y) = \{(x', y') \in E_X \times E_Y \mid f(x, y') - f(x', y) \leq 0\}.$$

By Example (i), because f is concave-convex, G is a KKM-map. Furthermore, because for each (x, y) the function $(x', y') \mapsto f(x, y') - f(x', y)$ is convex and l.s.c., the sets $G(x, y)$ are convex and closed. Consequently, by Theorem (1.5), there exists (x_0, y_0) such that $(x_0, y_0) \in G(x, y)$ for all $(x, y) \in X \times Y$; this means exactly that (x_0, y_0) is a saddle point for f □

From the above theorem we now obtain two important results in the theory of infinite systems of inequalities.

Let $X \subset E$ be compact convex in a linear topological space E, and let $\Phi = \{\varphi\}$ be a nonempty family of real-valued functions $\varphi : X \to \boldsymbol{R}$ that are convex and lower semicontinuous. To formulate a general result we let $[\Phi]$ be the convex hull of Φ in the vector space $\boldsymbol{R}^X$; we are concerned with the following two problems:

($\mathcal{P}_1$) There exists $x_0 \in X$ such that $\varphi(x_0) \leq 0$ for all $\varphi \in \Phi$.

($\mathcal{P}_2$) For each $\psi \in [\Phi]$ there exists $\widehat{x} \in X$ such that $\psi(\widehat{x}) \leq 0$.

(2.2) THEOREM. *The problems* ($\mathcal{P}_1$) *and* ($\mathcal{P}_2$) *are equivalent. In other words, either*

(a) *there is* $x_0 \in X$ *satisfying* $\varphi(x_0) \leq 0$ *for all* $\varphi \in \Phi$, *or*

(b) *there is* $\psi \in [\Phi]$ *such that* $\psi(x) > 0$ *for all* $x \in X$.

PROOF. Clearly, it is enough to show that ($\mathcal{P}_2$)⇒($\mathcal{P}_1$). Assume ($\mathcal{P}_2$) holds and let $S(\varphi) = \{x \in X \mid \varphi(x) \leq 0\}$. To establish our claim, we have to show that $\bigcap_{\varphi \in \Phi} S(\varphi)$ is not empty. Since the sets $S(\varphi)$ are convex, closed and nonempty (by ($\mathcal{P}_2$)), this reduces to showing that the family $\{S(\varphi) \mid \varphi \in \Phi\}$ has the finite intersection property. To this end, let $\varphi_1, \ldots, \varphi_n \in \Phi$. Set

$$\Lambda = \left\{(\lambda_1, \ldots, \lambda_n) \in \boldsymbol{R}^n \,\middle|\, \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{n} \lambda_i = 1\right\}$$

and on the product of two compact convex sets X and Λ, consider the

function $f : X \times \Lambda \to \boldsymbol{R}$ given by

$$f(x, \lambda) = \sum_{i=1}^{n} \lambda_i \varphi_i(x).$$

As is easily seen, f satisfies all the conditions of Theorem (2.1), and consequently it has a saddle point. Thus, there exists $(x_0, \widehat{\lambda}) \in X \times \Lambda$ such that $f(x_0, \lambda) \le f(x, \widehat{\lambda})$ for all $(x, \lambda) \in X \times \Lambda$. Said differently, there exist $x_0 \in X$ and $\psi = \sum_{i=1}^{n} \widehat{\lambda}_i \varphi_i \in [\Phi]$ such that $\varphi_i(x_0) \le \psi(x)$ for all $i \in [n]$ and $x \in X$. Now, by $(\mathcal{P}_2)$, there exists $\widehat{x} \in X$ such that $\psi(\widehat{x}) \le 0$, so $\varphi_i(x_0) \le 0$ for all $i \in [n]$; that gives $x_0 \in \bigcap_{i=1}^{n} S(\varphi_i)$, and the proof is complete. □

Let X be a set and $\Phi = \{\varphi\}$ a nonempty family of real-valued functions $\varphi : X \to \boldsymbol{R}$. We say that Φ is *concave in the sense of Fan* (or simply *F-concave*) provided for any convex combination $\sum_{i=1}^{n} \lambda_i \varphi_i$ of $\varphi_1, \dots, \varphi_n \in \Phi$ there is $\varphi \in \Phi$ such that $\varphi(x) \ge \sum_{i=1}^{n} \lambda_i \varphi_i(x)$ for each $x \in X$.

(2.3) COROLLARY. *Let X be a nonempty compact convex subset of a linear topological space E and $\Phi = \{\varphi\}$ an F-concave family of convex l.s.c. real-valued functions $\varphi : X \to \boldsymbol{R}$. Then the following conditions are equivalent:*

(A) *there exists $x_0 \in X$ such that $\varphi(x_0) \le 0$ for all $\varphi \in \Phi$,*

(B) *for each $\varphi \in \Phi$ there exists $\widehat{x} \in X$ such that $\varphi(\widehat{x}) \le 0$.*

PROOF. Clearly, it is enough to show that (B)⇒(A). To the contrary, suppose that (A) does not hold. Then by Theorem (2.2) there is a convex combination $\sum_{i=1}^{n} \lambda_i \varphi_i \in [\Phi]$ such that

$$\sum_{i=1}^{n} \lambda_i \varphi_i(x) > 0 \quad \text{for all } x \in X,$$

and hence, by definition of the F-concave family, we have, for some $\varphi \in \Phi$,

$$\varphi(x) \ge \sum_{i=1}^{n} \lambda_i \varphi_i(x) > 0 \quad \text{for all } x \in X.$$

This contradicts (B), and the proof is complete. □

3. Fixed Points of Affine Maps. Markoff–Kakutani Theorem

In this section we first obtain an elementary fixed point theorem for continuous affine maps and then establish the Markoff–Kakutani theorem.

Given a linear topological space E, we denote by E^* its conjugate space, i.e., the space of all continuous linear functionals on E. We say that E *has sufficiently many linear functionals* if the elements of E^* separate points of E, i.e., for every nonzero $x \in E$ there is an $l \in E^*$ such that $l(x) \ne 0$.

(3.1) THEOREM. *Let E be a linear topological space with sufficiently many linear functionals, $C \subset E$ a nonempty compact convex set, and $F : C \to E$ a continuous affine map. Suppose that for each $y \in C$ with $y \neq Fy$ the line segment $[y, Fy]$ contains at least two points of C. Then* $\text{Fix}(F) \neq \emptyset$.

PROOF. Let l be a given element of E^* We first treat the problem of solving in C the inequality

$$(*) \qquad l(Fy - y) \leq 0.$$

To this end, consider the continuous function $l|C : C \to \boldsymbol{R}$; let $y_0 \in C$ be a point at which it assumes a maximum in C. If $Fy_0 \neq y_0$, then by assumption there exists a $\lambda > 0$ such that the point $\lambda Fy_0 + (1 - \lambda)y_0$ is in C. Then

$$l[\lambda Fy_0 + (1 - \lambda)y_0] \leq l(y_0),$$

and thus $\lambda l(Fy_0 - y_0) \leq 0$, i.e., the point y_0 solves the inequality $(*)$.

Consider now on C the family $\Phi = \{\varphi\}$ of continuous convex functions $\varphi : C \to \boldsymbol{R}$ of the form $\varphi(y) = l(Fy - y)$, $y \in C$, where $l \in E^*$. To show that $\text{Fix}(F) \neq \emptyset$, it is clearly enough to find $y_0 \in C$ that solves the system of inequalities

$$l(Fy_0 - y_0) \leq 0 \quad \text{for all } l \in E^*.$$

By Theorem (2.2), and because the family Φ is convex, this problem reduces to solving $(*)$ for a given $l \in E^*$; but since this was already established, the proof is complete. □

As an immediate corollary we obtain at once one of the basic results in linear functional analysis:

(3.2) THEOREM (Markoff–Kakutani). *Let C be a nonempty compact convex set in a linear topological space with sufficiently many linear functionals, and let $\mathscr{F}$ be a commuting family of continuous affine maps of C into itself. Then $\mathscr{F}$ has a common fixed point.*

PROOF. By (3.1), $\text{Fix}(F) \neq \emptyset$ for each $F \in \mathscr{F}$; moreover, $\text{Fix}(F)$ is compact, being closed in the compact set C, and $\text{Fix}(F)$ is convex because F is affine. We must prove that $\bigcap\{\text{Fix}(F) \mid F \in \mathscr{F}\} \neq \emptyset$; because each set $\text{Fix}(F)$ is compact, it is sufficient to show that each finite intersection $\text{Fix}(F_1, \dots, F_n) \equiv \bigcap_{i=1}^n \text{Fix}(F_i)$ is nonempty.

We proceed by induction on n, the result being true for $n = 1$. Assume that $\text{Fix}(F_1, \dots, F_i) \neq \emptyset$ whenever $i < n$, and consider any n members $F_1, \dots, F_n$ of $\mathscr{F}$. Because $\mathscr{F}$ is commuting, we find that

$$F_n[\text{Fix}(F_1, \dots, F_{n-1})] \subset \text{Fix}(F_1, \dots, F_{n-1});$$

for if $x \in \mathrm{Fix}(F_1, \ldots, F_{n-1})$, then $F_i[F_n(x)] = F_n[F_i(x)] = F_n(x)$ for each $i < n$, so $F_n(x) \in \mathrm{Fix}(F_1, \ldots, F_{n-1})$. Since $\mathrm{Fix}(F_1, \ldots, F_{n-1})$ is a nonempty compact convex set, we conclude from (3.1) that $\mathrm{Fix}(F_1, \ldots, F_n) \neq \emptyset$. This completes the induction. □

The Markoff–Kakutani theorem has numerous applications, including a refined Hahn-Banach theorem the proof of which will be given in the next paragraph.

4. Fixed Points for Families of Maps. Theorem of Kakutani

If $\mathcal{G}$ is any family of maps of a space X into itself, by a *fixed point* for $\mathcal{G}$ is meant a point $x_0 \in X$ such that $g(x_0) = x_0$ for every $g \in \mathcal{G}$. Our aim in this section is to prove the theorem of Kakutani on fixed points for certain families of self-maps in normed linear spaces. The proof does not depend on the previous material of this paragraph; it is given to indicate another technique that is also based on the notion of convexity. The approach given here is due to W. Hurewicz; it relies on some simple properties of metric spaces and affine maps.

We begin with terminology and notation. Let $\mathcal{G}$ be a family of maps of a space X into itself. For each $x \in X$ we let $\mathcal{G}(x) = \{g(x) \mid g \in \mathcal{G}\}$ be the *orbit* of x under $\mathcal{G}$; a subset $A \subset X$ is called *$\mathcal{G}$-invariant* if $g(A) \subset A$ for all $g \in \mathcal{G}$. If (X, d) is a bounded metric space, $\mathcal{G}$ is said to be *equicontinuous* provided given any $\varepsilon > 0$, there is an $\eta > 0$ such that $d(x, x_0) \leq \eta$ implies $d(g(x), g(x_0)) \leq \varepsilon$ for all $g \in \mathcal{G}$.

(4.1) LEMMA. *Let (X, d) be a bounded metric space and $\mathcal{G}$ an equicontinuous family of maps $g : X \to X$. For each $x \in X$, let $D(x) = \delta(\mathcal{G}(x))$ be the diameter of $\mathcal{G}(x)$. Then the function $D : X \to \boldsymbol{R}$ is continuous.*

PROOF. Given $x_0 \in X$ and $\varepsilon > 0$, there is an $\eta > 0$ such that $d(g(x), g(x_0)) \leq \varepsilon$ for all $x \in B(x_0, \eta)$ and all $g \in \mathcal{G}$. This says that for any $g, g' \in \mathcal{G}$ and any $x \in B(x_0, \eta)$, since

$$d[g(x), g'(x)] \leq d[g(x), g(x_0)] + d[g(x_0), g'(x_0)] + d[g'(x_0), g'(x)],$$

we get $D(x) \leq \varepsilon + D(x_0) + \varepsilon$ and, in the usual way, $|D(x) - D(x_0)| \leq 2\varepsilon$. □

The crucial result in Hurewicz's approach is

(4.2) LEMMA. *Let E be a normed linear space, C a compact convex subset of E, containing more than one point, and let $\mathcal{G}$ be an equicontinuous group of affine transformations on C. Then there exists a nonempty $\mathcal{G}$-invariant subset $A \subset C$ with $\delta(A) < \delta(C)$ (strict inequality!).*

PROOF. By compactness and equicontinuity, there is an $\eta > 0$ such that $\|x - y\| \leq \eta$ implies $\|g(x) - g(y)\| < \frac{1}{2}\delta(C)$ for all $g \in \mathcal{G}$.

Let $c_1, \ldots, c_n$ be an η-dense set in C and note that given any $c \in C$ and $g \in \mathcal{G}$, there is at least one c_i such that

$$\text{(i)} \qquad \|c - g(c_i)\| \leq \frac{1}{2}\delta(C).$$

For consider the inverse $g^{-1} \in \mathcal{G}$; since the $\{c_i\}$ are η-dense, there is at least one c_i with $\|c_i - g^{-1}(c)\| \leq \eta$, so (i) follows from the equicontinuity.

Now let

$$y = \frac{c_1 + \cdots + c_n}{n}$$

Then, for any $c \in C$ and any $g \in \mathcal{G}$ we have

$$\|c - g(y)\| = \left\| c - g\left(\frac{1}{n}\sum_j c_j\right)\right\| \leq \frac{1}{n}\sum_j \|c - g(c_j)\|$$

$$\leq \frac{1}{n}\left[(n-1)\delta(C) + \frac{1}{2}\delta(C)\right]$$

because of (i), so $\|c - g(y)\| \leq (1 - 1/(2n))\delta(C)$.

Finally, choose $A = \mathcal{G}(y)$ to be the orbit of y under $\mathcal{G}$ and note that since $\mathcal{G}$ is a group, the set A is $\mathcal{G}$-invariant. Moreover, for any $g, g' \in \mathcal{G}$, taking $g'(y) = c$ in the above formula shows

$$\|g(y) - g'(y)\| \leq (1 - 1/(2n))\delta(C),$$

so that $\delta(A) < \delta(C)$, and the proof is complete. □

We are now in a position to prove the basic result.

(4.3) THEOREM (Kakutani). *Let C be a nonempty compact convex subset of a normed linear space E, and let $\mathcal{G}$ be a group of affine transformations of C into itself. If $\mathcal{G}$ is equicontinuous on C, then $\mathcal{G}$ has a fixed point.*

PROOF. For each $c \in C$, let $D(c) = \delta(\mathcal{G}(c))$. By Lemma (4.1), D is continuous on the compact set C, so it attains its minimum at some $c_0 \in C$. Assume $D(c_0) > 0$. Since $\mathcal{G}(c_0)$ is $\mathcal{G}$-invariant, and each g is affine, we infer that so is the set $\operatorname{Conv}\mathcal{G}(c_0)$; moreover, we have $\delta(\operatorname{Conv}\mathcal{G}(c_0)) = \delta(\mathcal{G}(c_0))$.

The hypotheses of Lemma (4.2) being satisfied, there exists a $\mathcal{G}$-invariant subset A of $\operatorname{Conv}\mathcal{G}(c_0)$ with $\delta(A) < \delta(\operatorname{Conv}\mathcal{G}(c_0))$. Choosing any $y_0 \in A$, we therefore have $\mathcal{G}(y_0) \subset A$ so

$$D(y_0) \leq \delta(A) < \delta(\operatorname{Conv}\mathcal{G}(c_0)) = \delta(\mathcal{G}(c_0)) = D(c_0),$$

contradicting the assumption that $D(c_0) = \min$. Thus $D(c_0) = 0$, completing the proof. □

An extension of the Kakutani theorem to locally convex linear topological spaces will be given in Chapter II.

5. Miscellaneous Results and Examples

A. Geometric KKM-theory

In this subsection a set will be called *compact convex* if it is a nonempty compact convex subset of a linear topological space.

(A.1) Let X be a set. Using the notation of Section 1, let $\mathcal{A}(X)$, $\mathcal{B}(X)$ be two classes of set-valued maps $X \to 2^X$ defined as follows:

$$S \in \mathcal{A}(X) \Leftrightarrow \text{ either } \bigcap\{Sx \mid x \in X\} \neq \emptyset \text{ or } x \in Sx \text{ for each } x \in X,$$

$$T \in \mathcal{B}(X) \Leftrightarrow \text{ either } T \text{ has a fixed point or } Tx_0 = \emptyset \text{ for some } x_0 \in X.$$

Let $S, T : X \to 2^X$ be two maps. Show:
(a) If $T^* \in \mathcal{A}(X)$, then $T \in \mathcal{B}(X)$.
(b) If $S^* \in \mathcal{B}(X)$, then $S \in \mathcal{A}(X)$.

(A.2) Let C be compact convex, and let $S : C \to 2^C$ have convex values, open fibers and convex cofibers. Prove: Either S has a fixed point or $Sx_0 = \emptyset$ for some $x_0 \in C$.

(A.3) (*Maximal elements of binary relations*) Let X be a set. We establish a one-to-one correspondence $T \leftrightarrow \prec_T$ between maps $T \quad X \to 2^X$ and binary relations $\prec$ on X by the assignments: $\prec \mapsto T$, where $Tx = \{y \in X \mid x \prec y\}$, and $T \mapsto \prec_T$, where $x \prec_T y \Leftrightarrow y \in Tx$; clearly, a point $x_0 \in X$ is maximal in $(X, \prec_T)$ (i.e., for no $y \in X$ with $y \neq x_0$ do we have $x_0 \prec_T y$) if $Tx_0 = \emptyset$.

Let C be compact convex and $T : C \to 2^C$ be such that:
(i) T has open fibers and convex cofibers,
(ii) $x \notin \operatorname{conv} Tx$ for each $x \in C$.
Show: There exists a maximal element in $(C, \prec_T)$.

(A.4) Let X be compact convex, and let $S, T : X \to 2^X$ satisfy:
(i) $Sx \subset Tx$ for all $x \in X$,
(ii) S has convex cofibers,
(iii) T has closed and convex values.
Show: If $x \in Sx$ for each $x \in X$, then the intersection $\bigcap\{Tx \mid x \in X\}$ is not empty.

(A.5) Let X be compact convex, and let $f, g : X \times X \to \boldsymbol{R}$ satisfy:
(i) $g(x, y) \leq f(x, y)$ for all $x, y \in X$,
(ii) $x \mapsto f(x, y)$ is quasi-concave on X for each $y \in X$,
(iii) $y \mapsto g(x, y)$ is l.s.c. and quasi-convex on X for each $x \in X$.
Prove:
(a) For any $\lambda \in \boldsymbol{R}$, either (i) there exists a $y_0 \in X$ such that $g(x, y_0) \leq \lambda$ for all $x \in X$, or (ii) there exists a $w \in X$ such that $f(w, w) > \lambda$.
(b) The following minimax inequality holds:

$$\inf_{x \in X} \sup_{y \in X} g(x, y) \leq \sup_{x \in X} f(x, x).$$

[For (a): define $S, T \quad X \to 2^X$ by $Sx = \{y \in X \mid f(x, y) \leq \lambda\}$ and $Tx = \{y \in X \mid g(x, y) \leq \lambda\}$, and apply (A.4).]

(A.6) Let X be compact convex, and let $f : X \times X \to \boldsymbol{R}$ satisfy:
(i) $x \mapsto f(x, y)$ is quasi-concave on X for each $y \in X$,
(ii) $y \mapsto f(x, y)$ is l.s.c. and convex on X for each $x \in Y$

Show:

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X} f(x,x).$$

(A.7) Let Y be compact convex, and let $\Phi = \{\varphi\}$ be an F-concave family of convex l.s.c. real-valued functions $\varphi : Y \to \boldsymbol{R}$. Show:

$$\alpha = \inf_{y\in Y}\sup_{\varphi\in\Phi} \varphi(y) = \sup_{\varphi\in\Phi}\inf_{y\in Y} \varphi(y) = \beta.$$

[As always $\beta \le \alpha$, it is enough to show $\alpha \le \beta$. Supposing the contrary, let λ be such that $\alpha > \lambda > \beta$, and let Ψ be the family of functions $\psi : X \to \boldsymbol{R}$ of the form $\psi = \varphi - \lambda$, where $\varphi \in \Phi$. Applying (2.3) to Ψ, get a contradiction.]

(A.8) (*Kneser–Fan theorem*) Let X and Y be two nonempty convex subsets of some linear topological spaces and assume that Y is compact. Let $f : X \times Y \to \boldsymbol{R}$ satisfy

(i) $x \mapsto f(x,y)$ is concave for each $y \in Y$,

(ii) $y \mapsto f(x,y)$ is l.s.c. and convex for each $x \in X$.

Show:

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} f(x,y)$$

(cf. Kneser [1952] and Fan [1957]).

[Apply (A.7) to the F-concave family $\Phi = \{\varphi_x \mid x \in X\}$ of convex l.s.c. functions $\varphi_x : Y \to \boldsymbol{R}$, where $\varphi_x(y) = f(x,y)$ for $y \in Y$.]

B. Invariant functionals, means, and measures

(B.1) Let X be a nonempty set and $B(X)$ the Banach space of bounded real-valued functions on X with $\|f\| = \sup_{x\in X}|f(x)|$. By E we denote a closed subspace of $B(X)$ containing the function $e = \{e(x)\} = \{1\}$, and we let E^* be the dual of E. A *mean* M on E is a linear functional $M \in E^*$ such that

(i) $M(f) \ge 0$ for $f \ge 0$,

(ii) $M(e) = 1$.

A mean $M \in E^*$ is *invariant* under a transformation $T : X \to X$ provided $M(f) = M(f \circ T)$ for all $f \in E$.

Let $\mathscr{T} = \{T\}$ be a family of transformations $T : X \to X$ such that $S \circ T = T \circ S$ for all $S, T \in \mathscr{T}$ Show: There is a mean $M \in E^*$ that is invariant under all $T \in \mathscr{T}$ (Mazur–Orlicz [1953]).

[Equip E^* with the weak* topology and consider the convex set $K = \{M \in E^* \mid M(f) \ge 0$ for $f \ge 0$ and $M(e) = 1\}$; for each T define an affine map $F_T\ \ K \to K$ by $M \mapsto M_T$, where $M_T(f) = M(f \circ T)$ for all $f \in E$; apply the Markoff–Kakutani theorem to the family $\{F_T\}_{T\in\mathscr{T}}$.]

(B.2) (*Generalized limits*) Show that there exists a method of assigning a "generalized limit" $\operatorname{Lim}_{n\to\infty}$ to every bounded real-valued sequence $\{x_n\}$ in such a way that

(i) $\operatorname{Lim}_{n\to\infty} x_n = \operatorname{Lim}_{n\to\infty} x_{n+1}$,

(ii) $\operatorname{Lim}_{n\to\infty}(\alpha x_n + \beta y_n) = \alpha \operatorname{Lim}_{n\to\infty} x_n + \beta \operatorname{Lim}_{n\to\infty} y_n$,

(iii) $\operatorname{Lim}_{n\to\infty} x_n \ge 0$ if $x_n \ge 0$,

(iv) $\liminf_{n\to\infty} x_n \le \operatorname{Lim}_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n$,

(v) if $\{x_n\}$ is a convergent sequence, then $\operatorname{Lim}_{n\to\infty} x_n = \lim_{n\to\infty} x_n$

(Mazur [1927]).

(B.3) Let X be a compact topological space, G an abelian group, and let $\{S_a\}_{a\in G}$ be a family of homeomorphisms of X onto X such that $S_a \circ S_b = S_{a+b}$ for all $a, b \in G$. A Borel measure μ on X is *invariant* under $\{S_a\}_{a\in G}$ if $\mu(A) = \mu(S_a A)$ for all $a \in G$ and all μ-measurable sets $A \subset X$. Show: There exists on X an invariant measure μ such that $\mu(X) = 1$ (Kryloff Bogoliouboff [1937]).

[First observe that using the Riesz representation theorem, it is enough to prove the existence of a positive linear functional on the space $C(X)$ that is invariant under $\{S_a\}_{a\in G}$; then use (B.1).]

(B.4) Let S be a semigroup and f $S \to \boldsymbol{R}$ any function. For a fixed $a \in S$ we let $L_a f(x) = f(ax)$ for each $x \in S$; $L_a f$ is the *left translate* of f by a. Let E be a linear subspace of $B(S)$ such that $x \in S$ and $f \in E$ imply $L_x f \in E$. A *left-invariant mean* M on E is a mean such that $M(L_x f) = M(f)$ for all $x \in S$ and $f \in E$. A semigroup S is *left-amenable* if there is a left-invariant mean on $B(S)$. Show: Every abelian semigroup is left-amenable (Day [1961]).

(B.5) Let G be a locally compact topological group and $CB(G)$ the Banach space of all continuous bounded functions on G with the sup norm. We say that a function $f \in CB(G)$ is *almost periodic* (written $f \in AP(G)$) provided its orbit under left translations $\mathcal{O}(f) = \{L_x f \mid x \in G\}$ is relatively compact in $CB(G)$. Prove: If $f \in AP(G)$, then $K = \operatorname{Conv}\mathcal{O}(f)$ contains a constant function.

[For each $a \in G$ define an affine map $T_a : K \to K$ by $T_a f = L_a f$, and apply the Kakutani theorem to the family $\{T_a\}_{a\in G}$.]

6. Notes and Comments

Geometric KKM-theory

The topological KKM-property was discovered by Knaster–Kuratowski–Mazurkiewicz [1929]. Really, though, the KKM-theory in the infinite-dimensional setting began with the articles of Fan [1961], [1964], [1966], in which the significance of the topological KKM-property was brought to light and diverse applications were given. Several of those results were simplified and codified by Browder [1968], who introduced to the theory the technique of selection of set-valued maps. The term "KKM-map" (introduced in Dugundji–Granas [1978]) was employed for the first time in a systematic way in Lassonde [1983]. In §3, we are concerned with the "geometric" part of the theory of KKM-maps; the "topological" part is treated in Chapter II. The presentation of results in §3, including the formulation of the "geometric KKM-principle" (1.5), follows Granas Lassonde [1991]. We remark that (1.5) (which appeared in a different form in Asakawa [1986]) can be shown to be in fact equivalent to the following theorem of Klee [1951] (see also Berge's monograph [1959]): *If $C_1, \dots, C_{n+1}$ are closed convex subsets of a Euclidean space $\boldsymbol{R}^s$, any n of which have nonempty intersection, and $\bigcup_{i=1}^{n+1} C_i$ is convex, then $\bigcap_{i=1}^{n+1} C_i \neq \emptyset$.*

The following are some applications of the geometric KKM-principle discussed in §§3 and 4:

1° theory of games (theorem of von Neumann (2.1));
2° systems of convex inequalities;
3° linear functional analysis (theorems of Markoff–Kakutani and Mazur–Orlicz);
4° variational inequalities (theorem of Stampacchia for bilinear forms, theorem of Hartman–Stampacchia);
5° maximal monotone operators.

Among aplications that can be obtained in the context of the geometric KKM-theory we mention the proof (due to Fan) of the following theorem of Hardy–Littlewood–Pólya: *If the real numbers* a_i, b_i $(1 \leq i \leq n)$ *satisfy:*

(i) $a_1 \geq \cdots \geq a_n \geq 0$, $b_1 \geq \cdots \geq b_n \geq 0$,

(ii) $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ $(1 \leq k \leq n)$,

then there exists a doubly stochastic matrix $P = (P_{ij})_{i,j=1}^n$ (*i.e.*, $P_{ij} \geq 0$, $\sum_{j=1}^n P_{ij} = 1$ *for each* i *and* $\sum_{i=1}^n P_{ij} = 1$ *for each* j) *such that*

$$a_i = \sum_{j=1}^{n} P_{ij} b_j \qquad (1 \leq i \leq n).$$

In the special case where $X \subset \boldsymbol{R}^n$, $Y \subset \boldsymbol{R}^m$ are simplices and the map f is bilinear, Theorem (2.1) was discovered by von Neumann [1928]; Theorems (2.2) and (2.3) are due to Fan [1957]; for earlier versions see Kneser [1952] and Nikaidô [1954].

Fixed points for families of maps

Theorem (3.2) was proved by Markoff [1936] with the aid of the Schauder–Tychonoff fixed point theorem. Kakutani [1938] found a direct elementary proof of (3.2) (valid in any linear topological space, not necessarily locally convex), and demonstrated the importance of the result by giving a number of applications; he also showed that (3.2) (together with the fact that a Tychonoff cube is compact) permits one to prove the Hahn–Banach principle. The following extension of the Markoff–Kakutani theorem is due to Day [1961]: *Let* K *be a compact convex set in a locally convex space, and let* S *be a left-amenable* (see (B.4)) *semigroup of continuous affine maps acting on* K. *Then there is a common fixed point under* S. Because every abelian semigroup is left-amenable and since in the formulation of (3.2) a "commuting family" can clearly be replaced by an "abelian semigroup", the result of Day contains the Markoff–Kakutani theorem as a special case.

The proof of Kakutani's theorem (4.3) is presented for normed linear spaces using the approach given in Hurewicz's lectures. An extension of the above theorem to locally convex spaces as well as other fixed point results for families of affine maps are given at the end of Chapter II.

Bruhat–Tits [1972] introduced a class of complete metric spaces satisfying the semi-parallelogram law and established in such spaces the existence of a common fixed point for a group of isometries having a bounded orbit. In spite of its elementary character, the Bruhat–Tits theorem turns out to be of importance in numerous problems of differential geometry (see Lang [1998]).

Invariant functionals, means and measures

Because of their numerous applications, fixed point theorems for families of affine maps provide a powerful tool in a variety of mathematical areas such as linear functional analysis, abstract harmonic analysis, and ergodic theory.

Let X be a set, G a semigroup acting on X, and E a linear topological space of real-valued functions on X and invariant under translations. An *invariant mean* is a positive linear functional on E that is invariant under the action of G. Invariant means were brought to use by Mazur [1927], who (before the discovery of the Hahn–Banach theorem) established the existence of an invariant mean on the additive semigroup N of natural numbers; this invariant mean (presently called a "generalized limit" or "Banach–Mazur limit") is a positive functional on the space l^∞ of bounded sequences that extends the ordinary limit functional from the space c over l^∞ (cf. (5.B.2)). The most important case in which invariant means occur is that of X a topological group and G the group of left, right, or two-sided translations on X. The corresponding means are called left, right, or two-sided. A group G is *amenable* if there is a left-invariant mean on $\mathrm{CB}(G)$ (the continuous bounded functions). *For locally compact groups amenability is equivalent to the existence of a fixed point for an arbitrary affine action of the group on a compact convex set in a locally convex space* (see Day [1961] and also the monograph of Greenleaf [1969]).

The consideration of an invariant mean on the space $C_0(G)$ (= continuous functions vanishing at infinity) leads to the important concept of Haar measure; when G is compact, we have $C_0(G) = C(G)$, and by the Riesz representation theorem such a mean must be of the form $M(f) = \int_G f(x)\,d\mu(x)$, where μ is a certain Borel measure. The existence of a Haar measure on an arbitrary compact group G can be established with the aid of the Kakutani theorem; when G is abelian, a much simpler proof can be given using the Markoff–Kakutani theorem. For applications in another area, we remark that the Markoff–Kakutani theorem has uses in ergodic theory: it yields, for example, a simple proof of the theorem of Kryloff–Bogolioubofff [1937] concerning the existence of an invariant measure of a dynamical system.

For more details about invariant means see "Miscellaneous Results and Examples", Dixmier [1950] and also the monographs by Greenleaf [1969] and Hewitt–Ross [1963], where further references can be found.

§4. Further Results and Applications

This paragraph is devoted primarily to applications of the main theorems in the text. We begin with two direct and simple applications of the Banach principle, first to establish some fixed point theorems for nonexpansive maps in Hilbert space and second to integral and differential equations. In Section 3, we give numerous applications of the elementary invariance of domain theorem. The last two sections are devoted to some further applications of the geometric KKM-principle.

1. Nonexpansive Maps in Hilbert Space

The Banach principle involves a contractive map in an arbitrary complete metric space. By giving the space a sufficiently rich structure, the contractiveness hypothesis on the map can be relaxed to nonexpansiveness; of course, uniqueness of the fixed point cannot be preserved, as the reflection of $\boldsymbol{R}^2$ in a line shows.

In this section we deal with (real) Hilbert space; the following proposition will play a basic role.

(1.1) PROPOSITION. *Let H be a Hilbert space, and let u, v be two elements of H. If there is an $x \in H$ such that $\|x - u\| \le R$, $\|x - v\| \le R$ and $\|x - (u+v)/2\| \ge r$, then $\|u - v\| \le 2\sqrt{R^2 - r^2}$*

PROOF. By the parallelogram law,

$$\begin{aligned}\|u - v\|^2 &= \|(x - v) - (x - u)\|^2 \\ &= 2\|x - v\|^2 + 2\|x - u\|^2 - \|x - v + x - u\|^2 \\ &= 2\|x - v\|^2 + 2\|x - u\|^2 - 4\left\|x - \frac{u+v}{2}\right\|^2,\end{aligned}$$

and the conclusion follows. □

We apply this proposition to study nonexpansive maps on bounded sets:

(1.2) LEMMA. *Let $C \subset H$ be a bounded set, and let $F : C \to C$ be nonexpansive. Assume that x, y and $a = (x+y)/2$ belong to C. If $\|x - F(x)\| \le \varepsilon$ and $\|y - F(y)\| \le \varepsilon$, then*

$$\|a - F(a)\| \le 2\sqrt{2\delta(C)}\sqrt{\varepsilon},$$

where $\delta(C) =$ diameter of C.

PROOF. Because

$$\|x - y\| \le \left\|x - \frac{a + F(a)}{2}\right\| + \left\|y - \frac{a + F(a)}{2}\right\|,$$

at least one of the terms on the right, say the first, must satisfy

$$\left\| x - \frac{a+F(a)}{2} \right\| \geq \frac{1}{2}\|x-y\|.$$

But also $\|x-a\| = \frac{1}{2}\|x-y\|$ and

$$\|x - F(a)\| \leq \|x - F(x)\| + \|F(x) - F(a)\| \leq \varepsilon + \|x - a\| = \varepsilon + \frac{1}{2}\|x-y\|.$$

By (1.1), we conclude that

$$\|a - F(a)\| \leq 2\sqrt{\left(\varepsilon + \tfrac{1}{2}\|x-y\|\right)^2 - \left(\tfrac{1}{2}\|x-y\|\right)^2} = 2\sqrt{\varepsilon}\sqrt{\varepsilon + \|x-y\|},$$

and since both ε and $\|x-y\|$ do not exceed $\delta(C)$, the proof is complete. □

This leads to the desired modification of the Banach theorem.

(1.3) THEOREM (Browder-Göhde–Kirk). *Let C be a nonempty closed bounded convex set in a Hilbert space. Then each nonexpansive map $F: C \to C$ has at least one fixed point.*

PROOF. There is no loss in generality to assume that $0 \in C$. For each integer $n = 2, 3, \ldots$ let $F_n = \left(1 - \frac{1}{n}\right)F$; because C is convex and contains the origin, each F_n maps C into itself. Moreover, each $F_n : C \to C$ is contractive, so by Banach's theorem, each F_n has a fixed point x_n, and

$$\|x_n - F(x_n)\| = \frac{1}{n}\|F(x_n)\| \leq \frac{1}{n}\delta(C).$$

For each $n \geq 2$, let $Q_n = \{x \in C \mid \|x - F(x)\| \leq \frac{1}{n}\delta(C)\}$; then $Q_2 \supset Q_3 \supset \cdots$ is a descending sequence of closed sets, and by what we have just shown, no Q_n is empty. We observe that if $x, y \in Q_{8n^2}$ and $a = (x+y)/2$, then according to the lemma

$$\|a - F(a)\| \leq 2\sqrt{2\delta(C)}\sqrt{\frac{\delta(C)}{8n^2}}, \quad \text{so that} \quad \frac{x+y}{2} \in Q_n.$$

Let $d_n = \inf\{\|x\| \mid x \in Q_n\}$; because the Q_n are descending, we see that $d_2 \leq d_3 \leq \cdots$ is a nondecreasing sequence of reals, which, being bounded by $\delta(C)$, converges to some d. Finally, let

$$A_n = Q_{8n^2} \cap \overline{B\left(0, d + \frac{1}{n}\right)}.$$

Then A_n is a descending sequence of nonempty closed sets. We calculate the diameter of A_n: if $x, y \in A_n$, then $\|0 - x\| \leq d + 1/n$, $\|0 - y\| \leq d + 1/n$, and by our observation above, $\|0 - (x+y)/2\| \geq d_n$; therefore, by (1.1)

we find

$$\|x-y\| \le 2\sqrt{\left(d+\frac{1}{n}\right)^2 - d_n^2} = 2\sqrt{2dn^{-1}+n^{-2}+(d^2-d_n^2)}.$$

The term on the right is therefore an upper bound for $\delta(A_n)$, and shows that $\delta(A_n) \to 0$ as $n \to \infty$.

By Cantor's theorem, there is an $x_0 \in \bigcap_n A_n$; since $x_0 \in \bigcap_n Q_{8n^2}$, we find

$$\|x_0 - F(x_0)\| \le \delta(C)/(8n^2) \quad \text{for all } n;$$

therefore, $\|x_0 - F(x_0)\| = 0$, and x_0 is a fixed point. □

Let C be a closed ball in a Hilbert space H; we will now consider the nonexpansive maps defined on C with values in H. For this purpose, we need the nonexpansiveness of the standard retraction of H on C:

(1.4) LEMMA. *Let H be a Hilbert space and C the closed ball $\{x \in H \mid \|x\| \le c\}$. Define a map $r : H \to C$ by*

$$r(x) = \begin{cases} x, & \|x\| \le c, \\ c\dfrac{x}{\|x\|}, & \|x\| \ge c. \end{cases}$$

Then $r : H \to C$ is nonexpansive.

PROOF. We first observe that if $u, v \neq 0$, then

$$(u - r(u), r(v) - r(u)) \le 0.$$

This is certainly true for $\|u\| \le c$ since $r(u) = u$; and if $\|u\| \ge c$, we have

$$(u-r(u), r(v)-r(u)) = \begin{cases} \left(1 - \dfrac{c}{\|u\|}\right)[(u,v) - c\|u\|], & \|v\| \le c, \\ \left(1 - \dfrac{c}{\|u\|}\right)\left[c\dfrac{(u,v)}{\|v\|} - c\|u\|\right], & \|v\| \ge c, \end{cases}$$

so that because $|(u,v)| \le \|u\|\|v\|$, our observation is established. To prove the lemma, write

$$x - y = r(x) - r(y) + x - r(x) + r(y) - y \equiv r(x) - r(y) + a;$$

then

$$\|x-y\|^2 = \|r(x)-r(y)\|^2 + \|a\|^2 + 2(a, r(x)-r(y));$$

because of our observation,

$$(a, r(x)-r(y)) = -(x - r(x), r(y) - r(x)) - (y - r(y), r(x) - r(y)) \ge 0,$$

so $\|x-y\|^2 \ge \|r(x)-r(y)\|^2$, and the proof is complete. □

This leads to the desired result:

(1.5) THEOREM (Nonlinear alternative for nonexpansive maps). *Let H be a Hilbert space and C the closed ball $\{x \in H \mid \|x\| \leq c\}$. Then each nonexpansive $F : C \to H$ has at least one of the following two properties:*
(a) *F has a fixed point,*
(b) *there exist $x \in \partial C$ and $\lambda \in (0,1)$ such that $x = \lambda F(x)$.*

PROOF. By (1.4), the map $r : H \to C$ is nonexpansive, therefore so also is $r \circ F : C \to C$, and by (1.3), we have $rF(x) = x$ for some $x \in C$. Now we repeat the reasoning of (0.2.3): If $F(x) \in C$, then $x = rF(x) = F(x)$, so F has a fixed point; if $F(x)$ does not belong to C, then $x = rF(x) = cF(x)/\|F(x)\|$, so $x \in \partial C$, and taking $\lambda = c/\|F(x)\| < 1$ completes the proof. □

Several fixed point theorems are obtained from (1.5) by imposing conditions that prevent occurrence of the second possibility:

(1.6) COROLLARY. *Let $C = \{x \in H \mid \|x\| \leq r\}$, and let $F : C \to H$ be nonexpansive. Assume that for all $x \in \partial C$, one of the following conditions holds:*
(a) $\|F(x)\| \leq \|x\|$,
(b) $\|F(x)\| \leq \|x - F(x)\|$,
(c) $\|F(x)\|^2 \leq \|x\|^2 + \|x - F(x)\|^2$,
(d) $(x, F(x)) \leq \|x\|^2$,
(e) $F(x) = -F(-x)$.
Then F has a fixed point.

The proof is strictly analogous to the proof of (1.4.2) and (1.4.3), and is left to the reader.

As a further application, we have

(1.7) COROLLARY. *Let H be a Hilbert space and $F : H \to H$ be nonexpansive. Assume $(x, x - F(x)) \geq \mu(\|x\|)\|x\|$, where $\mu(\|x\|) \to \infty$ as $\|x\| \to \infty$. Then the nonexpansive field $x \mapsto f(x) = x - F(x)$ is surjective.*

PROOF. Given a point $y_0 \in H$ let $g(x) = x - [F(x) + y_0]$ for $x \in H$. From

$$\frac{(x, g(x))}{\|x\|} = \frac{(x, f(x))}{\|x\|} - \frac{(x, y_0)}{\|x\|} \geq \mu(\|x\|) - \|y_0\|$$

it follows that for a sufficiently large $r > 0$,

$$(x, g(x)) \geq 0 \quad \text{for all } x \in H \text{ with } \|x\| = r.$$

By (1.6)(d), because $G(x) = F(x) + y_0$ is nonexpansive, we get $g(x_0) = 0$ for some x_0; hence $y_0 = x_0 - F(x_0) = f(x_0)$, and our assertion follows. □

2. Applications of the Banach Principle to Integral and Differential Equations

Use of the Banach principle requires that a given $F : Y \to Y$ be contractive relative to some complete metric d in Y If the given F is not contractive with respect to one metric, it may be possible to find another complete metric with respect to which F is contractive. For example, the (linear) map $(x, y) \mapsto \frac{1}{10}(8x + 8y, x + y)$ of $\boldsymbol{R}^2$ into itself is not contractive with respect to the usual metric

$$d[(x,y),(z,w)] = \sqrt{(x-z)^2 + (y-w)^2};$$

but it is contractive, with contraction constant $\frac{9}{10}$, relative to the (equivalent, and complete) metric

$$\widehat{d}\,[(x,y),(z,w)] = |x-z| + |y-w|.$$

Thus, each complete metric d in Y determines a class $\mathscr{F}(d)$ of maps $F : Y \to Y$ that are contractive with respect to d, and in general, $\mathscr{F}(d) \neq \mathscr{F}(\widehat{d}\,)$ even for equivalent metrics d and $\widehat{d}$.

If E is a Banach space, recall that two norms $|x|$ and $\|x\|$ are *equivalent* if there are constants $m, M > 0$ with $m\|x\| \le |x| \le M\|x\|$, so that a map Lipschitzian in one norm is Lipschitzian in any equivalent norm. Thus, in Banach spaces, to study a Lipschitzian map $F : E \to E$, it is frequently very fruitful to seek a norm under which F is contractive.

These considerations are illustrated in the following proof of the existence of solutions for the Volterra integral equation of the second kind.

(2.1) THEOREM. *Let $K : [0,T] \times [0,T] \times \boldsymbol{R} \to \boldsymbol{R}$ be continuous and satisfy a Lipschitz condition*

$$|K(t,s,x) - K(t,s,y)| \le L|x-y|$$

for all $(s,t) \in [0,T] \times [0,T]$, and $x, y \in \boldsymbol{R}$. Then for any $v \in C[0,T]$ the equation

$$u(t) = v(t) + \int_0^t K(t,s,u(s))\,ds \qquad (0 \le t \le T)$$

has a unique solution $u \in C[0,T]$. Moreover, if we define a sequence of functions $\{u_n\}$ inductively by choosing any $u_0 \in C[0,T]$ and setting

$$u_{n+1}(t) = v(t) + \int_0^t K(t,s,u_n(s))\,ds,$$

then the sequence $\{u_n\}$ converges uniformly on $[0,T]$ to the unique solution u.

PROOF. Let E be the Banach space of all continuous real-valued functions on $[0, T]$ equipped with the norm

$$|g| = \max_{0 \le t \le T} e^{-Lt} |g(t)|.$$

This norm is in fact equivalent to the sup norm $\|x\|$, since

$$e^{-LT}\|x\| \le |x| \le \|x\|;$$

and moreover, it is also complete.

Define $F : E \to E$ by

$$F(g)(t) = v(t) + \int_0^t K(t, s, g(s))\, ds;$$

to prove that the integral equation has a solution, it is enough to show that $F : E \to E$ has a fixed point. We prove that, in fact, F is contractive: for

$$\begin{aligned}
|F(g) - F(h)| &\le \max_{0 \le t \le T} e^{-Lt} \int_0^t |K(t, s, g(s)) - K(t, s, h(s))|\, ds \\
&\le L \max_{0 \le t \le T} e^{-Lt} \int_0^t |g(s) - h(s)|\, ds \\
&= L \max_{0 \le t \le T} e^{-Lt} \int_0^t e^{Ls} e^{-Ls} |g(s) - h(s)|\, ds \\
&\le L|g - h| \max_{0 \le t \le T} e^{-Lt} \int_0^t e^{Ls}\, ds \\
&= L|g - h| \max_{0 \le t \le T} e^{-Lt} \frac{e^{Lt} - 1}{L} \\
&\le (1 - e^{-LT})|g - h|.
\end{aligned}$$

Because $1 - e^{-LT} < 1$, the map $F : E \to E$ is contractive; Banach's principle therefore guarantees first a unique fixed point $u \in E$, and then that the sequence $\{u_n\}$ determined by the iterations described in the statement of the theorem converges uniformly in the norm $|x|$, therefore also in the sup norm $\|x\|$, to that fixed point. □

Observe that if we had used the sup norm $\|x\|$ rather than $|x|$, then F would be contractive when regarded as a map $C[0, \lambda] \to C[0, \lambda]$, where $\lambda < \min\{T, 1/L\}$. Thus, if $T > 1/L$, then the Banach principle with the usual sup norm would have guaranteed a unique solution only on a subinterval of $[0, T]$, whereas by modifying the norm we have shown that in fact there is a unique solution on the entire interval $[0, T]$.

(2.2) THEOREM. *Let $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ satisfy the Lipschitz condition*

$$|f(s,x) - f(s,y)| \le L|x-y|$$

for $s \in [0,T]$, $x, y \in \mathbf{R}$. Then the initial value problem

$$\frac{du}{ds} = f(s,u), \qquad u(0) = 0,$$

has exactly one solution u defined on the entire interval $[0,T]$.

PROOF. If $K(t,s,u) = f(s,u)$ and $v(t) = 0$ in (2.1), the Volterra equation becomes

$$u(t) = \int_0^t f(s,u(s))\,ds,$$

and the solution of this integral equation is precisely the solution of the present initial value problem. □

3. Applications of the Elementary Domain Invariance

We now give applications of the elementary domain invariance theorem in various fields such as linear functional analysis and the geometry of Banach spaces.

a. Domain invariance and invertibility of linear operators

We begin with two simple propositions:

(3.1) PROPOSITION. *Let $T : E \to E$ be a linear operator in a Banach space. If $\|I - T\| < 1$, then T is invertible, and*

$$\|T^{-1}\| \le \frac{1}{1 - \|I-T\|}.$$

PROOF. The map $I - T : E \to E$ is contractive, since

$$\|(I-T)(x-y)\| \le \|I-T\|\|x-y\|,$$

so by (1.2.2), the map $I - (I-T) = T$ is a homeomorphism, therefore invertible. The bound for the norm follows from

$$1 = \|TT^{-1}\| = \|T^{-1} - T^{-1}(I-T)\| \ge \|T^{-1}\| - \|T^{-1}\|\|I-T\|. \qquad \square$$

(3.2) PROPOSITION. *Let $T : E \to E$ be an invertible linear operator in a Banach space. Then each linear operator S with $\|T - S\| < 1/\|T^{-1}\|$ is invertible, and*

$$\|S^{-1}\| \le \frac{\|T^{-1}\|}{1 - \|I - ST^{-1}\|}.$$

PROOF. Because T is invertible, it is enough to show that $J = S \circ T^{-1}$ is invertible, for then $S = J \circ T$ has $T^{-1}J^{-1}$ as inverse. From

$$\|I - J\| = \|I - ST^{-1}\| = \|(T - S) \circ T^{-1}\| \leq \|T - S\|\|T^{-1}\| < 1$$

and (3.1) we find that J is in fact invertible; and since $S^{-1} = T^{-1}J^{-1}$, the norm estimate follows from (3.1). □

These results lead to

(3.3) THEOREM. *Let E be a Banach space and $\mathscr{A} \subset \mathscr{L}(E, E)$ the set of all invertible linear operators. Let* Inv : $\mathscr{A} \to \mathscr{A}$ *be the map $T \mapsto T^{-1}$ Then $\mathscr{A}$ is open in $\mathscr{L}(E, E)$ and* Inv *is a homeomorphism of $\mathscr{A}$ onto itself.*

PROOF. By (3.2), for each $T \in \mathscr{A}$, the ball $B(T, 1/(2\|T^{-1}\|))$ is also contained in $\mathscr{A}$; therefore $\mathscr{A}$ is open in $\mathscr{L}(E, E)$. To prove that Inv is continuous at any given $T \in \mathscr{A}$, it is enough to note that if $S \in B(T, 1/(2\|T^{-1}\|))$, so that $\|I - ST^{-1}\| < \frac{1}{2}$, then by (3.2) we have $S \in \mathscr{A}$ and

$$\begin{aligned}\|S^{-1} - T^{-1}\| &= \|T^{-1}(T - S)S^{-1}\| \\ &\leq \frac{\|T^{-1}\|^2}{1 - \|I - ST^{-1}\|}\|T - S\| \leq 2\|T^{-1}\|^2\|T - S\|;\end{aligned}$$

and since Inv ∘ Inv = id, it follows that Inv is a homeomorphism. □

Let E, F be Banach spaces and $S : E \to F$ a linear operator. If there is some $m > 0$ such that $\|Sx\|_F \geq m\|x\|_E$ for all $x \in E$, then it is immediate that S is injective; if such an S is also surjective, then it is invertible because $\|S^{-1}y\|_E \leq (1/m)\|y\|_F$ for all $y \in F$ shows that the inverse is continuous. The following result extends this observation to suitable perturbations of such operators, and is of importance in work with "a priori" estimates for linear differential operators.

(3.4) THEOREM (Schauder invertibility theorem). *Let E, F be Banach spaces and $S, T : E \to F$ two linear operators, with S invertible. Assume that there is an $m > 0$ such that for each $0 \leq t \leq 1$, the operator $L_t = (1-t)S + tT$ satisfies $\|L_t x\|_F \geq m\|x\|_E$ for all $x \in E$. Then L_t is invertible for all $0 \leq t \leq 1$, and in particular, T is invertible.*

PROOF. We begin by showing that if an operator L_s is invertible, then for each t in the open interval $J_s = \{t \mid |t - s| < m/\|T - S\|\}$, the operator L_t is invertible or, what is equivalent, that $L_s^{-1}L_t : E \to E$ is invertible for each $t \in J_s$. For this purpose, note that

$$L_t = S + s(T - S) + (t - s)(T - S) = L_s + (t - s)(T - S),$$

so that

$$L_s^{-1}L_t = I + (t-s)L_s^{-1}(T-S).$$

Because $\|L_s x\|_F \geq m\|x\|_E$, we have $\|L_s^{-1}\| \leq 1/m$, so for $t \in J_s$, we find

$$\|(t-s)L_s^{-1}(T-S)\| \leq |t-s|\frac{1}{m}\|T-S\|,$$

and by (3.1), the operator $L_s^{-1}L_t$ is therefore invertible.

To prove the theorem, let $\mathscr{J} = \{t \in [0,1] \mid L_t \text{ is invertible}\}$. By what we have just shown, $\mathscr{J}$ is an open set. If $t \notin \mathscr{J}$, then again by what we have just shown, no operator L_s with $s \in J_t$ can be invertible, so that $[0,1] - \mathscr{J}$ is also an open set. Because $[0,1]$ is connected and $\mathscr{J}$ is nonempty, we conclude that $\mathscr{J} = [0,1]$. □

b. The inverse function theorem

As another application we obtain a standard theorem in analysis:

(3.5) THEOREM (Inverse function theorem). *Let E be a Banach space, $U \subset E$ open, and $f : U \to E$ a C^1 map. Assume that at some $x_0 \in U$, the derivative $Df(x_0) : E \to E$ is an isomorphism. Then there exists a neighborhood V of x_0 and a neighborhood W of $f(x_0)$ such that:*

(1) *$Df(x) : E \to E$ is invertible for each $x \in V$,*
(2) *$f|V : V \to W$ is a homeomorphism of V onto W,*
(3) *the inverse $g : W \to V$ of $f|V$ is differentiable at each $w \in W$ and $Dg(w) = [Df(gw)]^{-1}$,*
(4) *the map $w \mapsto Dg(w)$ of W into $\mathscr{L}(E,E)$ is continuous.*

PROOF. We first consider the special case where $x_0 = 0$, $f(0) = 0$, and $Df(0) = I$. Because the set of invertible operators is open in $\mathscr{L}(E,E)$ and $x \mapsto Df(x)$ is continuous with $Df(0)$ invertible, we can find a ball B with $0 \in B \subset U$ on which $Df(x)$ is invertible.

Define $F : B \to E$ by $F(x) = x - f(x)$. Then F is a C^1 map, $DF(0) = I - Df(0) = 0$, and because F is continuously differentiable, there is a ball V with $0 \in V \subset B$ such that $M = \sup\{\|DF(x)\| \mid x \in V\} < \frac{1}{2}$. The map $F : V \to E$ is contractive: for, by the mean value theorem,

$$\|F(x_1) - F(x_2)\| \leq M\|x_1 - x_2\| \leq \tfrac{1}{2}\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in V.$$

Thus, by (1.2.1), the map $f : V \to E$ is a homeomorphism onto the open set $W = f(V)$ containing $f(0) = 0$, and the proof of both (1) and (2) is complete. We observe, for later reference, that if $x, a \in V$, then

$$\|x-a\| - \|f(x) - f(a)\| \leq \|F(x) - F(a)\| \leq \tfrac{1}{2}\|x-a\|,$$

so that

$$\|x - a\| \le 2\|f(x) - f(a)\|.$$

We now prove (3). Let $g : W \to V$ be the inverse of $f|V$. Given $y, b \in W$, let $g(b) = a$, $g(y) = x$, and write $Df(a) = T$ By the differentiability of f at a, we have

$$f(x) - f(a) = T(x - a) + \varphi(x, a),$$

where $\varphi(x, a)/\|x - a\| \to 0$ as $\|x - a\| \to 0$. Applying T^{-1} to this expression and noting that $f(x) = y$, $f(a) = b$, we find

$$T^{-1}(y - b) = g(y) - g(b) + T^{-1}\varphi[g(y), g(b)]$$

so it suffices to show that

$$R = \frac{\|T^{-1}\varphi[g(y), g(b)]\|}{\|y - b\|} \to 0 \quad \text{as } \|y - b\| \to 0.$$

Because $g|W : W \to V$ is bijective, we have

$$\begin{aligned} R &\le \frac{\|T^{-1}\| \|\varphi[g(y), g(b)]\|}{\|g(y) - g(b)\|} \frac{\|g(y) - g(b)\|}{\|y - b\|} \\ &\le 2\|T^{-1}\| \frac{\|\varphi[g(y), g(b)]\|}{\|g(y) - g(b)\|} = 2\|T^{-1}\| \frac{\|\varphi(x, a)\|}{\|x - a\|}, \end{aligned}$$

where we have used the observation above that $\|x - a\| \le 2\|f(x) - f(a)\|$ on V, i.e., $\|g(y) - g(b)\| \le 2\|y - b\|$ on W. Thus, as $\|y - b\| \to 0$, the continuity of g shows that $\|x - a\| \to 0$, and therefore $R \to 0$. This completes the proof of (3).

To prove (4), we note that $w \mapsto Dg(w)$ is the composition $\mathrm{Inv} \circ Df \circ g$ of three continuous maps, so it is continuous on W.

This completes the proof of the theorem in the special case where $f(0) = 0$ and $Df(0) = I$. To prove the theorem as it is stated, apply this special case to the function

$$h(x) = [Df(x_0)]^{-1}(f(x + x_0) - f(x_0)). \qquad \square$$

c. Stability of open embeddings and monotone operators

We now apply elementary domain invariance to stability of open embeddings into Banach spaces.

(3.6) THEOREM. *Let X be any space, E a Banach space, and $F : X \to E$ an embedding of X onto an open $U \subset E$. Let $G : X \to E$ be a map such that $G \circ F^{-1} : U \to E$ is contractive. Then $x \mapsto Fx + Gx$ is also an open embedding of X into E.*

PROOF. Consider $h = (F+G)\circ F^{-1} = I + GF^{-1} : U \to E$. By domain invariance this h maps U homeomorphically onto an open $h(U) \subset E$. Since $F + G = h \circ F$, the proof is complete. □

As an evident consequence we obtain the stability property of Lipschitzian open embeddings into Banach spaces.

(3.7) THEOREM. *Let X be a metric space and E a Banach space, and let $F : X \to E$ be an open embedding such that F^{-1} is Lipschitzian. Let $G : X \to E$ be a Lipschitzian map such that $L(G)L(F^{-1}) < 1$. Then $x \mapsto Fx + Gx$ is also an open embedding of X into E.*

PROOF. It is enough to observe that $L(G \circ F^{-1}) \le L(G)L(F^{-1})$ and to apply (3.6). □

We next derive some simple facts about monotone operators in Hilbert spaces.

Let H be a Hilbert space and $U \subset H$. A map $f : U \to H$ (not necessarily continuous) is said to be *monotone* if

(i) $$(fx - fy, x - y) \ge 0 \quad \text{for all } x, y \in U;$$

f is called *strongly monotone* if for some $C > 0$,

(ii) $$(fx - fy, x - y) \ge C\|x - y\|^2 \quad \text{for all } x, y \in U.$$

Clearly, every strongly monotone map is injective.

(3.8) THEOREM. *Let $U \subset H$ be open, and let $f : U \to H$ be Lipschitzian and strongly monotone, i.e.,*

$$\|fx - fy\| \le M\|x - y\| \quad \textit{for all } x, y \in U \textit{ and some } M > 0$$

and

$$(fx - fy, x - y) \ge C\|x - y\|^2 \quad \textit{for all } x, y \in U.$$

Then f is an open map, in particular $f(U)$ is open in H, and f is a homeomorphism of U onto $f(U)$.

PROOF. It is clearly enough to show that for a sufficiently small λ, the map λf is a contractive field. Given $\lambda > 0$, we have for $x, y \in U$,

$$\begin{aligned}
&\|(I - \lambda f)x - (I - \lambda f)y\|^2 \\
&\qquad = \|x - y\|^2 + \lambda^2\|fx - fy\|^2 - 2\lambda(fx - fy, x - y) \\
&\qquad \le \|x - y\|^2 + M^2\lambda^2\|x - y\|^2 - 2\lambda C\|x - y\|^2 \\
&\qquad \le (1 + M^2\lambda^2 - 2\lambda C)\|x - y\|^2.
\end{aligned}$$

Fix $\lambda < 2C/M^2$; then $1 + M^2\lambda^2 - 2\lambda C < 1$, and therefore $I - \lambda f$ is a contraction; our assertion now follows from (1.2.1). □

As a corollary we have

(3.9) THEOREM. *Let $f : H \to H$ be a Lipschitzian and strongly monotone map. Then f is a homeomorphism of H onto itself.*

PROOF. By (3.8) it is enough to show that $f(H)$ is closed in H. Let $fx_n \to y$; from (ii) and the Cauchy inequality we get

$$\|fx_n - fx_m\| \geq C\|x_n - x_m\| \quad \text{for all } n, m \geq 1,$$

and hence $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x$; then $fx_n \to fx$, and hence $y = fx$. □

d. Application to negligible sets

A subset B of a space Y is called *negligible* whenever $Y - B$ is homeomorphic to Y; a homeomorphism $h : Y - B \approx Y$ is called a *deleting homeomorphism.* We now show that any complete subset of a noncomplete normed linear space is negligible.

(3.10) THEOREM. *Let E be a noncomplete normed linear space and C a complete subset of E. Then there is a homeomorphism $h : E - C \approx E$ with $h(x) = x$ whenever $d(x, C) \geq 1$.*

PROOF. Let $\widehat{E}$ be the completion of E; taken with the natural extension of the given norm, $\widehat{E}$ is a Banach space. Let $\{x_n\}$ be a Cauchy sequence in E converging to some point in $\widehat{E} - E$, with $\|x_1\| + \sum_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty$; since no scalar multiple of a point in $\widehat{E} - E$ can belong to E, replacing all the x_n by a suitable scalar multiple, we can assume that $\{x_n\} \subset E$ converges to $y_0 \in \widehat{E} - E$ and $\|x_1\| + \sum_{n=1}^{\infty} \|x_n - x_{n+1}\| = \frac{1}{2}$.

Let $x_0 = 0$, and let $L \subset E$ be the jagged line in E consisting of the segments $[x_0, x_1] \cup [x_1, x_2] \cup \cdots$; we construct a piecewise linear map φ of the unit interval $[0, 1]$ onto $L \cup \{y_0\}$ as follows: let $s_0 = 0$ and for $n \geq 1$, let s_n be the nth partial sum of the series with sum $\frac{1}{2}$; for each $n \geq 0$, map the interval $[2s_n, 2s_{n+1}]$ linearly onto the segment $[x_n, x_{n+1}]$ and set $\varphi(1) = y_0$. It is clear that $|\varphi(t) - \varphi(t')| = \frac{1}{2}|t - t'|$ whenever t, t' belong to a common interval $[2s_n, 2s_{n+1}]$, so by the triangle inequality, we have $|\varphi(t) - \varphi(t')| \leq \frac{1}{2}|t - t'|$ for all $t, t' \in I$. Extend φ to a map of $(-\infty, 1]$ into E by $\varphi(t) = 0$ for $t < 0$.

Now let $H : \widehat{E} \to \widehat{E}$ be the map $x \mapsto \varphi[1 - d(x, C)]$. This map is contractive, because

$$\|\varphi[1 - d(x, C)] - \varphi[1 - d(z, C)]\| \leq \tfrac{1}{2}\|d(z, C) - d(x, C)\| \leq \tfrac{1}{2}\|x - z\|;$$

therefore, by (1.2.1), the map $h(x) = x - H(x)$ is a homeomorphism of $\widehat{E}$ onto itself.

It is clear that $h(E-C) \subset E$; for if $x \in E-C$, then $d(x,C) > 0$, so $H(x) \in E$, and therefore $h(x) = x - H(x)$ also belongs to E. To establish the converse inclusion, assume $x \notin E - C$, so that $x \in (\widehat{E} - E) \cup C$; if $x \in \widehat{E} - E$, then $d(x,C) > 0$, so $H(x) \in E$, while if $x \in C$, then $x \in E$ and $H(x) = y_0 \notin E$; in both cases, exactly one of $x, H(x)$ belongs to E, so we conclude that $h(x) = x - H(x) \notin E$. Thus $h(E-C) \approx E$. □

The class of linear spaces for which such deleting homeomorphisms exist is very broad because of

(3.11) LEMMA. *Every infinite-dimensional Banach space $(E, \|\cdot\|)$ admits a noncomplete norm $|\!|\;|\!|$ with $|\!|x|\!| \leq \|x\|$ for all $x \in E$.*

PROOF. Assume first that E is separable. Choose a countable separating family $\{f_n \mid n = 1, 2, \ldots\}$ of continuous linear functionals such that $|f_n(x)| \leq (1/n)\|x\|$ for each n, and define $T : E \to l^2$ by $T(x) = \{f_n(x)\}$. This is a continuous linear operator, and $T : E \to T(E)$ is bijective because the family $\{f_n\}$ is separating. If the linear subspace $T(E) \subset l^2$ were complete, then because a bijective continuous linear map of Banach spaces is a homeomorphism, the inverse $T^{-1} : T(E) \to E$ would be continuous; but this is impossible because $\overline{T\{x \mid \|x\| \leq 1\}}$ is easily seen to be compact, and E is infinite-dimensional. Thus, the l^2 norm on $T(E)$ is not complete, and defining $\|x\|_0 = \|T(x)\|$ gives an incomplete norm on E. We note that $\|x\|_0 \leq \sqrt{\sum(1/n^2)}\|x\|$, so that $\|\cdot\|_0$ is continuous.

Now let E be arbitrary. Pick a separable infinite-dimensional closed linear subspace $L \subset E$ and an incomplete norm $\|\cdot\|_0$ on L. Let $A = \{x \in L \mid \|x\|_0 < 1\}$; because $\|\cdot\|_0$ is continuous, there is an $\varepsilon > 0$ such that $L \cap \{x \mid \|x\| < \varepsilon\} \subset A$. Let $C = \operatorname{conv}[A \cup \{x \mid \|x\| < \varepsilon\}]$; since C is a symmetric convex body with no rays, the Minkowski functional φ_C gives a norm, and because $\varphi_C(x) = \|x\|_0$ for $x \in L$, that norm is not complete. Finally, the continuity of φ_C implies that

$$b = \sup\{\varphi_C(x) \mid \|x\| \leq 1\}$$

is finite, so $|\!|x|\!| = b^{-1}\varphi_C(x)$ is an incomplete norm with $|\!|x|\!| \leq \|x\|$. □

Combining this with (3.10) gives

(3.12) THEOREM (Klee). *Let E be an arbitrary infinite-dimensional normed linear space, and $C \subset E$ compact. Then there is a homeomorphism $h : E - C \approx E$.*

PROOF. We can assume that E is a Banach space, else the result follows directly from (3.10). By (3.11), there is an incomplete norm $|\!|x|\!| \leq \|x\|$; denote $(E, |\!|\cdot|\!|)$ by $\widehat{E}$; the identity map $j : E \to \widehat{E}$ is therefore con-

tinuous, so $\widehat{C} = j(C)$ is compact, therefore complete, in $\widehat{E}$. By (3.10), there is a deleting homeomorphism $\widehat{h} : \widehat{E} - \widehat{C} \approx \widehat{E}$ given by $\widehat{h}(\widehat{x}) = \widehat{x} - \widehat{\varphi}[1 - \widehat{d}(\widehat{x}, \widehat{C})]$, where $\widehat{d}$ is the norm-induced metric and $\widehat{\varphi}$ is a piecewise linear map $\widehat{\varphi} : (-\infty, 1) \to \widehat{E}$.

Because $\widehat{\varphi}$ is piecewise linear, it is continuous with any linear topology in the range space, so regarding it as a map $\varphi : (-\infty, 1) \to E$ we have φ continuous and $j \circ \varphi = \widehat{\varphi}$. Now define $h : E - C \to E$ by

$$h(x) = x - \varphi[1 - \widehat{d}(j(x), \widehat{C})];$$

this map is clearly continuous and makes the diagram

$$\begin{tikzcd} E - C \arrow[r, "h"] \arrow[d, "j"'] & E \arrow[d, "j"] \\ \widehat{E} - \widehat{C} \arrow[r, "\widehat{h}"'] & \widehat{E} \end{tikzcd}$$

commutative. Finally, define $g : E \to E - C$ by

$$g(x) = x + \varphi[1 - \widehat{d}(\widehat{h}^{-1}j(x), \widehat{C})];$$

this map is also continuous. We have

$$\begin{aligned} g[h(x)] &= h(x) + \varphi[1 - \widehat{d}(\widehat{h}^{-1}jh(x), \widehat{C})] \\ &= \{x - \varphi[1 - \widehat{d}(j(x), \widehat{C})]\} + \varphi[1 - \widehat{d}(\widehat{h}^{-1}\widehat{h}j(x), \widehat{C})] \\ &= x, \end{aligned}$$

and similarly $h \circ g = \mathrm{id}$. Thus, $h : E - C \to E$ is a homeomorphism having g as its inverse. □

4. Elementary KKM-Principle and Its Applications

In this section we present a version of the geometric KKM-principle that permits us to establish in an elementary manner a large number of important results in Hilbert space theory. The approach, in which neither the weak topology nor compactness are used, is based on some simple intersection property of convex sets in Hilbert spaces.

a. Basic intersection property of convex sets in Hilbert spaces

Let $(H, \|\cdot\|)$ be a Hilbert space. From the parallelogram equality it follows immediately that the norm $\|\cdot\|$ in H is *uniformly convex*, i.e., if $\{x_n\}$ and $\{y_n\}$ are sequences in H such that the numerical sequences $\|x_n\|$, $\|y_n\|$, and $\frac{1}{2}\|x_n + y_n\|$ converge to 1, then the sequence $\{\|x_n - y_n\|\}$ tends to 0.

The following preliminary result will be of importance.

(4.1) LEMMA. *Let $(H, \|\cdot\|)$ be a Hilbert space and $\{C_n\}$ be a decreasing sequence of nonempty closed convex subsets of H. Suppose that $d = \sup_n d(0, C_n)$ is finite. Then there exists a unique point $x \in \bigcap C_n$ such that $\|x\| = d$.*

PROOF. Letting $P_n = C_n \cap K(0, d + 1/n)$ for each $n = 1, 2, \ldots$, we obtain a decreasing sequence $\{P_n\}$ of nonempty closed and convex sets and we show that $\delta(P_n) \to 0$, where $\delta(P_n)$ is the diameter of P_n. Indeed, for any n, let $x_n, y_n \in P_n$ be such that $\delta(P_n) \le \|x_n - y_n\| + 1/n$; since the point $\frac{1}{2}(x_n + y_n)$ also belongs to P_n, the values $\|x_n\|$, $\|y_n\|$, and $\frac{1}{2}\|x_n + y_n\|$ lie between $d(0, C_n)$ and $d + 1/n$, and therefore the three sequences converge to d. By the uniform convexity of the norm, we infer that $\|x_n - y_n\| \to 0$, and consequently $\delta(P_n) \to 0$. Applying now the Cantor theorem, we get a unique point $x \in \bigcap P_n$; for this point we have $d(0, C_n) \le \|x\| \le d + 1/n$ for each n, implying that $\|x\| = d$. □

We are now in a position to prove the desired result:

(4.2) THEOREM (Intersection property). *Let $\{C_i \mid i \in I\}$ be a family of closed convex sets in a Hilbert space H with the finite intersection property. If C_{i_0} is bounded for some $i_0 \in I$, then the intersection $\bigcap\{C_i \mid i \in I\}$ is not empty.*

PROOF. Let $\langle I \rangle$ be the set of all finite subsets of I containing i_0. For any $J \in \langle I \rangle$, let $C_J = \bigcap\{C_j \mid j \in J\}$ and note that since each C_J is a nonempty closed convex subset of C_{i_0}, the supremum $d = \sup_{J \in \langle I \rangle} d(0, C_J)$ is finite.

Let $\{J_n\}$ be an increasing sequence of sets in $\langle I \rangle$ with $d(0, C_{J_n}) \ge d - 1/n$. Then $\{C_{J_n}\}$ is a decreasing sequence of nonempty closed convex sets in H such that $d = \sup_n d(0, C_{J_n})$. Applying Lemma (4.1), we get a unique point $x \in \bigcap_n C_{J_n}$ with $\|x\| = d$.

Next, let $J \in \langle I \rangle$ be arbitrary, and set $C_n = C_J \cap C_{J_n}$. Again, $\{C_n\}$ is a decreasing sequence of nonempty closed convex sets in H such that $d = \sup_n d(0, C_n)$, so by Lemma (4.1), there is a unique point $x' \in \bigcap_n C_n = C_J \cap \bigcap_n C_{J_n}$ with $\|x'\| = d$. By the uniqueness it is evident that $x = x'$ belongs to C_J.

Finally, we observe that the point x belongs to C_J for all $J \in \langle I \rangle$, which proves that the set $\bigcap\{C_i \mid i \in I\} \supset \bigcap\{C_J \mid J \in \langle I \rangle\}$ is not empty. □

Using (4.2) and the basic geometric property (3.1.4) of KKM-maps, we obtain the desired version of the geometric KKM-principle:

(4.3) THEOREM (Elementary KKM-principle). *Let H be a Hilbert space, X a nonempty subset of H, and $G : X \to 2^H$ a KKM-map with closed convex values such that Gx_0 is bounded for some $x_0 \in X$. Then the intersection $\bigcap\{Gx \mid x \in X\}$ is not empty.* □

In the remaining part of this section we give a number of applications of Theorem (4.3).

b. Theorem of Stampacchia

A function $\varphi : X \to \boldsymbol{R}$ on a subset of a normed linear space is said to be *coercive* if $\{x \in X \mid \varphi(x) \le r\}$ is bounded for each $r \in \boldsymbol{R}$. If H is a Hilbert space, a bilinear form $a : H \times H \to \boldsymbol{R}$ is *coercive* if the function $x \mapsto a(x,x)/\|x\|$ is coercive on $H - \{0\}$.

(4.4) THEOREM (Stampacchia). *Let C be a nonempty closed convex subset of a Hilbert space H, $a : H \times H \to \boldsymbol{R}$ a continuous coercive bilinear form, and $l : H \to \boldsymbol{R}$ a continuous linear form. Then there exists a unique point $y_0 \in C$ such that $a(y_0, y_0 - x) \le l(y_0 - x)$ for all $x \in C$.*

PROOF. It follows from the coercivity of a that $a(x,x) > 0$ for all $x \neq 0$, so there can be at most one solution. Consider the map $G : C \to 2^C$ given by

$$Gx = \{y \in C \mid a(y, y - x) \le l(y - x)\}.$$

It is easy to check that the values of G are closed, convex (since $x \mapsto a(x,x)$ is continuous and convex) and bounded (because a is coercive); furthermore, since $x \in Gx$, and the cofibers of G are convex, it follows from (3.1.2) that G is a KKM-map. Therefore, by (4.3), there is a $y_0 \in \bigcap_{x\in C} Gx$, which was to be proved. □

We note that the special case $C = H$ of (4.4) yields

(4.5) THEOREM (Lax–Milgram–Vishik). *Let $a : H \times H \to \boldsymbol{R}$ be a continuous coercive bilinear form. Then for any continuous linear form $l : H \to \boldsymbol{R}$ there exists a unique point $y_0 \in H$ such that $a(y_0, x) = l(x)$ for all $x \in H$.*

PROOF. By the Stampacchia theorem, because $C = H$, there exists a unique $y_0 \in H$ such that $a(y_0, z) \le l(z)$ for all $z \in H$; replacing in this inequality z by $-z$ we obtain $a(y_0, z) \ge l(z)$ for all $z \in H$, and the conclusion follows. □

c. Variational inequalities. Theorem of Hartman–Stampacchia

We now extend the above results to a certain class of nonlinear operators that we describe below. Let H be a Hilbert space and C be any subset of H. We recall that an operator $f : C \to H$ is said to be *monotone* on C

if $(f(y) - f(x), y - x) \geq 0$ for all $x, y \in C$. We say that $f : C \to H$ is *hemicontinuous* if the function $[0,1] \ni t \mapsto (f(y + t(x-y)), x - y)$ is continuous at 0 for all $x, y \in C$, and *coercive* if for some $x_0 \in C$ the function $x \mapsto (f(x), x - x_0)/\|x - x_0\|$ is coercive on $C - \{x_0\}$.

(4.6) THEOREM (Hartman–Stampacchia). *Let C be a nonempty closed convex subset of H, $f : C \to H$ monotone coercive hemicontinuous, and $l : H \to \boldsymbol{R}$ a continuous linear form. Then there exists a point $y_0 \in C$ such that $(f(y_0), y_0 - x) \leq l(y_0 - x)$ for all $x \in C$.*

PROOF. We consider only the case of a bounded C; an easy proof of the general case is left to the reader. Define $G, F : C \to 2^C$ by

$$Gx = \{y \in C \mid (f(y), y - x) \leq l(y - x)\},$$
$$Fx = \{y \in C \mid (f(x), y - x) \leq l(y - x)\}.$$

Because f is monotone, we have

$$(f(y), y - x) \geq (f(x), y - x) \quad \text{for all } x, y \in C,$$

and therefore $Gx \subset Fx$ for each $x \in C$. Observe that $x \in Gx$ and that the cofibers of G are convex; thus, by (3.1.2), G is a KKM-map; consequently, so is the map F. Since by definition the values of F are convex and closed, we infer by (4.3) that for some $y_0 \in C$ we have $y_0 \in \bigcap_{x \in C} Fx$, and thus

$$(f(z), y_0 - z) \leq l(y_0 - z) \quad \text{for all } z \in C.$$

Choose any $x \in C$ and let $z_t = y_0 + t(x - y_0)$ for $t \in [0,1]$. We have

$$(f(y_0 + t(x - y_0)), y_0 - x) \leq l(y_0 - x) \quad \text{for } t > 0.$$

Now let $t \to 0$; the hemicontinuity of f gives $(f(y_0), y_0 - x) \leq l(y_0 - x)$. Since x was arbitrary, the conclusion follows. □

As an immediate consequence we obtain

(4.7) THEOREM (Minty–Browder). *Let $f : H \to H$ be a monotone coercive hemicontinuous operator. Then for any continuous linear form $l : H \to \boldsymbol{R}$ there exists a point $y_0 \in H$ such that $(f(y_0), x) = l(x)$ for all $x \in H$.* □

As another consequence of Theorem (4.6), we get a version of the fixed point theorem of Browder–Göhde–Kirk:

(4.8) THEOREM. *Let C be a nonempty closed convex bounded subset of H, and let $F : C \to H$ be nonexpansive (i.e., $\|F(x) - F(y)\| \leq \|x - y\|$ for all $x, y \in C$). Suppose that for each $x \in C$ with $x \neq F(x)$ the line segment $[x, F(x)]$ contains at least two points of C. Then F has a fixed point.*

PROOF. Since F is nonexpansive, the operator $f(x) = x - F(x)$ from C to H is monotone continuous. Applying Theorem (4.6) we get a point $y_0 \in C$ such that $(y_0 - F(y_0), y_0 - x) \leq 0$ for all $x \in C$. Since for some $t > 0$ the point $y_0 + t(F(y_0) - y_0)$ lies in C, we can insert that value into the above inequality to get $(y_0 - F(y_0), F(y_0) - y_0) \geq 0$, showing that y_0 is a fixed point for F. □

d. Maximal monotone operators

We conclude this section by deriving some basic facts in the theory of maximal monotone operators in a Hilbert space H. A set-valued operator $T : H \to 2^H$ is said to be *monotone* if $(y^* - x^*, y - x) \geq 0$ whenever $x^* \in Tx$ and $y^* \in Ty$, and *maximal monotone* if it is monotone and maximal in the set of all monotone operators from H into 2^H ordered by $S \leq T$ if $Sx \subset Tx$ for all $x \in H$. In what follows, we denote by $D(T)$ the domain of T, i.e., $D(T) = \{y \in H \mid Ty \neq \emptyset\}$.

It is clear from the definitions that:

(1) if T is monotone, then

$$\sup_{x^* \in Tx} (x^*, y - x) < \infty \quad \text{for all } x \in D(T) \text{ and } y \in \operatorname{Conv} D(T),$$

(2) if T is maximal monotone, then $y^* \in Ty$ whenever

$$(x^* - y^*, x - y) \geq 0 \quad \text{for all } x \in D(T) \text{ and } x^* \in Tx.$$

For the proof of our main result, we need

(4.9) LEMMA. *Let E be a vector space, $C \subset E$ convex, and D an arbitrary subset of C. Let $g : D \times C \to \mathbf{R}$ be a function such that:*
(a) *$g(x, y) + g(y, x) \leq 0$ for all $(x, y) \in D \times D$,*
(b) *$y \mapsto g(x, y)$ is convex on C for each $x \in D$.*
Then the map $\mathcal{G} : D \to 2^C$ given by $\mathcal{G}x = \{y \in C \mid g(x, y) \leq 0\}$ is a KKM-map.

PROOF. Let $A = \{x_1, \ldots, x_n\} \subset D$, and let $y_0 = \sum_{i=1}^n \lambda_i x_i$ be a convex combination of the x_i's; we are going to show that $y_0 \in \mathcal{G}(A)$.

In view of (a), we have

$$g(x_i, x_j) + g(x_j, x_i) \leq 0 \quad \text{for all } i, j \in [n].$$

So, multiplying by λ_i and summing over i, we find

$$\sum_{i=1}^n \lambda_i g(x_i, x_j) + \sum_{i=1}^n \lambda_i g(x_j, x_i) \leq 0 \quad \text{for every } j \in [n],$$

and therefore, because $y \mapsto g(x, y)$ is convex, we get

$$\sum_{i=1}^n \lambda_i g(x_i, x_j) + g(x_j, y_0) \leq 0 \quad \text{for every } j \in [n].$$

By multiplying each of the above inequalities by λ_j, then summing over all j, and using the convexity of $y \mapsto g(x, y)$, we finally get

$$\sum_{i=1}^{n} \lambda_i g(x_i, y_0) + \sum_{j=1}^{n} \lambda_j g(x_j, y_0) \leq 0.$$

This implies that $g(x_i, y_0) \leq 0$ for at least one point x_i, i.e., $y_0 \in \bigcup_{i=1}^{n} \mathcal{G}x_i$, as asserted. □

We are now able to prove the desired result:

(4.10) THEOREM. *Let $T : H \to 2^H$ be a monotone set-valued operator and $u : H \to H$ be a single-valued, linear, monotone, and bounded operator. Set $D = D(T)$ and $C = \operatorname{Conv} D(T)$. Assume that for some $x_0 \in D$ the set*

$$\{y \in C \mid \sup_{x^* \in Tx_0} (u(y) + x^*, y - x_0) \leq 0\}$$

is bounded. Then there is a point $y_0 \in C$ such that

$$\sup_{x^* \in Tx} (u(y_0) + x^*, y_0 - x) \leq 0 \quad \textit{for all } x \in D.$$

PROOF. We show that the map $G : D \to 2^C$ defined by

$$Gx = \{y \in C \mid \sup_{x^* \in Tx} (u(y) + x^*, y - x) \leq 0\} \quad \text{for } x \in D$$

satisfies all the conditions of the elementary KKM-principle (4.3).

First, we show that G is a KKM-map. To this end consider the function $f : C \times D \times C \to \boldsymbol{R}$ given by

$$f(\xi, x, y) = \sup_{x^* \in Tx} (u(\xi) + x^*, y - x).$$

Because T is monotone, f is well defined and satisfies the following conditions:

(a) $f(\xi, x, y) + f(\xi, y, x) \leq 0$ for all $(x, y) \in D \times D$ and all $\xi \in C$,

(b) $y \mapsto f(\xi, x, y)$ is convex on C for each $x \in D$ and each $\xi \in C$.

Now, observe that using f, we can equivalently describe $G : D \to 2^C$ as

$$Gx = \{y \in C \mid f(y, x, y) \leq 0\}.$$

To show that G is KKM, let $A = \{x_1, \ldots, x_n\} \subset D$ and let $y_0 \in [A]$. Define $g : A \times [A] \to \boldsymbol{R}$ by $g(x, y) = f(y_0, x, y)$. It follows from (a) and (b) that g satisfies the conditions of (4.9), so the map $\mathcal{G} : A \to 2^{[A]}$ given by $\mathcal{G}x = \{y \in [A] \mid g(x, y) \leq 0\}$ is KKM. This implies in particular that $y_0 \in \mathcal{G}x_i$ for some $x_i \in A$, which means that $f(y_0, x_i, y_0) = g(x_i, y_0) \leq 0$, that is, $y_0 \in Gx_i$. The proof that G is KKM is complete.

On the other hand, the values of G are closed and convex (because the function $y \mapsto (u(y), y)$ is continuous and convex on C), and the value Gx_0 is bounded by assumption. By the elementary KKM-principle (4.3), we get $\bigcap\{Gx \mid x \in D\} \neq \emptyset$. $\square$

(4.11) COROLLARY (Minty). *Let $T : H \to 2^H$ be a maximal monotone operator. Then:*

(a) $I + T$ is surjective (I denotes the identity operator on H),

(b) if $D(T)$ is bounded, then T is surjective.

PROOF. (a) It is clearly enough to show that $0 \in (I+T)(H)$. By (4.10) with $u = I$, there is $y_0 \in C = \operatorname{Conv} D(T)$ such that

$$(x^* - (-y_0), x - y_0) \geq 0 \quad \text{for all } x \in D(T) \text{ and } x^* \in Tx.$$

Because T is maximal monotone, we derive that $-y_0 \in Ty_0$, or equivalently, that $0 \in y_0 + Ty_0$.

(b) As in (a), it is sufficient to show that $0 \in T(H)$. Since T is maximal, $D(T)$ is not empty, and therefore C is closed, convex, bounded, and nonempty. By (4.10) with $u = 0$, we find $y_0 \in C$ such that

$$(x^*, x - y_0) \geq 0 \quad \text{for all } x \in D(T) \text{ and } x^* \in Tx.$$

Since T is maximal monotone, this implies that $0 \in Ty_0$. $\square$

5. Theorems of Mazur–Orlicz and Hahn–Banach

In this section, using the Markoff–Kakutani theorem, Theorem (3.2.2) and the fact that a Tychonoff cube is compact, we derive some basic facts of linear functional analysis.

Let E be a vector space and E' the algebraic dual of E. We recall that a functional $p : E \to \boldsymbol{R}$ is said to be *sublinear* if

(i) $$p(x + y) \leq p(x) + p(y) \quad \text{for all } x, y \in E,$$

(ii) $$p(\alpha x) = \alpha p(x) \quad \text{for all } \alpha \geq 0 \text{ and } x \in E.$$

Note that if p is sublinear, then $0 = p(0) = p(x + (-x)) \leq p(x) + p(-x)$, and therefore

(iii) $$-p(-x) \leq p(x) \quad \text{for each } x \in E.$$

(5.1) LEMMA (Banach). *Let $p : E \to \boldsymbol{R}$ be a sublinear functional. Then there exists an $f \in E'$ such that $f(x) \leq p(x)$ for all $x \in E$.*

PROOF. Let $X = \boldsymbol{R}^E$ be the linear topological space of maps $E \to \boldsymbol{R}$ equipped with the product topology; clearly, X has sufficiently many linear

functionals (the evaluation maps $f \mapsto f(x)$ are in fact linear and continuous from $\boldsymbol{R}^E$ to $\boldsymbol{R}$). Consider now the sets

$$\begin{aligned} X_0 &= \prod_{x\in E} [-p(-x), p(x)], \\ X_1 &= \{g \in X_0 \mid -p(-x) \le g(x+y) - g(y) \le p(x) \text{ for all } x, y \in E\}; \end{aligned}$$

clearly, both X_0 and X_1 are nonempty (because from $-p(y-x) \le p(x)-p(y) \le p(x-y)$ it follows that $p \in X_1$). They are both convex and compact (by the Tychonoff theorem).

We define a family $\{T_y \mid y \in E\}$ of maps $T_y : X_1 \to X_1$ by

$$(T_y g)(x) = g(x+y) - g(y) \quad \text{for } x \in E.$$

Clearly, the family $\{T_y\}$ consists of continuous affine maps and is commuting. By the Markoff–Kakutani fixed point theorem, there exists an $f \in X_1$ such that

$$T_y f = f \quad \text{for all } y \in E,$$

i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in E$.

Note that the additivity of f gives $f(rx) = rf(x)$ for each $r \in \boldsymbol{Q}$. Let λ be any real number, and $\{r_n\}$ be a sequence of rational numbers such that $r_n \to \lambda$ and $r_n < \lambda$. Because f is in X_1, we have

$$-(\lambda - r_n)p(-x) \le f(\lambda x) - f(r_n x) \le (\lambda - r_n)p(x),$$

and therefore $f(\lambda x) = \lim f(r_n x) = \lim r_n f(x) = \lambda f(x)$. Thus, $f \in E'$ and $f(x) \le p(x)$ for all $x \in E$. □

(5.2) LEMMA. *Let $p : E \to \boldsymbol{R}$ be a sublinear functional and $x_0 \in E$. Then there exists an $f \in E'$ such that $f(x_0) = p(x_0)$ and $f(x) \le p(x)$ for all $x \in E$.*

PROOF. Define $p^* : E \to \boldsymbol{R}$ by

$$p^*(x) = \inf\{p(x + \lambda x_0) - \lambda p(x_0) \mid \lambda \ge 0\}, \qquad x \in E.$$

Clearly, because $-p(-x) \le p^*(x) \le p(x)$ for all $x \in E$, p^* is well defined and $p^* \le p$. Since for each $\alpha > 0$,

$$\begin{aligned} p^*(\alpha x) &= \inf\{p(\alpha x + \lambda x_0) - \lambda p(x_0) \mid \lambda \ge 0\} \\ &= \inf\{\alpha[p(x + (\lambda/\alpha)x_0) - (\lambda/\alpha)p(x_0)] \mid \lambda \ge 0\} \\ &= \alpha \inf\{p(x + \lambda' x_0) - \lambda' p(x_0) \mid \lambda' = \lambda\alpha^{-1} \ge 0\} = \alpha p^*(x), \end{aligned}$$

p^* is positively homogeneous. Now, for $x_1, x_2 \in E$ and fixed $\varepsilon > 0$ take $\lambda_1, \lambda_2 \ge 0$ so that $p^*(x_i) \ge p(x_i + \lambda_i x_0) - \lambda_i p(x_0) - \varepsilon$ for $i = 1, 2$. Letting

$\mu = \lambda_1 + \lambda_2$, and adding the above inequalities, we obtain

$$\begin{aligned} p^*(x_1) + p^*(x_2) &\geq p(x_1 + \lambda_1 x_0) + p(x_2 + \lambda_2 x_0) - \mu p(x_0) - 2\varepsilon \\ &\geq p(x_1 + x_2 + \mu x_0) - \mu p(x_0) - 2\varepsilon \\ &\geq p^*(x_1 + x_2) - 2\varepsilon, \end{aligned}$$

and this (because ε was arbitrary) implies that p^* is sublinear. By Lemma (5.1), there is a linear functional $f \in E'$ such that

$$f(x) \leq p^*(x) \leq p(x) \quad \text{for all } x \in E.$$

Then

$$-f(x_0) = f(-x_0) \leq p^*(-x_0) \leq p(-x_0 + \lambda x_0) - \lambda p(x_0)$$

for all $\lambda \geq 0$, which implies by putting $\lambda = 1$ that $-f(x_0) \leq -p(x_0)$, and hence $f(x_0) = p(x_0)$. □

We are now in a position to prove the following fundamental result:

(5.3) THEOREM (Mazur-Orlicz). *Let $p : E \to \boldsymbol{R}$ be a sublinear functional. Assume that we are given a family $\{x_t \mid t \in T\}$ of points in E and a family $\{\beta_t \mid t \in T\}$ of real numbers, both indexed by the same abstract set T. Then the following two conditions are equivalent:*

(A) *there exists a linear functional $f \in E'$ such that $f(x) \leq p(x)$ for all $x \in E$ and $\beta_t \leq f(x_t)$ for all $t \in T$,*

(B) *for every convex combination $\sum_{i=1}^n \lambda_i x_{t_i}$ of points $x_{t_1}, \dots, x_{t_n}$ in E,*

$$\sum_{i=1}^n \lambda_i \beta_{t_i} \leq p\Big(\sum_{i=1}^n \lambda_i x_{t_i}\Big).$$

PROOF. Clearly, (A)⇒(B); we are going to show that (B) implies (A). Consider the convex sets

$$X_0 = \prod_{x \in E} [-p(-x), p(x)],$$
$$Y = \{f \in E' \mid -p(-x) \leq f(x) \leq p(x) \text{ for all } x \in E\}.$$

By Lemma (5.1), Y is nonempty, and because Y is closed in X_0, it is also compact.

Consider now the family $\Phi = \{\varphi_t \mid t \in T\}$ of continuous affine functions $\varphi_t : Y \to \boldsymbol{R}$ defined by $\varphi_t(f) = \beta_t - f(x_t)$ for $f \in Y$, and examine the following two conditions:

(C) there exists an $f \in Y$ such that

$$\varphi_t(f) = \beta_t - f(x_t) \leq 0 \quad \text{for all } \varphi_t \in \Phi,$$

(D) for every convex combination $\psi = \sum \lambda_i \varphi_{t_i} \in [\Phi] = \operatorname{conv} \Phi$ of elements of Φ, there is an $f \in Y$ such that

$$\psi(f) = \sum \lambda_i [\beta_{t_i} - f(x_{t_i})] \leq 0.$$

We observe that (C) is clearly equivalent to (A), and by (3.2.2), also (C) and (D) are equivalent.

It follows that it is enough to show that (B) implies (D). So assume that (B) is true, and let $\psi = \sum \lambda_i \varphi_{t_i} \in \operatorname{conv} \Phi$. By assumption we have $\sum \lambda_i \beta_{t_i} \leq p(\sum \lambda_i x_{t_i})$. Let $x_0 = \sum \lambda_i x_{t_i}$; by Lemma (5.2), there is an $f \in Y$ such that $f(x_0) = p(x_0)$, so we have

$$\sum \lambda_i \beta_{t_i} \leq f\Big(\sum \lambda_i x_{t_i}\Big).$$

and thus $\psi(f) = \sum \lambda_i [\beta_{t_i} - f(x_{t_i})] \leq 0$; hence (D) is true, and the proof is complete. □

As an immediate consequence we obtain a refined version of the Hahn Banach theorem:

(5.4) THEOREM. *Let $p : E \to \boldsymbol{R}$ be a sublinear functional, C a convex subset of E and $g : C \to \boldsymbol{R}$ a concave function such that $g(y) \leq p(y)$ for all $y \in C$. Then there is a linear functional $f \in E'$ such that $g(y) \leq f(y)$ for all $y \in C$ and $f(x) \leq p(x)$ for all $x \in E$.*

PROOF. Take $T = C$, $\beta_t = g(t)$, and $x_t = t$ for $t \in C$; then condition (B) of (5.3) is satisfied. Consequently, by the Mazur–Orlicz theorem, there exists a linear functional $f \in E'$ such that $f(x) \leq p(x)$ for all $x \in E$ and $g(t) \leq f(t)$ for all $t \in C$. □

Another consequence is concerned with an extended version of the classical moments problem:

(5.5) THEOREM. *Let E be a normed linear space, $\{x_m\}_{m=1}^{\infty}$ a given sequence in E, and $\{c_m\}_{m=1}^{\infty}$ a sequence of real numbers. Then the following two conditions are equivalent:*

(A) *there exists a linear functional $f \in E^*$ such that $f(x_m) \geq c_m$ for all $m = 1, 2, \ldots$ and $\|f\| \leq M$, where $M > 0$,*

(B) *for every convex combination $\sum_{i=1}^{n} \lambda_i x_i$ of the points $x_1, \ldots, x_n$ we have*

$$\sum_{i=1}^{n} \lambda_i c_i \leq M \Big\| \sum_{i=1}^{n} \lambda_i x_i \Big\|.$$

PROOF. Clearly, it is enough to show that (B)⇒(A). To this end, letting $p(x) = M\|x\|$ for $x \in E$, we apply (5.3) to the set $T = N$ and the sequences $\{x_m\}$ and $\beta_m = c_m$. □

6. Miscellaneous Results and Examples

A. Applications of the Banach theorem and of related results to analysis

(A.1) (*Systems of linear equations*) Consider the infinite system of linear equations

(*) $$x_i = \sum_{j=1}^{\infty} a_{ij} x_j + b_i, \quad i = 1, 2, \ldots, \quad b_i, a_{ij} \in \boldsymbol{R},$$

and assume that one of the following conditions is satisfied:

(a) for some constants $0 \le \alpha < 1$ and $\beta > 0$ we have

$$\sum_{j=1}^{\infty} |a_{ij}| \le \alpha, \quad |b_i| \le \beta \quad \text{for each } i = 1, 2, \ldots.$$

(b) for some $p > 1$ we have

$$\sum_{i=1}^{\infty} \Big(\sum_{j=1}^{\infty} |a_{ij}|^{p/(p-1)} \Big)^{p-1} < 1, \quad \sum_{i=1}^{\infty} |b_i|^p < \infty.$$

(c) letting $c_i = \sup\{|a_{ij}| \mid j = 1, 2, \ldots\}$, we have

$$\sum_{i=1}^{\infty} c_i < 1, \quad \sum_{i=1}^{\infty} |b_i| < \infty.$$

Prove: The system (*) has a unique solution in the space, respectively: m (the space of bounded sequences with the sup norm), l^p and l^1

(A.2) (*Integral equations*) Let $K : [a, b] \times [a, b] \to \boldsymbol{R}$ be a measurable and square integrable function. Assume that for a real parameter λ,

$$|\lambda| \Big(\int K(s,t)^2 \, ds\, dt \Big)^{1/2} < 1.$$

Show: The integral equation $u(s) = f(s) + \lambda \int K(s,t) u(t)\, dt$ $(a \le s \le b)$, where $f \in L^2[a, b]$, has a unique solution $u \in L^2[a, b]$.

(A.3) (*Application of the nonlinear alternative for contractive maps*) We seek the solutions to the initial value problem

(P) $$\begin{cases} x'(t) = f(t, x(t)), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

where $f : [0, T] \times \boldsymbol{R} \to \boldsymbol{R}$ is continuous. Suppose

(a) for each $r > 0$ there is an $l_r \in \boldsymbol{R}$ such that

$$|f(t, x) - f(t, y)| \le l_r |x - y| \quad \text{for all } t \in [0, T] \text{ and } x, y \in [-r, r],$$

(b) there is a continuous function $\varphi : [0, \infty) \to (0, \infty)$ such that

$$|f(t, x)| \le \varphi(|x|) \quad \text{for all } t \in [0, T] \text{ and } x \in \boldsymbol{R}.$$

Prove: If $T < \int_0^\infty ds/\varphi(s)$, then the initial value problem (P) has a unique solution $x \in C^1([0, T])$.

[Consider the family of problems

$$(\mathrm{P}_\lambda) \qquad \begin{cases} x'(t) = \lambda f(t, x(t)), & t \in [0, T], \\ x(0) = 0, \end{cases}$$

depending on a parameter $\lambda \in [0, 1]$. Fix an M so that $T < \int_0^M ds/\varphi(s)$ and show that if x is a solution of the problem (P_λ) for some λ, then $|x(t)| < M$ for all $t \in [0, T]$. Let $L = l_M$ be the constant given in (a), and define the norm $\| \ \|_L$ in $C([0, T])$ by

$$\|x\|_L = \sup\{e^{-tL}|x(t)| \mid t \in [0, T]\}.$$

Set $U = \{x \in C([0, T]) \mid |x(t)| < M \text{ for all } t \in [0, T]\}$ and prove that $G \cdot \overline{U} \to C([0, T])$ given by $G(x)(t) = \int_0^t f(s, x(s))\, ds$ is contractive in the norm $\| \ \|_L$; conclude the argument by showing that λG has no fixed points on the boundary of U.]

B. Nonexpansive maps and monotone operators in Hilbert space

(B.1) Let $f : A \to H$ be a map (not necessarily continuous) on a subset $A \subset H$.
(a) f is *monotone* $\Leftrightarrow$ $(fx - fy, x - y) \geq 0$ for all $x, y \in A$.
(b) f is *strictly monotone* $\Leftrightarrow$ $(fx - fy, x - y) > 0$ for all $x, y \in A$, $x \neq y$.
(c) f is *strongly monotone* $\Leftrightarrow$ $(fx - fy, x - y) \geq C\|x - y\|^2$ for all $x, y \in A$ and some $C > 0$.

Show:
(i) Every contractive field is strictly monotone.
(ii) Every nonexpansive field is monotone.
(iii) Every strongly monotone map is injective.
(iv) f is strongly monotone with constant C if and only if $\frac{1}{C}f - I$ is monotone.
(v) If f is strongly monotone, then $\|fx - fy\| \geq C\|x - y\|$.
(vi) If $f : H \to H$ is differentiable, then f is strongly monotone if and only if $(Df(x)h, h) \geq C\|h\|^2$ for some $C > 0$ and all $h \in H$.

(B.2) (*Minimizing convex functionals*) Let $C \subset H$ be a closed convex set.
(a) Let $\varphi : C \to \boldsymbol{R}$ be a quasi-convex l.s.c. coercive function. Prove: φ attains its minimum at some $y_0 \in C$.
[For each $x \in C$, let $\Gamma x = \{y \in C \mid \varphi(y) \leq \varphi(x)\}$ and apply (4.2).]
(b) Let $a : H \times H \to \boldsymbol{R}$ be a coercive continuous bilinear form, and let $l \ \ H \to \boldsymbol{R}$ be a continuous linear form. Show: There is a unique $y_0 \in C$ such that the following equivalent properties hold:
(i) $\frac{1}{2}[a(y_0, y_0 - x) + a(y_0 - x, y_0)] \leq l(y_0 - x)$ for all $x \in C$,
(ii) the quadratic form $x \mapsto \varphi(x) = \frac{1}{2}a(x, x) - l(x)$ on C attains its minimum at y_0.
[Verify first that (i)$\Leftrightarrow$(ii); to prove (ii), apply (a) to the coercive convex function φ.]

(B.3) (*Nikodym theorem*) Let $C \subset H$ be closed convex. Show: There exists a retraction $r : H \to C$ with the following properties:
(i) If $x_0 \in H$, then $r(x_0)$ is the unique point in C with $\|x_0 - r(x_0)\| = \inf\{\|x_0 - x\| \mid x \in C\} = d(x_0, C)$.
(ii) For each $x_0 \in H$, the point $r(x_0)$ is a solution of the variational inequality $(r(x_0) - x_0, r(x_0) - x) \leq 0$ for all $x \in C$.
(iii) The retraction r is nonexpansive.
(iv) If $C = H_0$ is a linear subspace of H, then $r : H \to H_0$ is the orthogonal projection (i.e., for each $x \in H$, $(x - r(x), y) = 0$ for all $y \in H_0$).

(Parts (i) and (iv) are due to Nikodym [1931].)

(B.4) Let C be a closed convex set in R^n and $f : C \to R$ be a C^1 function that attains its minimum at $y_0 \in C$. Let $F(x) = \operatorname{grad} f(x)$. Show: $(F(y_0), y_0 - x) \leq 0$ for all $x \in C$.

(B.5) (*Complementarity problem*) Let $R^n_+ = \{x = (x_1, \ldots, x_n) \in R^n \mid x_i \geq 0 \text{ for all } i\}$, and let $F : R^n \to R^n$. The *complementarity problem* is to find $y_0 \in R^n_+$ such that $F(y_0) \in R^n_+$ and $(F(y_0), y_0) = 0$.

(a) Show: The following statements are equivalent: (i) $y_0 \in R^n_+$ is a solution of the complementarity problem, (ii) $(F(y_0), y_0 - x) \leq 0$ for all $x \in R^n_+$.

[For (ii)⇒(i), note that if $y_0 \in R^n_+$ solves (ii), then letting $e_i \in R^n$ be the standard unit basis, we have $x = y_0 + e_i \in R^n_+$ for each $i \in [n]$, and therefore $F(y_0) \in R^n_+$ by (ii); deduce that $(F(y_0), y_0) \leq 0$ and finally that $(F(y_0), y_0) = 0$.]

(b) Let $F : R^n \to R^n$ be strongly monotone and continuous on the cone R^n_+. Show: The complementarity problem for F has a unique solution.

(The above results are due to Karamardian [1972].)

(B.6) Let $C = \{x \in H \mid \|x\| \leq r\}$ and $f : C \to H$ be monotone and hemicontinuous. Prove: if $f(y) \neq \lambda y$ for all $\lambda < 0$ and $\|y\| = r$, then there is $y_0 \in C$ such that $f(y_0) = 0$.

[Apply (4.6) to get $y_0 \in C$ with $(fy_0, y_0 - x) \leq 0$ for all $x \in X$, consider two cases: $\|y_0\| = r$ and $\|y_0\| < r$.]

(B.7) Let $f : H \to H$ be monotone and hemicontinuous. Prove: If $(fx, x)/\|x\| \to \infty$ uniformly as $\|x\| \to \infty$, then f is surjective (G. Minty).

[Given $y_0 \in H$ consider $x \mapsto g(x) = f(x) - y_0$ and apply (B.6) to the map $g : H \to H$ on a sufficiently large ball.]

(B.8) Let $C \subset H$ be a closed convex set, and let $r : H \to C$ be the map sending each $x \in H$ to its nearest point in C. Show: $r : H \to C$ is nonexpansive.

[Use the reasoning in the latter half of (1.4).]

(B.9) Show: The nonlinear alternative (1.5) for nonexpansive maps and its corollaries (1.6)(a)–(d) remain valid if in (1.5) and (1.6) the closed ball in H is replaced by any closed convex bounded subset $C \subset H$ with $0 \in \operatorname{Int}(C)$.

(B.10) Let C be a bounded closed convex subset of a Hilbert space and $\mathscr{F}$ be a family of commuting nonexpansive maps of C into C. Show: The maps in $\mathscr{F}$ have a common fixed point (F. Browder).

C. Nonexpansive maps in Banach spaces

(C.1) A Banach space is *uniformly convex* if there is a monotone increasing surjection $\varphi : [0, 2] \to [0, 1]$ continuous at 0, with $\varphi(0) = 0, \varphi(2) = 1$, such that $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$ implies $\|(x + y)/2\| \leq 1 - \varphi(\varepsilon)$. Let $\eta : [0, 1] \to [0, 2]$ be the inverse of φ.

(a) Let E be uniformly convex, and u, v two elements of E. Assume that there is an $x \in E$ with $\|x - u\| \leq R$, $\|x - v\| \leq R$, $\|x - (u + v)/2\| \geq r > 0$. Prove:

$$\|u - v\| \leq R\eta\left[\frac{R - r}{R}\right].$$

[Write $r = (1 - (R - r)/R)R$.]

(b) Let E be a uniformly convex Banach space and $C \subset E$ a closed bounded convex set. Prove: Every nonexpansive $F : C \to C$ has a fixed point (Browder [1965], Göhde [1965], Kirk [1965]).

(c) Let C be a closed convex set in a uniformly convex Banach space E. Show: For each $x_0 \in E$, there is a unique $u \in C$ with $\|x_0 - u\| = \inf_{c \in C} \|x_0 - c\|$.

(C.2) Let $K^\infty = \{x = \{x_i\} \in c_0 \mid \|x\| = \sup_{1 \le i < \infty} |x_i| \le 1\}$ be the unit ball in c_0. Show that $\varphi : K^\infty \to K^\infty$ given by $(x_1, x_2, \ldots) \mapsto (1, x_1, x_2, \ldots)$ is a nonexpansive map without fixed points (Beals).

(C.3) Let E be a Banach space. A set $S \subset E$ is *star-shaped* if there is some $p \in S$ such that $tx + (1-t)p \in S$ for all $x \in S$ and $0 \le t \le 1$. A set $A \subset S$ is called an *attractor* for a map $F : S \to S$ provided

$$\overline{\bigcup_{n \ge 1} F^n(x)} \cap A \ne \emptyset \quad \text{for each } x \in S.$$

(a) Let S be a compact star-shaped subset of a Banach space. Prove: Every nonexpansive $F : S \to S$ has a fixed point.

(b) Let S be a star-shaped subset of a Banach space and $F : S \to S$ a nonexpansive map with a compact attractor. Show: F has a fixed point (Göhde [1965]).

[Assuming, without loss of generality, that $0 \in S$, establish that given $\lambda \in (0,1)$ there is a point $x_\lambda \in S$ satisfying $\|x_\lambda - Fx_\lambda\| \le d(1-\lambda)$, where $d = \delta(S)$: then find a point y_λ in a compact attractor A of F such that $\|y_\lambda - F^{n(\lambda)}(x_\lambda)\| \le (1-\lambda)$ for a sufficiently large integer $n(\lambda)$. Establish the inequality $\|y_\lambda - Fy_\lambda\| \le (1-\lambda)(d+2)$ and use compactness of A to conclude the proof.]

(C.4) Let E be a Banach space, and $A \subset E$ any nonempty subset. Let

$$\begin{aligned} r_a(A) &= \inf\{r \mid A \subset B(a,r)\} \quad (a \in A), \\ r(A) &= \inf\{r_a(A) \mid a \in A\}, \\ \check{C}(A) &= \{a \in A \mid r_a(A) = r(A)\}. \end{aligned}$$

(a) Let A be a bounded closed set. Show: $\delta[\check{C}(A)] \le r(A)$.

(b) Let K be a bounded closed convex set and $T : K \to K$ nonexpansive. Prove: If $\operatorname{Conv} T(K) = T(K)$, then $T[\check{C}(K)] \subset \check{C}(K)$.

(c) A convex set K in a Banach space is said to have *normal structure* if $r(D) < \delta(D)$ for each bounded closed convex $D \subset K$ with $\delta(D) > 0$.

Let E be a reflexive Banach space, and K a nonempty bounded closed convex set with normal structure. Prove: If $\delta(K) > 0$, then $\check{C}(K)$ is a nonempty proper closed convex subset of K (Brodskiĭ–Milman [1948]).

[Observe $K \subset B(u,r) \Leftrightarrow u \in \bigcap\{\overline{B}(x,r) \mid x \in K\}$. Next note that for each $\varepsilon > 0$, the set $C_\varepsilon(K) = \bigcap\{\overline{B}(x, r(K)+\varepsilon) | x \in K\} \ne \emptyset$ and that $\check{C}(K) = \bigcap\{C_\varepsilon(K) \mid \varepsilon > 0\}$. Now use the Mazur–Šmulian theorem.]

(d) The Hilbert space l^2 renormed by $\|x\| = \sup_n\{\frac{1}{2}\sqrt{\sum_i x_i^2}, |x_n|\}$ is reflexive. Let $K = \{x \mid \|x\| \le 1 \text{ and } x_i \ge 0 \text{ for all } i\}$. Show: K is a closed bounded convex set that does not have normal structure.

(e) Prove: If E is a uniformly convex Banach space, then every bounded closed convex set has normal structure (Brodskiĭ–Milman [1948]).

(C.5) Let E be a reflexive Banach space, and K a nonempty bounded closed convex set with normal structure. Prove: Every nonexpansive $T : K \to K$ has a fixed point (Kirk [1965]).

[Use the Kuratowski–Zorn lemma to find a minimal nonempty closed convex $K_0 \subset K$ with $T(K_0) \subset K_0$; show that $\operatorname{Conv} T(K_0) = K_0$; then apply (C.4)(b) and (c).]

(C.6) The following example of Alspach [1981] shows that if C is a weakly compact convex set in a Banach space, then a nonexpansive $T : C \to C$ need not have a fixed point.

Let $T : I^2 \to I^2$ be the "baker's transformation"

$$T(x, y) = \begin{cases} (x/2, 2y), & 0 \le y \le \frac{1}{2}, \\ (x/2 + 1/2, 2y - 1), & \frac{1}{2} < y \le 1, \end{cases}$$

which can be visualized as first squeezing I^2 into the rectangle $\{(x, y) \mid 0 \le x \le \frac{1}{2},\ 0 \le y \le 2\}$, then cutting off the top half and placing it next to the lower half. It is known that T is measure-preserving, i.e., $\mu(TA) = \mu(A)$ for every measurable $A \subset I^2$

In $L^1(I)$ with the usual norm $\|f\| = \int_I |f|$, consider the weakly compact subset

$$C = \{f \in L^1(I) \mid 0 \le f \le 1,\ \textstyle\int_I f = \frac{1}{2}\}.$$

For each $f \in C$, let

$$\widehat{T}f(x) = \begin{cases} \min[2f(2x), 1], & 0 \le x \le \frac{1}{2}, \\ \max[2f(2x - 1) - 1, 0], & \frac{1}{2} < x \le 1 \end{cases}$$

(the graph of $\widehat{T}f$ is that obtained from the graph of f after the top half of the squeezed rectangle is placed next to the lower half). Prove: (a) $\widehat{T}$ is an isometry $C \to C$, (b) $\widehat{T}$ has no fixed point.

[(a) Observe that if A_f is the ordinal set $\{(x, y) \in I^2 \mid y \le f(x)\}$, then $\|f - g\| =$ the measure of the symmetric difference $A_f \Delta A_g$ of the ordinal sets, and recall that T is measure-preserving. (b) If $\widehat{T}f = f$, then either $f = 0$ or $f = 1$ a.e.; but then $\int_I f \ne \frac{1}{2}$.]

(C.7) Let C be a compact convex set in a normed linear space and $\mathscr{F}$ be a family of commuting nonexpansive maps of C into itself. Prove: There is a common fixed point for the family $\mathscr{F}$ (DeMarr [1964]).

D. Geometric and elementary KKM-theory

(D.1) (*Intersection property in superreflexive spaces*) Let E be a Banach space. We call E *superreflexive* if it admits an equivalent uniformly convex norm. Let $\{C_i \mid i \in I\}$ be a family of closed convex sets in a superreflexive Banach space with the finite intersection property. Show: If C_0 is bounded for some $i_0 \in I$, then $\bigcap\{C_i \mid i \in I\} \ne \emptyset$.

[First assume E to be uniformly convex; follow the proof of (4.1) and (4.2).]

(D.2) (*Mazur Schauder theorem*) Let E be a reflexive Banach space and C a closed convex subset of E. Let $\varphi : C \to \boldsymbol{R}$ be a lower semicontinuous, quasi-convex, and coercive (i.e., $\varphi(x) \to \infty$ as $\|x\| \to \infty$) functional on C. Show: The functional φ attains its minimum at some $x_0 \in C$ (Mazur Schauder [1936]).

[For E superreflexive, use (D.1); in the general case equip E with the weak topology.]

(D.3) (*KKM-maps in superreflexive spaces*) Let E be a superreflexive Banach space and $X \subset E$. Let $G : X \to 2^E$ be a KKM-map with closed convex values such that one the sets Gx_0 is bounded. Show: $\bigcap\{Gx \mid x \in X\} \ne \emptyset$.

[Use (3.1.4) and (D.1).]

(D.4) (*Hartman Stampacchia theorem in reflexive Banach spaces*) Let E be a Banach space, E^* its dual space and for $(\xi, \nu) \in E^* \times E$ denote $\xi(\nu)$ by $\langle \xi, \nu \rangle$. A map $f : C \to E^*$ defined on a subset $C \subset E$ is called *monotone* if $\langle f(x) - f(y), x - y \rangle \ge 0$ for all $x, y \in C$; f is *hemicontinuous* if for all $x, y \in C$ the mapping $[0, 1] \ni t \mapsto \langle f(y + t(x - y)), x - y \rangle$ is continuous at 0.

Let E be a reflexive Banach space, C a nonempty closed bounded convex subset of E, and let $f : C \to E^*$ be monotone and hemicontinuous. Prove: There exists a $y_0 \in C$ such that $\langle f(y_0), y_0 - x\rangle \leq 0$ for all $x \in C$ (Hartman–Stampacchia [1965]).

[Equip E with the weak topology and using the geometric KKM-principle follow the proof of (4.6).]

(D.5) Let C be a nonempty bounded closed convex subset of a superreflexive Banach space E, and let $S, T : C \to 2^C$ be such that:

(i) $Sx \subset Tx$ for all $x \in C$,

(ii) S has convex cofibers,

(iii) T has closed and convex values.

Show: If $x \in Sx$ for each $x \in C$, then $\bigcap\{Tx \mid x \in C\} \neq \emptyset$.

(D.6) Let C be a nonempty bounded closed convex subset of a superreflexive Banach space E, and let $f, g : C \times C \to \boldsymbol{R}$ satisfy:

(i) $g(x, y) \leq f(x, y)$ for all $x, y \in C$,

(ii) $x \mapsto f(x, y)$ is quasi-concave on C for each $y \in C$,

(iii) $y \mapsto g(x, y)$ is l.s.c. and quasi-convex on C for each $x \in C$.

Prove:

(a) For any $\lambda \in \boldsymbol{R}$, either (i) there exists a $y_0 \in C$ such that $g(x, y_0) \leq \lambda$ for all $x \in C$, or (ii) there exists a $w \in C$ such that $f(w, w) > \lambda$.

(b) The following minimax inequality holds:

$$\inf_{y\in C}\sup_{x\in C} f(x, y) \leq \sup_{x\in C} g(x, x).$$

[For (a), define $S, T : C \to 2^C$ by $Sx = \{y \in C \mid f(x, y) \leq 0\}$ and $Tx = \{y \in C \mid g(x, y) \leq 0\}$ and apply (D.5).]

(D.7) (*Maximal monotone operators in reflexive spaces*) Let E be a reflexive Banach space. A set-valued operator $T : E \to 2^{E^*}$ is *monotone* if $\langle y^* - x^*, y - x\rangle \geq 0$ whenever $x^* \in Tx$ and $y^* \in Ty$; T is called *maximal monotone* if it is monotone and maximal in the set of all monotone operators from E into 2^{E^*}. Show: If $T : E \to 2^{E^*}$ is maximal and $D(T) = \{x \in E \mid Tx \neq \emptyset\}$ is bounded, then T is surjective (F. Browder).

[Follow the proof of (4.10), (4.11). If E is superreflexive, use (4.3); if not, equip E with the weak topology and apply the geometric KKM-principle.]

E. Selected results

(E.1) (*Elementary implicit function theorem*) Let $(E_1, \|\ \|)$ and $(E_2, |\ |)$ be Banach spaces, and let $U \subset E_1$ be open and $V \subset E_2$ be open connected. Assume $H : \overline{U} \times V \to E_1$ is a continuous map with the following properties:

(i) $\|H(u, v) - H(u', v)\| \leq \alpha\|u - u'\|$, where $0 \leq \alpha < 1$ and α is independent of v,

(ii) $H(u, v) \neq u$ for any $(u, v) \in \partial U \times V$,

(iii) for some $v_0 \in V$ the equation $H(u, v_0) = u$ has a unique solution $u \in U$.

Show:

(a) For each $v \in V$ the equation $H(u, v) = u$ has a unique solution u_v.

(b) The assignment $v \mapsto u_v$ is a continuous map from V to U.

(c) If the restriction $H|(U \times V) : U \times V \to E_1$ is C^1, then the assignment $v \mapsto u_v$ is a C^1 map from V to U.

(E.2) (*Miranda theorem*) Let E_1, E_2 be Banach spaces and $H : E_1 \times [0, 1] \to E_2$ be a continuous map with the following properties:

(i) $(x,t) \mapsto H(x,t)$ admits a continuous partial derivative $H_x : E_1 \times [0,1] \to \mathscr{L}(E_1, E_2)$ with respect to $x \in E_1$,
(ii) the set $\{x \in E_1 \mid H(x,t) = 0 \text{ for some } t \in [0,1]\}$ is compact,
(iii) if $H(x,t) = 0$ for some x and t, then the linear operator $H_x(x,t)$ is invertible,
(iv) for some $t_0 \in [0,1]$ the equation $H(x,t_0) = 0$ has a unique solution.

Show: The equation $H(x,t) = 0$ has a unique solution for each $t \in [0,1]$ (Miranda [1971]).

(E.3) (*Hartman theorem*) Let $(E, |\ |)$ be a Banach space and $(\mathscr{C}(E,E), \|\ \|)$ the Banach space of bounded uniformly continuous functions $p : E \to E$ with the sup norm $\|\ \|$. Let $L \in \mathrm{GL}(E)$ be a given hyperbolic isomorphism: $E = E_1 \oplus E_2$, $L|E_i = L_i \in \mathrm{GL}(E_i)$, $i = 1,2$, with $\|L_1\| < 1$ and $\|L_2^{-1}\| < 1$. We assume that $\alpha = \max(\|L_1\|, \|L_2^{-1}\|) < 1$ and that E is given the norm $|x_1 + x_2| = \max(|x_1|, |x_2|)$ for $x_i \in E_i$, $i = 1,2$.

(a) For $\nu > 0$ let

$$\mathscr{L}_\nu(L) = \{A = L + \lambda \mid \lambda \in \mathscr{C}(E,E) \text{ is bounded by } \nu \text{ and Lipschitz with constant } \leq \nu\}.$$

Call $\mathscr{L}_\nu(L)$ *admissible* if its elements are Lipschitz isomorphisms of E onto itself. Show: 1° $\mathscr{L}_\nu(L) \subset \mathscr{C}(E,E)$ is a complete metric space. 2° $\mathscr{L}_\nu(L)$ is admissible for small ν.

[For 2°, use elementary domain invariance.]

(b) Let $\mathscr{L}_\nu(L)$ be admissible and assume that $A = L + \lambda$ and $A' = L + \lambda'$ are two elements of $\mathscr{L}_\nu(L)$. Consider the equation

$$\text{(i)} \qquad hA = A'h \quad \text{for } h \in \mathscr{H},$$

where $\mathscr{H} = \{h = 1 + p \mid p \in \mathscr{C}(E,E)\}$ with $1 = \mathrm{id}_E$. Show: The equation (i) is equivalent to the equation

$$\text{(ii)} \qquad p = L^{-1}[pA + \lambda - \lambda'(1+p)] \quad \text{for } p \in \mathscr{C}(E,E).$$

(c) Show: The equation (ii) is equivalent to the system

$$\text{(iii)} \qquad \begin{cases} p_1 = [L_1 p_1 + \lambda_1'(1+p) - \lambda_1]A^{-1}, \\ p_2 = L_2^{-1}[p_2 A + \lambda_2 - \lambda_2'(1+p)], \end{cases}$$

where p_i, L_i, λ_i, λ_i' $(i = 1,2)$ are the components of p, L, λ, λ', respectively, with respect to the splitting $E = E_1 \oplus E_2$.

(d) Define a map $H : \mathscr{C}(E,E) \times \mathscr{L}_\nu(L) \times \mathscr{L}_\nu(L) \to \mathscr{C}(E,E)$ by

$$H((p_1,p_2), A, A') = ([L_1 p_1 + \lambda_1'(1+p) - \lambda_1]A^{-1}, L_2^{-1}[p_2 A + \lambda_2 - \lambda_2'(1+p)]).$$

Show: If $\alpha + \nu < 1$, then H is $(\alpha+\nu)$-contractive with respect to (p_1, p_2) and continuous with respect to (A, A').

(e) Assume $\mathscr{L}_\nu(L)$ is admissible with $\alpha + \nu < 1$. Show: To each pair $A, A' \in \mathscr{L}_\nu(L)$ there corresponds a unique homeomorphism $h_{AA'} = h \in \mathscr{H}$ such that $hA = A'h$; this homeomorphism depends continuously on A, A'.

[Use (b), (c), (d) and the parametrized version of the Banach theorem (1.6.A.2).]

(The above proof of a theorem of Hartman [1964] is due to Pugh [1969].)

(E.4) (*Bruhat Tits theorem*) Let (X, d) be a complete metric space that satisfies the following *semiparallelogram law*: for any $a, b \in X$ there is a point $z \in X$ such that for all $x \in X$,

$$(*) \qquad d(a,b)^2 + 4d(x,z)^2 \leq 2d(x,a)^2 + 2d(x,b)^2$$

(a) Prove: The point z in $(*)$ is the midpoint between a and b, i.e., $d(a,z)+d(b,z)=\frac{1}{2}d(a,b)$.

(b) Let A be a bounded subset of X. Show: There exists a unique closed ball $K(a,r)$ in X of minimal radius containing A (J.-P. Serre).

[For uniqueness, use (a); for existence, consider a sequence of closed balls $K(a_n, r_n) \supset A$ with $r_n \to r$. Prove that $\{a_n\}$ is a Cauchy sequence and that $K(a,r)$, where $a = \lim a_n$, is the desired ball.]

(c) Let $\mathscr{G}$ be a group of isometries of X. Show: If $\mathscr{G}$ has a bounded orbit $\mathscr{G}(x)$, then $\mathscr{G}$ has a common fixed point (Bruhat–Tits [1972]).

[Letting $K(x,r)$ be the unique closed ball of minimal radius containing $\mathscr{G}(x)$, prove that x is a common fixed point of $\mathscr{G}$.]

7. Notes and Comments

Fixed points for nonexpansive maps

Nonexpansive maps appear for the first time in Kolmogoroff [1933], where they were used in the axiomatic treatment of measure theory. Pontrjagin and Schnirelmann [1932] used the notion in dimension theory and established the following result: *If X is a compact metric space with* $\dim X \geq r$, *then there exists a nonexpansive map* $\varphi : X \to \boldsymbol{R}^r$ *such that* $\dim \varphi(X) = r$.

In Section 1 we give only a few theorems that are related to the contraction principle. Theorem (1.3) is a special case of more general results (see (C.1(b))) obtained independently by Browder [1965], Göhde [1965], and Kirk [1965]. Earlier, a general fixed point result for isometries was obtained by Brodskiĭ–Milman [1948]. All the above authors used weak-topology arguments in the proofs of their results; the elementary proof of (1.3) given in the text is due to Goebel [1969]. We remark that in (1.3), the sequence of iterates $\{F^n(x)\}$ does not necessarily converge to a fixed point of F; it can be proved, however, that for each $x \in C$ the sequence $\frac{1}{n}(x+Fx+\cdots+F^n x)$ converges weakly to a fixed point of F (Baillon [1975]).

We also remark that the nonlinear alternative for nonexpansive maps (1.5) remains valid for nonexpansive set-valued maps in uniformly convex spaces (Frigon [1995]). For single-valued maps, (1.5) and its corollaries (1.6) can be easily deduced (as observed by Z. Guennoun) from the following result given in Browder's survey [1976]: *Let E be a uniformly convex Banach space, $C \subset E$ be closed, convex, and bounded, and $F : C \to E$ be nonexpansive. Then the nonexpansive field $f(x) = x - Fx$ is demiclosed on C, i.e., if $\{x_n\}$ in C converges weakly to x and $\{f(x_n)\}$ converges strongly to y, then $x \in C$ and $f(x) = y$.*

For further results on nonexpansive maps (including some applications as well as some iterative techniques for approximating fixed points) the reader is referred to "Miscellaneous Results and Examples", the surveys of Opial [1967], Petryshyn [1975], Browder [1976], and to the books by Goebel–Reich [1984] and Goebel–Kirk [1991].

Applications of the Banach theorem

The fundamental idea of applying fixed point results to produce theorems in analysis is due to Poincaré [1884], [1912] and was developed further in the works of Birkhoff [1913], Birkhoff-Kellogg [1922] and then Schauder [1927a], [1927b], [1930]. Systematic applications of the Banach principle to various existence theorems in analysis were initiated by Caccioppoli [1930]. An expository account of many such applications may be found in the surveys by Niemytzki [1936] and Miranda [1949]. For applications to differential and integral equations the reader is referred to Pogorzelski's book [1966] and to Griffel [1985]. The renorming technique used in Section 2 was introduced by Bielecki [1956]. Numerous (and diverse) applications of the Banach theorem are given in "Miscellaneous Results and Examples"

Applications of the elementary domain invariance

Elementary domain invariance permits a simple and unified treatment of a number of familiar results in various fields. Theorem (3.4), established by Schauder [1934], is an abstraction (in the linear case) of Poincaré's method of continuation of solutions along a parameter and underlies the general idea due to Bernstein that obtaining suitable a priori bounds for solutions of a class of problems is frequently sufficient to establish their existence. For many uses of Theorem (3.4) in partial differential equations the reader is referred to the book of Gilbarg–Trudinger [1977].

The proof of the inverse function theorem presented in the text is an adaptation of that given in H. Cartan's book [1967].

Monotone operators were introduced independently by Kachurovskiĭ [1960], Zarantonello [1960], and Minty [1962]. Kachurovskiĭ observed that the gradient maps of convex functions are monotone and introduced the term "monotonicity" Theorem (3.9), due to Zarantonello [1960], is one of the simplest results of the theory that is related to the contraction principle. More information on monotone operators and their applications to integral and differential equations can be found in the surveys by Kachurovskiĭ [1968] and Browder [1976] and also in Brézis's book [1973].

Theorem (3.12) is due to Klee [1956], who was the first to study the negligibility of sets in Banach spaces. The method of proof presented in the text is based on the noncomplete norm technique due to Bessaga. By refining this technique, Bessaga [1966] proved that *every infinite-dimensional Hilbert space is diffeomorphic to its unit sphere,* and as a consequence established the following theorem: *There exists a C^∞ retraction of the closed unit ball in an infinite-dimensional Hilbert space onto its boundary.* The last result implies the existence of a fixed point free C^∞ self-map of the closed unit ball

in an infinite-dimensional Hilbert space. For more details on negligibility of sets the reader is referred to the book of Bessaga–Pełczyński [1975].

Other invertibility results

We mention some global invertibility theorems that are not proved in the text. In the differentiable case the following result is due to Hadamard [1906]: *If $f : E \to F$ is a C^1 map between finite-dimensional Banach spaces and if f is a local homeomorphism such that $\|[f'(x)]^{-1}\| \le M$ for some $M > 0$ and all $x \in E$, then f is a diffeomorphism.* For a proof of the Hadamard theorem for arbitrary Banach spaces, the reader is referred to the lecture notes by J.T. Schwartz [1969].

S. Mazur and S. Ulam, Lwów, 1935

A general invertibility theorem is due to Banach–Mazur [1934]: *Let X and Y be metric spaces, where X is connected and Y is locally arcwise connected and simply connected. Let $f : X \to Y$ be a proper map. Then f is invertible if and only if it is a local homeomorphism.*

A special case of the Banach–Mazur theorem was established earlier by Caccioppoli [1932]: *Let $f : E \to F$ be a C^1 proper map between Banach spaces. Then f is a diffeomorphism if and only if it is a local diffeomorphism.*

The proofs of the above two results can be found in Berger's book [1977].

For some other related results the reader is referred to Carathéodory–Rademacher [1917] and Ambrosetti–Prodi [1972].

Applications of the elementary KKM-principle

The presentation in Section 4 follows Granas–Lassonde [1995]. Variational inequalities (the systematic study of which began around 1965) are of importance in many applied problems (see Kinderlehrer–Stampacchia [1980], where an introductory account of the theory and further references can be found). Theorem (4.6) is due to Hartman–Stampacchia [1966]; its proof is a simplification of the one in Dugundji–Granas [1978]. Theorem (4.7) is due to Minty [1962]. The significance of maximality of set-valued monotone operators was brought to light by Minty [1965], to whom the theory of maximal monotone operators in a Hilbert space is due.

For more general or related results the reader is referred to Brézis's book [1973], Browder's survey [1976] and also to "Miscellaneous Results and Examples".

Mazur–Orlicz theorem

Kakutani [1938] proved the Hahn–Banach theorem using the Markoff–Kakutani theorem and the compactness of the Tychonoff cube. The same idea is used in §4 for the proof of Banach's lemma (5.1) on which the proof of the Mazur–Orlicz theorem (5.3) is based. This proof follows Granas–Lassonde [1991] and is due to F. C. Liu (unpublished). We remark that the formulation of (5.3) (obtained from the original one by replacing "linear combinations" with "convex combinations") permits getting at once a refined version of the Hahn–Banach theorem (5.4). Theorem (5.5), due to Mazur–Orlicz [1953], is a generalization of the classical "moments problem" theorem. The classical separation theorems of Mazur and Eidelheit (called by Bourbaki the "geometric forms of the Hahn–Banach theorem") also follow at once from Theorem (5.3).

S. Mazur observed that the Mazur–Orlicz and Hahn–Banach theorems remain valid if in their formulation a sublinear functional p is replaced by a convex functional; the corresponding proofs can be found in Alexiewicz [1969]. For more recent applications of the Mazur–Orlicz theorem the reader is referred to Liu [1993].

II.

Theorem of Borsuk and Topological Transversality

In this chapter we provide an easily accessible and unified account of some of the most fundamental results in fixed point theory. Among them, the antipodal theorem of Borsuk and the theorem on topological transversality occupy the central position; all the other results in this chapter are their consequences. The chapter ends with diverse applications to various fields.

§5. Theorems of Brouwer and Borsuk

Our aim in this paragraph is to establish the theorem of Borsuk and its immediate consequence, the Brouwer fixed point theorem. We obtain these results by first establishing the Lusternik–Schnirelmann–Borsuk theorem about the n-sphere. Our approach is elementary, in that it involves only some simple simplicial decompositions of the sphere and a combinatorial lemma.

1. Preliminary Remarks

Let E be a normed linear space. We recall that a finite set of $s+1$ points in E is said to be *affinely independent* if it is not contained in any $(s-1)$-flat of E.

(1.1) DEFINITION. Let $\{p_0, p_1, \ldots, p_s\}$ be an affinely independent set of $s+1$ points in E. Their convex hull

$$\left\{x \in E \,\middle|\, x = \sum_{i=0}^{s} \lambda_i p_i;\ 0 \leq \lambda_i \leq 1,\ \sum_{i=0}^{s} \lambda_i = 1\right\}$$

is called the (closed) *s-simplex* with vertices $p_0, \ldots, p_s$ and is denoted by $[p_0, \ldots, p_s]$. If the vertices do not have to be explicitly stated, a simplex is denoted by σ or σ^s, the upper index indicating its dimension.

The k-simplex spanned by any $k+1$ of the vertices $p_0, \ldots, p_s$ is called a *k-face* of σ^s; the only s-face of σ^s is σ^s itself. The *boundary*, $\partial\sigma^s$ (which is not necessarily the topological boundary of σ^s in E), is the union of all faces of dimension $\leq s-1$; the *open s-simplex* is $\sigma^s - \partial\sigma^s$ The 0-faces of σ^s are its vertices, and the 1-faces $[p_i, p_j]$ are called the *edges* of σ^s; because σ^s is convex, it is easy to see that the diameter $\delta(\sigma^s)$ of σ^s is the length of its longest edge.

A simplex is obviously a compact metric space. Moreover, because the set of its vertices is affinely independent, each $x \in \sigma^s = [p_0, \ldots, p_s]$ can be written uniquely as $x = \sum_{i=0}^{s} \lambda_i(x) p_i$, where $\sum_{i=0}^{s} \lambda_i(x) = 1$ and $0 \leq \lambda_i(x) \leq 1$ for each $x \in \sigma^s$ and $i = 0, \ldots, s$; the $(s+1)$-tuple $(\lambda_0(x), \ldots, \lambda_s(x))$ of real numbers is called the *barycentric coordinates* of $x \in \sigma$, and each $\lambda_i : \sigma \to [0,1]$ is called the ith *barycentric coordinate function* of σ.

(1.2) PROPOSITION. *Any two s-simplices are affinely homeomorphic. Furthermore, for any simplex σ, each of its barycentric coordinate functions $\lambda_i : \sigma \to [0,1]$ is continuous.*

PROOF. Let $\boldsymbol{R}^{s+1}$ be the $(s+1)$-dimensional Euclidean space and $\Delta^s \subset \boldsymbol{R}^{s+1}$ be the s-simplex having the unit points $e_0 = (1, 0, \ldots, 0), \ldots, e_s = (0, 0, \ldots, 1)$ in $\boldsymbol{R}^{s+1}$ as vertices; Δ^s is called the *standard s-simplex*. Observe that the barycentric coordinates of any $x \in \Delta^s$ are precisely the Euclidean coordinates of x, so that

$$\Delta^s = \left\{ (\lambda_0, \ldots, \lambda_s) \in \boldsymbol{R}^{s+1} \,\middle|\, 0 \leq \lambda_i \leq 1, \sum_{i=0}^{s} \lambda_i = 1 \right\}.$$

We now show that any given $\sigma^s = [p_0, \ldots, p_s] \subset E$ is affinely homeomorphic to Δ^s Let $h : \boldsymbol{R}^{s+1} \to E$ be the map $h(\lambda_0, \ldots, \lambda_s) = \sum_{i=0}^{s} \lambda_i p_i$; this is clearly continuous, and $g = h|\Delta^s$ maps Δ^s onto σ^s; since g is also a bijective map of the compact Δ^s onto σ^s, we conclude that g is an affine homeomorphism of Δ^s onto σ^s. To prove that the barycentric coordinate functions are continuous, let $\pi_i : \boldsymbol{R}^{s+1} \to \boldsymbol{R}$ be the projection onto the ith coordinate space; since $\lambda_i = \pi_i \circ g^{-1}$, and both π_i, g^{-1} are continuous, the proof is complete. □

2. Basic Triangulation of S^n

Because we will be using simplices throughout, it is convenient to work with an equivalent norm for Euclidean space under which the unit sphere can be regarded as the union of geometric simplices.

Let E be the normed space of all those sequences $x = \{x_1, x_2, \ldots\}$ of real numbers having at most finitely many $x_n \neq 0$, with the norm $\|x\| = \sum |x_i|$.

The subset $\{x \in E \mid x_i = 0 \text{ for all } i > n\}$ is denoted by E^n; the (closed) *unit n-ball* is

$$K^n = \{x \in E^n \mid \|x\| \leq 1\}.$$

The *unit n-sphere* is $S^n = \{x \in E^{n+1} \mid \|x\| = 1\}$; its upper hemisphere is $S^n_+ = \{x \in S^n \mid x_{n+1} \geq 0\}$, and its lower hemisphere is $S^n_- = \{x \in S^n \mid x_{n+1} \leq 0\}$; clearly, $S^n = S^n_+ \cup S^n_-$. Observe that for any $k < n$, we have

$$S^k = \{x \in S^n \mid x_{k+2} = \cdots = x_{n+1} = 0\}$$

and that $S^{n-1} = S^n_+ \cap S^n_-$.

Given an n-dimensional normed linear space L^n it is easy to see that there is a homeomorphism of L^n onto E^n sending points symmetric with respect to the origin in L^n onto points symmetric with respect to the origin in E^n and mapping the unit sphere in L^n onto S^n Therefore, the results that will be established in this paragraph in E^n remain valid for any finite-dimensional normed linear space L^n

By a triangulation of S^n is meant a decomposition of S^n into simplices that are pasted together along common faces in an orderly manner. Precisely:

(2.1) DEFINITION. A finite family $\mathscr{S}^n = \{\sigma\}$ of simplices in S^n is called a *triangulation* of S^n provided:
- (i) the intersection of any two simplices in $\mathscr{S}^n$ is either empty or a common face of each,
- (ii) if $\sigma \in \mathscr{S}^n$ then every face of σ is in $\mathscr{S}^n$,
- (iii) $S^n = \bigcup\{\sigma \mid \sigma \in \mathscr{S}^n\}$.
- (iv) each $(n-1)$-simplex of $\mathscr{S}^n$ is the common face of exactly two n-simplices in $\mathscr{S}^n$.

We remark that (iv) can be deduced from properties (i)-(iii).

The following triangulation of S^n is important for our purposes: For each $i = 1, \ldots, n+1$, let $e_i = \{\delta^i_1, \delta^i_2, \ldots\} \in E$, where δ^i_j is the Kronecker delta; clearly, the unit ball K^{n+1} is precisely the convex hull of the set $\{e_1, \ldots, e_{n+1}, -e_1, \ldots, -e_{n+1}\}$. It is easy to see that the set of all n-simplices $[\pm e_1, \ldots, \pm e_{n+1}]$ and all their faces provides a triangulation of S^n, called the *basic triangulation*; this triangulation is denoted by Σ^n. Note that each simplex of Σ^n has a unique representation (called its *standard form*) by a symbol $[\pm e_{i_0}, \ldots, \pm e_{i_s}]$, where $i_0 < \cdots < i_s$.

Let $\alpha : S^n \to S^n$ be the *antipodal map* $x \mapsto -x$; two elements of any sort (points, simplices, sets) corresponding under α will be called *antipodal*. Note that for each $k \leq n$, the restriction $\alpha|S^k$ is the antipodal map of S^k. It is clear that no simplex of Σ^n contains a pair of antipodal vertices, and

that for each simplex $\sigma^k \in \Sigma^n$ the set $\alpha(\sigma^k)$ is also a simplex of Σ^n. Since we need to consider triangulations of S^n other than Σ^n that have these two properties, we make the formal

(2.2) DEFINITION. A triangulation $\mathscr{S}^n$ of S^n is called *symmetric* if:
(a) for each $k \leq n$, the k-sphere S^k is a union of k-simplices of $\mathscr{S}^n$,
(b) for each $k \leq n$, and each simplex $\sigma^k \in \mathscr{S}^n$, the set $\alpha(\sigma^k)$ is also a k-simplex of $\mathscr{S}^n$

We have already observed that Σ^n is a symmetric triangulation; moreover, symmetric triangulations of S^n with arbitrarily small simplices clearly exist; a formal inductive proof of this evident geometric fact could be based on the observations that a symmetric small simplex triangulation of $S^n = \partial S^{n+1}_-$ can be extended to a small simplex simplicial decomposition of S^{n+1}_-, and that

$$\{\sigma^{n+1}, \alpha(\sigma^{n+1}) \mid \sigma^{n+1} \in S^{n+1}_-\}$$

is then a symmetric triangulation of S^{n+1}.

3. A Combinatorial Lemma

Let $\mathscr{S}^k$, respectively $\mathscr{S}^n$, be triangulations of S^k, respectively S^n. A map f of the vertices of $\mathscr{S}^k$ to the vertices of $\mathscr{S}^n$ is called a *simplicial vertex map* if for each simplex $[p_0, \ldots, p_s]$ of $\mathscr{S}^k$, the points $f(p_0), \ldots, f(p_s)$ are the vertices of a (possibly lower-dimensional) simplex of $\mathscr{S}^n$. Clearly, f extends to a map $S^k \to S^n$ (denoted also by f) sending simplices of $\mathscr{S}^k$ into simplices of $\mathscr{S}^n$.

(3.1) DEFINITION. Let $\mathscr{S}^k$ be an arbitrary triangulation of S^k, and let $f : \mathscr{S}^k \to \Sigma^n$ be a simplicial vertex map. An r-simplex $[p_0, \ldots, p_r]$ of $\mathscr{S}^k$ is called *positive* if:
(i) the vertices $f(p_0), \ldots, f(p_r)$ span an r-simplex $\sigma^r \in \Sigma^n$,
(ii) the standard form of σ^r is "alternating in sign",

$$\sigma^r = [+e_{i_0}, -e_{i_1}, \ldots, (-1)^r e_{i_r}],$$

with the first vertex positive.
An r-simplex of $\mathscr{S}^k$ is *negative* if its f-image is an r-simplex of Σ^n which, in standard form, is alternating in sign and has negative first vertex.
An r-simplex of $\mathscr{S}^k$ that is neither positive nor negative is called *neutral*.

For any simplicial vertex map $f : \mathscr{S}^k \to \Sigma^n$ and any subset $L \subset S^k$, the number of positive r-simplices in L under f is denoted by $p(f, L, r)$.

The main result of this section relies on the following

(3.2) PROPOSITION. *Let $k \leq n$, and let $f : \mathscr{S}^k \to \Sigma^n$ be a simplicial vertex map of a symmetric triangulation of S^k into Σ^n If $\alpha \circ f = f \circ \alpha$, then*

$$p(f, S^k, k) \equiv p(f, S^{k-1}, k-1) \bmod 2.$$

PROOF. Consider the upper hemisphere S^k_+ of S^k, and decompose the set of k-simplices in S^k_+ into three disjoint classes:

$$\mathscr{A}_+ = \{s^k \subset S^k_+ \mid s^k \text{ is positive}\},$$
$$\mathscr{A}_- = \{s^k \subset S^k_+ \mid s^k \text{ is negative}\},$$
$$\mathscr{A}_0 = \{s^k \subset S^k_+ \mid s^k \text{ is neutral}\}.$$

Consider the sum

$$T = \sum_{s^k \in \mathscr{A}_+} p(f, s^k, k-1) + \sum_{s^k \in \mathscr{A}_-} p(f, s^k, k-1) + \sum_{s^k \in \mathscr{A}_0} p(f, s^k, k-1);$$

we will determine the parity of T.

First note that, because each $p(f, s^k, k-1)$ is the number of positive $(k-1)$-faces of s^k, the sum T involves all the positive s^{k-1} in S^k_+. Observe next that each positive s^{k-1} not in S^{k-1} will occur twice in the sum T, since it is a face of exactly two s^k; because each positive s^{k-1} on S^{k-1} is the face of only one $s^k \in S^k_+$, we conclude that $T \equiv p(f, S^{k-1}, k-1) \bmod 2$.

We now develop another expression for T. Consider any neutral s^k. Since s^k can have no positive $(k-1)$-face (hence make no contribution to the sum), unless $\dim f(s^k) \geq k-1$, we can write $f(s^k) = [\pm e_{i_0}, \ldots, \pm e_{i_k}]$ with $i_0 \leq \cdots \leq i_k$, in which there is either one repeated vertex, or all the vertices are distinct but the signs do not alternate. In each case, a positive $(k-1)$-face can occur only if there is at most one pair of adjacent vertices with the same sign; and if removal of one of these vertices gives a positive face, so also will removal of the adjacent one. Thus, $p(f, s^k, k-1)$ is even for each $s^k \in \mathscr{A}_0$, so that

$$T \equiv \sum_{s^k \in \mathscr{A}_+} p(f, s^k, k-1) + \sum_{s^k \in \mathscr{A}_-} p(f, s^k, k-1) \bmod 2.$$

Noting now that each positive s^k has exactly one positive $(k-1)$-face, as also does each negative s^k, we find

$$\sum_{s^k \in \mathscr{A}_+} p(f, s^k, k-1) = \operatorname{card} \mathscr{A}_+, \qquad \sum_{s^k \in \mathscr{A}_-} p(f, s^k, k-1) = \operatorname{card} \mathscr{A}_-,$$

and therefore

$$T \equiv (\operatorname{card} \mathscr{A}_+ + \operatorname{card} \mathscr{A}_-) \bmod 2.$$

Finally, as $\alpha \circ f = f \circ \alpha$, it follows that an $s^k \in S_+^k$ is negative if and only if $\alpha(s^k) \in S_-^k$ is positive, so that $\operatorname{card} \mathscr{A}_- = \operatorname{card}\{s^k \in S_-^k \mid s^k \text{ is positive}\}$; therefore $\operatorname{card} \mathscr{A}_+ + \operatorname{card} \mathscr{A}_- = p(f, S^k, k)$, and the proof is complete. □

Proposition (3.2) leads to the main result of this section:

(3.3) THEOREM (Combinatorial lemma). *Let $f : \mathscr{S}^n \to \Sigma^n$ be a simplicial vertex map of a symmetric triangulation of S^n. If $\alpha \circ f = f \circ \alpha$, then f maps an odd number of simplices of S^n onto*

$$\sigma_0^n = [e_1, -e_2, \dots, (-1)^n e_{n+1}].$$

PROOF. According to the definition, an $s^n \in \mathscr{S}^n$ is positive if and only if its image, in standard form, is σ_0^n. According to the lemma,

$$p(f, S^n, n) \equiv p(f, S^{n-1}, n-1) \equiv \cdots \equiv p(f, S^0, 0) \bmod 2.$$

As S^0 consists of exactly two vertices and $f|S^0$ maps them onto a pair of antipodal vertices, it is clear that $p(f, S^0, 0) = 1$, completing the proof. □

4. The Lusternik–Schnirelmann–Borsuk Theorem

The combinatorial lemma will be applied to obtain the Lusternik-Schnirelmann–Borsuk theorem about the n-sphere, which is equivalent to the Borsuk antipodal theorem.

(4.1) LEMMA (Lebesgue). *Let $\{M_1, \dots, M_n\}$ be a family of closed nonempty sets in a compact metric space X, with $M_1 \cap \cdots \cap M_n = \emptyset$. Then there exists an $\varepsilon > 0$ with the property: any subset $A \subset X$ meeting every M_i must have $\delta(A) \geq \varepsilon$.*

PROOF. Let Z be the compact metric space $M_1 \times \cdots \times M_n$ and consider the continuous $\lambda : Z \to \boldsymbol{R}$ defined by

$$(x_1, \dots, x_n) \mapsto \max\{d(x_i, x_j) \mid 1 \leq i \leq j \leq n\}.$$

Because $M_1 \cap \cdots \cap M_n = \emptyset$, the map λ is never zero; consequently, it assumes a minimum, $\varepsilon > 0$. If $A \subset X$ meets each M_i, there is an $x_i \in A \cap M_i$ for each $i = 1, \dots, n$; since $\lambda(x_1, \dots, x_n) \geq \varepsilon$, at least one $d(x_i, x_j) \geq \varepsilon$, so $\delta(A) \geq \varepsilon$. This completes the proof. □

As an immediate consequence, we have

(4.2) THEOREM (Lebesgue). *Let $\{M_1, \dots, M_n\}$ be a closed covering of a compact metric space X. Then there exists a $\lambda > 0$ (a Lebesgue number of the covering) with the property: if any set A of diameter $< \lambda$ meets $M_{i_1}, \dots, M_{i_r}$, then*

$$M_{i_1} \cap \cdots \cap M_{i_r} \neq \emptyset. \qquad \square$$

With these preliminaries, we are ready to establish the fundamental

(4.3) LEMMA. *Let $M_1, \ldots, M_{n+1}$ be $n+1$ closed sets on S^n, no one of which contains a pair of antipodal points. If the family*

$$\{M_1, \ldots, M_{n+1}, \alpha(M_1), \ldots, \alpha(M_{n+1})\}$$

covers S^n, then $M_1 \cap \cdots \cap M_{n+1} \neq \emptyset$.

PROOF. Denote $\alpha(M_i)$ by M_{-i}; since M_i does not contain any pair of antipodal points, we have $d(M_i, M_{-i}) = \varepsilon_i > 0$ for each $i = 1, \ldots, n+1$.

Linearly order the covering by

$$M_1, M_{-1}; M_{-2}, M_2; M_3, M_{-3}; M_{-4}, M_4; \ldots,$$

and let λ be a Lebesgue number for this closed covering.

Let $\mathscr{S}^n$ be a symmetric triangulation of S^n with the diameter of each simplex $< \varepsilon = \min(\lambda, \varepsilon_1, \ldots, \varepsilon_{n+1})$. We first construct a simplicial vertex map $f : \mathscr{S}^n \to \Sigma^n$ as follows:

For each vertex $p \in \mathscr{S}^n$, let M_j be the first set of the ordered covering containing p, and set

$$f(p) = (\operatorname{sign} j)(-1)^{j+1} e_{|j|}.$$

This is, in fact, a simplicial vertex map: since Σ^n can be described as the set of all simplices $[\pm e_{i_0}, \ldots, \pm e_{i_s}]$ with no two entries antipodal, it is enough to show that no two vertices p_i, p_j of a simplex of $\mathscr{S}^n$ can map to antipodal vertices, and this follows from $d(p_i, p_j) < \varepsilon$ and the definition of ε.

It is evident, from the definition of f, that $\alpha \circ f = f \circ \alpha$, so from (3.3), there is some simplex $[p_1, \ldots, p_{n+1}]$ such that

$$f[p_1, \ldots, p_{n+1}] = [e_1, -e_2, \ldots, (-1)^n e_{n+1}].$$

This means that each $p_i \in M_i$, so that $[p_1, \ldots, p_{n+1}] \cap M_i \neq \emptyset$ for $i = 1, \ldots, n+1$. Since $\delta([p_1, \ldots, p_{n+1}]) < \lambda$, it follows that $M_1 \cap \cdots \cap M_{n+1} \neq \emptyset$, and the proof is complete. □

We now establish the main result of this section.

(4.4) THEOREM (Lusternik–Schnirelmann–Borsuk). *In any closed covering $\{M_1, \ldots, M_{n+1}\}$ of S^n by $n+1$ sets, at least one set M_i must contain a pair of antipodal points.*

PROOF. We argue by contradiction. Assume that no M_i contains a pair of antipodal points; then (4.3) applied to the covering $\{M_1, \ldots, M_{n+1}, \alpha(M_1), \ldots, \alpha(M_{n+1})\}$ would show $M_1 \cap \cdots \cap M_{n+1} \neq \emptyset$; since any $x_0 \in M_1 \cap \cdots \cap M_{n+1}$ must also be in some set $\alpha(M_j)$ of the covering $\{\alpha(M_1), \ldots, \alpha(M_{n+1})\}$ of S^n, this means that M_j would contain a pair of antipodal points, contradicting our hypothesis and completing the proof. □

5. Equivalent Formulations. The Borsuk–Ulam Theorem

Results equivalent to the Lusternik Schnirelmann Borsuk theorem use the notions of extendability and homotopy in their formulation. For the convenience of the reader, and to establish the terminology, we recall the relevant definitions. By space we understand a Hausdorff space; unless specifically stated otherwise, a map is a continuous transformation.

(a) Let X, Y be two spaces and $A \subset X$. A map $f : A \to Y$ is called *extendable* over X if there is a map $F : X \to Y$ with $F|A = f$.

(b) Two maps $f, g : X \to Y$ are called *homotopic* if there is a map $H : X \times I \to Y$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for each $x \in X$. The map H is called a *homotopy* (or *continuous deformation*) of f to g, and written $H : f \simeq g$. For each t, the map $x \mapsto H(x, t)$ is denoted by $H_t : X \to Y$; clearly the family $\{H_t : X \to Y\}_{0 \le t \le 1}$ determines H and vice versa.

Recall that the relation of homotopy is an equivalence relation in the set of all continuous maps of X into Y; for reflexivity, note $H(x, t) \equiv f(x)$ shows $f \simeq f$; for symmetry, observe that if $H : f \simeq g$ then $(x, t) \mapsto H(x, 1-t)$ gives $g \simeq f$; if $H : f \simeq g$ and $G : g \simeq h$, then

$$D(x,t) = \begin{cases} H(x, 2t), & 0 \le t \le \frac{1}{2}, \\ G(x, 2t-1), & \frac{1}{2} \le t \le 1, \end{cases}$$

is continuous and shows that $f \simeq h$, establishing transitivity. Thus, the relation of homotopy decomposes the set of all maps of X into Y into pairwise disjoint classes called *homotopy classes*. An $f : X \to Y$ homotopic to a constant map is called *nullhomotopic*; in this case we write $f \simeq 0$. A space X is called *contractible* if $\mathrm{id}_X : X \to X$ is nullhomotopic.

Observe that to establish a homotopy $H : f \simeq g$ is essentially an extendability problem: one has given a map $h : (X \times 0) \cup (X \times 1) \to Y$ and seeks an extension over $X \times I$. For the special case of maps of spheres into arbitrary spaces, their nullhomotopy is equivalent to a simpler extendability property, which is very frequently used:

(5.1) THEOREM. *A map $f : S^n \to Y$ is nullhomotopic if and only if f is extendable to an $F : K^{n+1} \to Y$.*

PROOF. Assume $f : S^n \to Y$ is extendable to $F : K^{n+1} \to Y$. For $x \in S^n$ and $0 \le t \le 1$, set $H(x, t) = F(tx)$ to see that $H : 0 \simeq f$. Conversely, if $H : S^n \times I \to Y$ shows $0 \simeq f$, define an extension $F : K^{n+1} \to Y$ of f by

$$F(y) = \begin{cases} H(S^n, 0), & 0 \le \|y\| \le \frac{1}{2}, \\ H(y/\|y\|, 2\|y\| - 1), & \frac{1}{2} \le \|y\| \le 1. \end{cases}$$

This completes the proof. □

We say that $f : S^n \to S^k$ is *antipode-preserving* if $f(-x) = -f(x)$ for all $x \in S^n$. With this terminology we now prove Borsuk's antipodal theorem and also show that it is equivalent to various geometric results about the n-sphere. A fixed point version of the theorem will be derived in the next section.

(5.2) THEOREM. *The following statements are equivalent:*

(1) *The Lusternik–Schnirelmann–Borsuk theorem.*
(2) *There is no antipode-preserving map* $f : S^n \to S^{n-1}$
(3) (Borsuk's antipodal theorem) *An antipode-preserving map* $f : S^{n-1} \to S^{n-1}$ *is not nullhomotopic.*
(4) (Borsuk–Ulam) *Every continuous* $f : S^n \to E^n$ *sends at least one pair of antipodal points to the same point.*

PROOF. (1)⇒(2). Supose $f : S^n \to S^{n-1}$ is antipode-preserving. Decompose S^{n-1} into $n+1$ closed sets $A_1, \ldots, A_{n+1}$ by projecting the boundary of an n-simplex centered at 0 onto S^{n-1} and letting A_i be the images of the $(n-1)$-faces. Clearly, no A_i contains a pair of antipodal points.

Let $M_i = f^{-1}(A_i)$, $i = 1, \ldots, n+1$. The M_i are closed and cover S^n, so by (1), there is an $x \in M_i \cap \alpha M_i$ for some i. Because f is antipode-preserving, this means that $f(x)$ and $f\alpha(x) = \alpha f(x)$ both belong to A_i, which is a contradiction.

(2)⇒(3). Suppose some antipode-preserving $g : S^{n-1} \to S^{n-1}$ were nullhomotopic. Then g would be extendable to a $G : K^n \to S^{n-1}$. Regarding K^n as S^n_+, we define $\varphi : S^n \to S^{n-1}$ by

$$\varphi(x) = \begin{cases} G(x), & x \in S^n_+, \\ -G\alpha(x), & x \in S^n_- \end{cases}$$

This is consistently defined on $S^n_+ \cap S^n_-$, and is an antipode-preserving map of S^n to S^{n-1}, contradicting (2).

(3)⇒(4). Assume $f : S^n \to E^n$ is such that $f(x) \neq f(-x)$ for every $x \in S^n$ Define $F : S^n \to S^{n-1}$ by

$$F(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

Then $F|S^{n-1} : S^{n-1} \to S^{n-1}$ is antipode-preserving, and since $F|S^n_+$ is an extension over $K^n, F|S^{n-1}$ would be nullhomotopic, contradicting (3).

(4)⇒(1). Assume there were some closed covering $M_1, \ldots, M_{n+1}$ of S^n with no M_i containing a pair of antipodal points, i.e., $M_i \cap \alpha(M_i) = \emptyset$ for each i. Let $g_i : S^n \to I$ be an Urysohn function with $g_i|M_i = 0$ and $g_i|\alpha(M_i) = 1$ for each $i = 1, \ldots, n$, and define $g : S^n \to E^n$ by

$$g(x) = (g_1(x), \ldots, g_n(x)).$$

According to (4), there must be a $z \in S^n$ with $g(z) = g\alpha(z)$, so that $g_i(z) = g_i(\alpha(z))$ for $i = 1, \ldots, n$, and therefore $z \in S^n - \bigcup_{i=1}^n M_i - \bigcup_{i=1}^n \alpha(M_i)$. Since both $\{M_i\}_{i=1}^{n+1}$ and $\{\alpha(M_i)\}_{i=1}^{n+1}$ cover S^n, the point z must belong to both M_{n+1} and $\alpha(M_{n+1})$, which is the desired contradiction. □

6. Some Simple Consequences

We give two consequences of (5.2) that are particularly useful for our later work. The first relaxes the condition $-f(x) = f(-x)$ in (5.2)(3) to simply $f(x) \neq f(-x)$;

(6.1) THEOREM. *A map $f : S^n \to S^n$ with $f(x) \neq f\alpha(x)$ for each x is not nullhomotopic.*

PROOF. Since $f(x) \neq f\alpha(x)$, the map $g : S^n \to S^n$ given by

$$x \mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

is continuous and clearly antipode-preserving. Now, $f, g : S^n \to S^n$ are never antipodal; for if $g(z) = -f(z)$ for some $z \in S^n$, then

$$[1 + \|f(z) - f(-z)\|]f(z) = f(-z),$$

so since $\|f(z)\| = \|f(-z)\| = 1$, we would have $1 + \|f(z) - f(-z)\| = 1$, which is impossible. Since f and g are never antipodal,

$$h_t(x) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is a well defined pont of S^n for all $(t, x) \in [0, 1] \times S^n$ and hence $\{h_t\}_{0 \leq t \leq 1}$ is a homotopy joining f and g. From this, since g is not nullhomotopic by (5.2)(3), we infer that neither is f. This completes the proof. □

The reader can easily show that in fact, (6.1) is equivalent to (5.2)(3).

The second consequence, which we shall use frequently, is Borsuk's fixed point theorem:

(6.2) THEOREM (Borsuk). *Let U be a bounded symmetric convex open neighborhood of the origin in E^n, and let $F : \overline{U} \to E^n$ be antipode-preserving on ∂U, i.e., $-F(a) = F(-a)$ for each $a \in \partial U$. Then F has a fixed point.*

PROOF. Let $p : E^n \to \boldsymbol{R}$ be the Minkowski functional for U, and let E be the set E^n with the norm $\|x\|_1 = p(x)$. The identity map $h : E^n \to E$ is a homeomorphism mapping $\overline{U}$ onto the unit ball K_1^n of E. Considering the map $g = h \circ F \circ h^{-1} : K_1^n \to E$, which is antipode-preserving on ∂K_1^n, we

first show that g has a fixed point; for if $g(x) \neq x$ in K_{I}^n, then

$$f(x) = \frac{g(x) - x}{\|g(x) - x\|_{\mathrm{I}}}$$

would be a continuous map $f : K_{\mathrm{I}}^n \to \partial K_{\mathrm{I}}^n$, so that $f|\partial K_{\mathrm{I}}^n : \partial K_{\mathrm{I}}^n \to \partial K_{\mathrm{I}}^n$ would be nullhomotopic; but $f|\partial K_{\mathrm{I}}^n$ is easily seen to be antipode-preserving and this is a contradiction. Therefore, $hFh^{-1}(x) = x$ for some $x \in K_{\mathrm{I}}^n$, so that $Fh^{-1}(x) = h^{-1}(x)$ and F has a fixed point. □

7. Brouwer's Theorem

The following special case of Borsuk's theorem (5.2)(3) is basic in fixed point theory.

(7.1) THEOREM. *The identity map* id : $S^n \to S^n$ *is not homotopic to a constant map.*

PROOF. Since id : $S^n \to S^n$ is antipode-preserving, (5.2)(3) applies. □

This result has many equivalent formulations; it is in fact equivalent to Brouwer's fixed point theorem:

(7.2) THEOREM. *The following statements are equivalent:*
- (1) S^n *is not contractible in itself.*
- (2) (Bohl) *Every continuous* $F : K^{n+1} \to E^{n+1}$ *has at least one of the following properties:*
  - (a) F *has a fixed point,*
  - (b) *there are* $x \in \partial K^{n+1}$ *and* $\lambda \in (0,1)$ *such that* $x = \lambda F(x)$.
- (3) (Brouwer) *Every continuous* $F : K^{n+1} \to K^{n+1}$ *has at least one fixed point.*
- (4) (Borsuk) *There is no retraction* $r : K^{n+1} \to S^n$, *i.e., there is no continuous* $r : K^{n+1} \to S^n$ *that keeps each* $x \in S^n$ *fixed.*

PROOF. (1)⇒(2). Suppose $F(x) \neq x$ for all $x \in K^{n+1}$, and $y \neq tF(y)$ for all $0 < t < 1$, $y \in \partial K^{n+1}$; then $y \neq tF(y)$ also for $t = 0$, and by our first hypothesis, for $t = 1$. Let $r : E^{n+1} - \{0\} \to S^n$ be the map $x \mapsto x/\|x\|$. Then $H : S^n \times I \to S^n$ defined by

$$H(y,t) = \begin{cases} r(y - 2tF(y)), & 0 \leq t \leq \frac{1}{2}, \\ r[(2-2t)y - F\{(2-2t)y\}], & \frac{1}{2} \leq t \leq 1, \end{cases}$$

would show that id : $S^n \to S^n$ is homotopic to a constant.

(2)⇒(3). The second possibility in (2) cannot occur, because $F(S^n) \subset K^{n+1}$.

(3)⇒(4). If there were a retraction, the map $x \mapsto -r(x)$ would be a fixed point free map of K^{n+1} into itself.

(4)$\Rightarrow$(1). Assume $h : 0 \simeq \mathrm{id}$, where $h(S^n, 0) = x_0 \in S^n$. Defining $r : K^{n+1} \to S^n$ by

$$r(x) = \begin{cases} x_0, & \|x\| \leq \frac{1}{2}, \\ h(x/\|x\|, 2\|x\| - 1), & \|x\| \geq \frac{1}{2}, \end{cases}$$

would give a retraction of K^{n+1} onto S^n. □

The following example shows that Brouwer's theorem (7.2)(3) cannot be extended to infinite-dimensional normed linear spaces.

EXAMPLE. Let E be a noncomplete normed linear space, and K its closed unit ball. By (4.3.10) there is a deleting homeomorphism $h : E \approx E - \{0\}$ such that $h(x) = x$ for all $x \in \partial K$. Consider the map $r : K \to \partial K$ given by $x \mapsto h(x)/\|h(x)\|$, which is well defined because $h(y) \neq 0$ for all $y \in E$. If $x \in \partial K$, then $h(x) = x$, so $r|\partial K = \mathrm{id}$ and $r : K \to \partial K$ is a retraction; the map $x \mapsto -r(x)$ is therefore a fixed point free map of K into itself.

This example shows that to obtain any generalization of the Brouwer fixed point theorem valid in infinite-dimensional spaces, it is necessary to restrict the type of map $F : K \to K$ that will be considered. We will show in the next paragraph that every compact map $F : K \to K$ (i.e., a continuous map such that $\overline{F(K)}$ is compact) of the unit ball K of any normed linear space has a fixed point. Note that this statement, valid in all normed linear spaces, is precisely the Brouwer theorem whenever the space E is finite-dimensional, since in that case a continuous $F : K \to K$ is necessarily compact. On this basis, it appears that for infinite-dimensional normed linear spaces, the natural analogue of a continuous map in finite-dimensional normed spaces is that of a compact (rather than simply continuous) map; maps of this type arise naturally in many problems of analysis.

8. Topological KKM-Principle

Among the results equivalent to Brouwer's fixed point theorem, the theorem of Knaster–Kuratowski–Mazurkiewicz occupies a special place: it admits an infinite-dimensional version which, as shown by Ky Fan, is particularly suitable for applications.

Let E be a vector space and $X \subset E$ an arbitrary subset. Recall that a set-valued map $G : X \to 2^E$ is called a *KKM-map* provided $\mathrm{conv}\{x_1, \ldots, x_s\} \subset \bigcup_{i=1}^{s} Gx_i$ for each finite subset $\{x_1, \ldots, x_s\} \subset X$; G is called *strongly KKM* provided $x \in Gx$ for each $x \in X$ and the cofibers of G (i.e., the sets $\{x \in X \mid y \notin Gx\}$ for $y \in E$) are all convex.

The basic topological property of KKM-maps is given in

(8.1) THEOREM. *Let E be a linear topological space, X an arbitrary subset of E, and $G : X \to 2^E$ a KKM-map such that each Gx is finitely*

closed. Then the family $\{Gx \mid x \in X\}$ has the finite intersection property.

PROOF. We argue by contradiction, so assume $\bigcap_{i=1}^n Gx_i = \emptyset$. Working in the finite-dimensional flat L spanned by $\{x_1, \dots, x_n\}$, let d be the Euclidean metric in L and $C = \operatorname{conv}\{x_1, \dots, x_n\} \subset L$; note that because each $L \cap Gx_i$ is closed in L, we have $d(x, L \cap Gx_i) = 0$ if and only if $x \in L \cap Gx_i$. Since $\bigcap_{i=1}^n L \cap Gx_i = \emptyset$ by assumption, the function $\lambda : C \to \boldsymbol{R}$ given by $c \mapsto \sum_{i=1}^n d(c, L \cap Gx_i)$ is not zero for any $c \in C$, and we can define a continuous $f : C \to C$ by setting

$$f(c) = \frac{1}{\lambda(c)} \sum_{i=1}^n d(c, L \cap Gx_i) \cdot x_i.$$

By Brouwer's theorem, f would have a fixed point $c_0 \in C$. Let

$$I = \{i \mid d(c_0, L \cap Gx_i) \neq 0\}.$$

Then the fixed point c_0 cannot belong to $\bigcup\{Gx_i \mid i \in I\}$; however,

$$c_0 = f(c_0) \in \operatorname{conv}\{x_i \mid i \in I\} \subset \bigcup\{Gx_i \mid i \in I\},$$

and with this contradiction, the proof is complete. □

As an immediate consequence of (8.1), we obtain the following fundamental result:

(8.2) THEOREM (Topological KKM-principle). *Let E be a linear topological space, $X \subset E$ an arbitrary subset, and $G : X \to 2^E$ a KKM-map. If all the sets Gx are closed in E, and if one of them is compact, then $\bigcap\{Gx \mid x \in X\} \neq \emptyset$.* □

Clearly, the topological KKM-principle contains as a special case the geometric KKM-principle (3.1.5). We now give two simple applications that do not follow from the geometric principle.

(8.3) THEOREM. *Let C be a nonempty compact convex set in a Hilbert space H with scalar product $(\ ,\ )$, and let $f : C \to H$ be continuous. Then there exists a $y_0 \in C$ such that*

$$(f(y_0), y_0 - x) \leq 0 \quad \textit{for all } x \in C.$$

PROOF. Define $G : C \to 2^C$ by

$$Gx = \{y \in C \mid (f(y), y - x) \leq 0\}.$$

Clearly, G is strongly KKM, and hence, because C is convex, it is a KKM-map (see (3.1.2)). Since f is continuous, the sets Gx are closed, therefore compact. By the topological KKM-principle, we find a point $y_0 \in C$ such that $y_0 \in Gx$ for all $x \in C$, which is the required conclusion. □

As an immediate consequence, we have

(8.4) THEOREM. *Let C be a nonempty compact convex set in a Hilbert space H. Let $F : C \to H$ be continuous and such that for each $x \in C$ with $x \neq F(x)$ the line segment $[x, F(x)]$ contains at least two points of C. Then F has a fixed point.*

PROOF. Define $f : C \to H$ by $f(x) = x - F(x)$ for $x \in C$. By (8.3) we find a point $y_0 \in C$ such that

$$(*) \qquad (y_0 - F(y_0), y_0 - x) \leq 0 \quad \text{for all } x \in C.$$

We show that y_0 is a fixed point of F. Indeed, if not, then the segment $[y_0, F(y_0)]$ must contain a point of C other than y_0, say $x = ty_0 + (1-t)F(y_0)$ for some $0 < t < 1$; then from $(*)$ we get $(1-t)(y_0 - F(y_0), y_0 - F(y_0)) \leq 0$, and since $t < 1$, we must have $y_0 = F(y_0)$. □

The theorem just proved implies that any continuous self-map of a compact convex set in a Hilbert space has a fixed point, thus showing in particular that the Brouwer fixed point theorem is equivalent to the topological KKM-principle.

Numerous applications of the topological KKM-principle will be given in §7.

We conclude by observing that as a special case of (8.1) we obtain one of the basic results in fixed point theory:

(8.5) THEOREM (Knaster–Kuratowski–Mazurkiewicz). *Let $X = \{x_0, \ldots, x_n\}$ be the set of vertices of a simplex $\sigma^n \subset R^n$ and $G : X \to 2^{R^n}$ a KKM-map assigning to each $x_i \in X$ a compact set $Gx_i \subset \sigma^n$. Then the intersection of the sets $Gx_0, \ldots, Gx_n$ is not empty.* □

We leave to the reader an easy proof that (8.1) and (8.5) are in fact equivalent.

9. Miscellaneous Results and Examples

A. Homotopy, retraction, and extendability of maps

(A.1) Let $f, g : X \to S^n$ be two maps. Show:
(a) If $f(x) \neq -g(x)$ for all $x \in X$, then $f \simeq g$.
(b) If $f(x) \neq g(x)$ for all $x \in X$, then $f \simeq -g$.
(c) If $f(x) \perp g(x)$ for all $x \in X$, then $f \simeq g$.

(A.2) Let $f : S^n \to S^n$ be a map. Prove:
(a) If $f(x) \neq -x$ for all $x \in S^n$, then $f \simeq \mathrm{id}_{S^n}$.
(b) If $f(x) \neq x$ for all $x \in S^n$, then $f \simeq \alpha$, where $\alpha : S^n \to S^n$ is the antipodal map.

(A.3) Let $f, g : X \to S^n$ be two maps such that $f(x) \neq \pm g(x)$ for $x \in X$. Show: There is $h : X \to S^n$ such that $g \simeq h$ and $f(x) \perp h(x)$ for each $x \in X$.

[Take
$$h(x) = \frac{g(x) - (g(x), f(x))f(x)}{\|g(x) - (g(x), f(x))f(x)\|}.]$$

(A.4) Let X be any space and $A \subset X$. Show: A is a retract of X if and only if for every space Y, each $f : A \to Y$ is extendable over X.

(A.5) Let K^n be the unit ball in R^n with boundary S^{n-1}. Show: $K^n \times \{0\} \cup S^{n-1} \times [0,1]$ is a retract of $K^n \times [0,1]$.

[Consider $r : K^n \times [0,1] \to K^n \times \{0\} \cup S^{n-1} \times [0,1]$ given by
$$r(x,t) = \begin{cases} \left(\dfrac{x}{\|x\|}, 2 - \dfrac{2-t}{\|x\|}\right), & \|x\| \geq 1 - \dfrac{t}{2}, \\ \left(\dfrac{2x}{2-t}, 0\right), & \|x\| \leq 1 - \dfrac{t}{2}.] \end{cases}$$

(A.6) Let Y be any space and $f_0, f_1 : S^n \to Y$ be homotopic. Show: If f_0 has an extension over K^{n+1}, then so also does f_1.

[Use (A.4) and (A.5).]

B. Borsuk's antipodal theorem

(B.1) Let $M_1, \ldots, M_{n+2}$ be a closed covering of S^n by $n+2$ nonempty sets. Show: If no M_i contains a pair of antipodal points, then $M_1 \cap \cdots \cap M_{n+2} = \emptyset$ and any $(n+1)$-element subfamily has a nonempty intersection.

[If $x \in M_1 \cap \cdots \cap M_{n+2}$ then $x \in \alpha(M_j)$ for some j, contradicting the assumption on M_j. For the second part: To show, say, $M_1 \cap \cdots \cap M_{n+1} \neq \emptyset$, note that
$$\{M_1, \ldots, M_{n+1}, \alpha(M_1), \ldots, \alpha(M_{n+1})\}$$
must be a covering of S^n, since an x not in the union must belong to both M_{n+2} and $\alpha(M_{n+2})$; then apply (4.3).]

(B.2) Prove: In each decomposition of K^{n+1} into $n+1$ closed sets, at least one of the sets must have diameter equal to 2.

[Use (4.4).]

(B.3) Prove the "invariance of dimension number" theorem: R^m is not homeomorphic to R^n whenever $n \neq m$.

[Suppose $n > m$ and $h : R^n \to R^m$ is a homeomorphism. Consider $h|S^{n-1}$ and apply (5.2)(4).]

(B.4) Let $f_1, \ldots, f_n$ be n continuous real-valued functions on S^n Show: There exists at least one $p \in S^n$ with $f_i(p) = f_i(-p)$ for all $i \in [n]$.

[Consider the map $x \mapsto (f_1(x), \ldots, f_n(x))$ of S^n into R^n.]

(B.5) Let $f_1, \ldots, f_n$ be n continuous real-valued functions on S^n such that $-f_i(p) = f_i(-p)$ for every $p \in S^n$. Show: The f_i have a common zero on S^n

[Consider the proof of (5.2)(3)⇒(4).]

(B.6) Let $f_1, \ldots, f_{n+2}$ be $n+2$ continuous real-valued functions on S^n and assume that for each $p \in S^n$ there is at least one f_i with $f_i(p) = 0 \neq f_i(-p)$. Show: The $f_1, \ldots, f_{n+2}$ have no common zero, but any $n+1$ of them do.

[Use (B.1) with $M_i = f_i^{-1}(0)$.]

(B.7) Let $f: S^n \to S^n$ be nullhomotopic. Show: f has a fixed point and sends some x to its antipode, i.e., $f(x_0) = -x_0$ for some x_0.

[If f has no fixed point, then $f \simeq -\mathrm{id}$. If no point maps to its antipode, then $f \simeq \mathrm{id}$, which is not nullhomotopic.]

C. Theorem of Brouwer and related results

(C.1) Let X be a compact subset of R^n with nonempty interior. Show: There is no retraction of X onto its boundary (Borsuk [1931]).

(C.2) (*Theorem on partitions*) Let J^n be the n-cube $\{(x_1, \ldots, x_n) \mid |x_i| \leq 1 \text{ for } i \in [n]\}$; the ith face $\{x \in J^n \mid x_i = 1\}$ is denoted by J_i^+, and the opposite face $\{x \in J^n \mid x_i = -1\}$ by J_i^-. For each $i \in [n]$ let A_i be a closed set separating J_i^+ and J_i^- (i.e., $J^n - A_i = U_i^+ \cup U_i^-$, where the U_i^+, U_i^- are disjoint open sets with $J_i^+ \subset U_i^+$ and $J_i^- \subset U_i^-$). Prove: $\bigcap_{i=1}^n A_i \neq \emptyset$ (Eilenberg-Otto [1938]; see also the book by Hurewicz-Wallman [1941]).

[For each i define

$$h_i(x) = \begin{cases} -d(x, A_i), & x \in U_i^+, \\ d(x, A_i), & x \in U_i^-, \end{cases}$$

and show that the map $x \mapsto x + (h_1(x), \ldots, h_n(x))$ maps J^n into itself.]

(C.3) (*Miranda theorem*) Let $f_1, \ldots, f_n$ be continuous real-valued functions on J^n such that for each $i \in [n]$,

$$f_i(x) \geq 0 \quad \text{for } x \in J_i^+, \qquad f_i(x) \leq 0 \quad \text{for } x \in J_i^-$$

Show: There exists $\widehat{x} \in J^n$ such that $f_i(\widehat{x}) = 0$ for each $i \in [n]$ (Miranda [1940]).

(C.4) Let $f: (K^{n+1}, S^n) \to (K^{n+1}, S^n)$ be a continuous map such that one of the following conditions holds: (a) $\mathrm{Fix}(f|S^n) = \emptyset$, (b) $f(x) = -x$ for each $x \in S^n$, (c) $f|S^n: S^n \to S^n$ is not nullhomotopic. Show: $f: K^{n+1} \to K^{n+1}$ is surjective (Kuratowski-Steinhaus).

(C.5) (*Theorem of Frum-Ketkov and Nussbaum*) Let E be a Banach space and D a closed ball about the origin in E.

(a) Let $f: D \to E$ be continuous with $f(\partial D) \subset D$. Show: If $P: E \to E$ is a finite-dimensional linear projection with $\|P\| = 1$, then $P \circ f: D \to E$ has a fixed point.

(b) Let K be a compact subset of E. By an *approximating sequence* for K is meant a sequence $\{P_n\}$ of finite-dimensional linear projections in E such that (i) $\|P_n\| = 1$ for all n; (ii) $P_n x \to x$ for each $x \in K$. Show: If $\{P_n\}$ is an approximating sequence for K, then for each $\varepsilon > 0$ there exists an integer n_ε such that $P_n(K) \subset B_\varepsilon(K)$ for all $n \geq n_\varepsilon$.

(c) Let $f: D \to E$ be continuous with $f(\partial D) \subset D$ and assume that there exists a compact subset K of E that admits an approximating sequence $\{P_n\}$ of projections and satisfies $d(f(x), K) \leq \alpha d(x, K)$ for some $\alpha < 1$ and all $x \in D$. Prove: f has a fixed point (Frum-Ketkov [1967], Nussbaum [1972a]).

[Establish successively the following assertions: for each n there is a fixed point x_n of $P_n \circ f$; given $\varepsilon > 0$, we have $(1 - \alpha) d(x_n, K) < \varepsilon$ for large n (use the definition of α and (b)), implying $d(x_n, K) \to 0$; there is a subsequence $\{x_{n_i}\}$ converging to some $x_0 \in K$. Then $f(x_0) = x_0$.]

D. Sperner's lemma and the Knaster Kuratowski-Mazurkiewicz theorem

Let $\Delta^n = [p_0, \ldots, p_n]$ be an n-simplex. By a *subdivision* $\mathscr{S}$ of Δ^n is meant a decomposition of Δ^n into finitely many nonoverlapping n-simplices $\sigma_1, \ldots, \sigma_k$ such that (1) the

intersection of any two simplices in $\mathscr{S}$ is either empty, or a common face of each, and (2) each $(n-1)$-simplex in $\mathscr{S}$ that is not on $\partial\Delta^n$ is the common face of exactly two n-simplices of $\mathscr{S}$. The *mesh* of $\mathscr{S}$ is $\max\{\operatorname{diam}\sigma_i^\nu \mid i \in [k]\}$; the *carrier* of a vertex $v \in \mathscr{S}$ is the lowest-dimensional face $[p_{i_0}, \dots, p_{i_s}]$ of Δ^n that contains v.

(D.1) Prove: Subdivisions of Δ^n having arbitrarily small mesh exist.

[Use repeated barycentric subdivision (cf. III, 8.2).]

(D.2) (*Sperner's lemma*) Let $\mathscr{S}$ be a subdivision of Δ^n A labeling of the vertices of $\mathscr{S}$ that assigns to each vertex $v \in \mathscr{S}$ one of the letters $\{p_{i_0}, \dots, p_{i_s}\}$ whenever $[p_{i_0}, \dots, p_{i_s}]$ is the carrier of v, is called a *Sperner labeling* of $\mathscr{S}$. Given a Sperner labeling of $\mathscr{S}$, an n-simplex $\sigma_i \in \mathscr{S}$ is called *complete* if its vertices are labeled $p_0, \dots, p_n$. Prove: In any Sperner labeling of $\mathscr{S}$, the number of complete simplices is odd (and therefore at least one will exist) (Sperner [1928]).

[The result being trivial for $n = 0$, proceed by induction, assuming that it is true for every Δ^{n-1} Given a Sperner labeling of a subdivision $\mathscr{S}$ of Δ^n, consider the set of $(n-1)$-simplices labeled $(p_0, \dots, p_{n-1})$. These arise from the α n-simplices of $\mathscr{S}$ labeled $(p_0, \dots, p_{n-1}, p_i)$, $p_i \neq p_n$, and the β complete simplices. Each of the α simplices has 2 faces labeled $(p_0, \dots, p_{n-1})$ and each complete n-simplex only 1, so counting the n-simplices of $\mathscr{S}$, we get a total of $2\alpha + \beta$ simplices labeled $(p_0, \dots, p_{n-1})$. This total is precisely that of the γ $(n-1)$-simplices $(p_0, \dots, p_{n-1})$ in the interior of Δ^n, each counted twice, and the δ such simplices on $\partial\Delta^n$, so $2\alpha + \beta = 2\gamma + \delta$. Now, the δ simplices necessarily belong to the face $(p_0, \dots, p_{n-1})$ of Δ^n, and are the complete simplices in a Sperner labeling of a subdivision of that face. By the induction hypothesis, δ is therefore odd, so β is odd. This completes the inductive step.]

(D.3) (*Knaster-Kuratowski–Mazurkiewicz theorem*) Let $\Delta^n = [p_0, \dots, p_n]$ be an n-simplex, and let $M_0, \dots, M_n$ be $n+1$ closed sets such that $[p_{i_0}, \dots, p_{i_s}] \subset M_{i_0} \cup \dots \cup M_{i_s}$ for each subset $\{i_0, \dots, i_s\} \subset \{0, \dots, n\}$. Prove: $\bigcap_{i=0}^n M_i \neq \emptyset$ (Knaster–Kuratowski- Mazurkiewicz [1929]).

[$\{M_0, \dots, M_n\}$ is a closed covering of Δ^n and $p_i \in M_i$ for each i. Let $\lambda > 0$ be a Lebesgue number for $\{M_i\}$, so that if $A \subset \Delta^n$ is any set with $\delta(A) < \lambda$, then $\bigcap\{M_i \mid M_i \cap A \neq \emptyset\} \neq \emptyset$. Let $\mathscr{S}$ be a subdivision of Δ^n with mesh $< \lambda$. If v is any vertex of $\mathscr{S}$, choose the carrier $[p_{i_0}, \dots, p_{i_s}]$ of v, and give v any one of the labels $\{p_{i_0}, \dots, p_{i_s}\}$ such that $v \in M_{i_r}$. Apply (D.2).]

(D.4) Let $\Delta^n = [p_0, \dots, p_n]$ be an n-simplex, and let $\{M_i \mid i = 0, \dots, n\}$ be a closed covering of Δ^n such that $[p_0, \dots, \widehat{p}_i, \dots, p_n] \cap M_i = \emptyset$ for each $i = 0, \dots, n$. Prove: $\bigcap_{i=0}^n M_i \neq \emptyset$.

[Let $X = \{p_0, \dots, p_n\} \subset \boldsymbol{R}^n$ and show that $p_i \mapsto M_i$ is a KKM-map.]

(D.5) Let $\Delta^n = [p_0, \dots, p_n]$ be an n-simplex, and let $\{M_i \mid i = 0, \dots, n\}$ be a closed covering such that $[p_0, \dots, \widehat{p}_i, \dots, p_n] \subset M_i$ for each $i = 0, \dots, n$. Prove: $\bigcap_{i=0}^n M_i \neq \emptyset$ (Alexandroff–Pasynkoff [1957]).

[For each $i = 0, \dots, n-1$, let $M_i' = M_{i+1}$ and $M_n' = M_0$. Apply (D.3) by showing that $p_i \mapsto M_i'$ is KKM.]

(D.6) Let $\Delta^n = [p_0, \dots, p_n]$ be an n-simplex. Show that there exists a $\lambda > 0$ with the property: If $\mathscr{H}$ is any finite closed covering of Δ^n and if each $H \in \mathscr{H}$ has diameter $< \lambda$, then there are at least $n+1$ sets in $\mathscr{H}$ that have a nonempty intersection (Lebesgue).

[Let $\Delta_i = [p_0, \dots, \widehat{p}_i, \dots, p_n]$; since $\bigcap_{i=0}^n \Delta_i = \emptyset$, they have a Lebesgue number $\lambda > 0$: any set in Δ^n of diameter $< \lambda$ does not meet at least one Δ_i. Assume that the sets of $\mathscr{H}$ have diameter $< \lambda$. Define Φ_0 to be all the sets in $\mathscr{H}$ that do not meet Δ_0, and

proceeding recursively, let Φ_k be the family of all sets in $\mathscr{H} - \bigcup_{i=1}^{k-1}\Phi_i$ that do not meet Δ_k, $k = 0, 1, \ldots, n$. Then $\bigcup_{i=0}^{n}\Phi_i = \mathscr{H}$, each $H \in \mathscr{H}$ belongs to at least one Φ_i, and the $n+1$ closed sets $M_i = \bigcup\{H \mid H \in \Phi_i\}$ satisfy the conditions in (D.4).]

(D.7) Prove: $\partial\Delta^n$ is not a retract of Δ^n

[If $r : \Delta^n \to \partial\Delta^n$ were a retraction, consider the sets $M_i = r^{-1}[p_0, \ldots, \widehat{p}_i, \ldots, p_n]$ and apply (D.5).]

(D.8) Let $\Delta^n = [p_0, \ldots, p_n]$ be an n-simplex, and let $U_0, U_1, \ldots, U_n$ be $n+1$ open sets in Δ^n such that $[p_{i_0}, \ldots, p_{i_s}] \subset U_{i_0} \cup \cdots \cup U_{i_s}$ for each subset $\{i_0, \ldots, i_s\} \subset \{0, \ldots, n\}$. Prove: $\bigcap_{i=0}^{n} U_i \neq \emptyset$ (Lassonde [1990]).

E. Universal maps

(E.1) Let X, Y be topological spaces. A map $f : X \to Y$ is called *universal* if for any $g : X \to Y$ there exists $x \in X$ such that $f(x) = g(x)$. Recall that a space X is called a *fixed point space* if every continuous $f : X \to X$ has a fixed point. Show:

(a) If $f : X \to Y$ is universal, then $f(X) = Y$
(b) If $f : X \to Y$ is universal, then Y is a fixed point space.
(c) X is a fixed point space if and only if id_X is universal.
(d) If gf is universal then so is g.
(e) Let $A \subset X$ and $f : X \to Y$. If $f|A : A \to Y$ is universal then so is f.
(f) Any continuous map $f : X \to [0,1]$ of a connected space onto $[0,1]$ is universal.
(g) Let Y be a connected linearly ordered space with the interval topology, with the minimal and maximal elements. Then any continuous map $f : X \to Y$ of a connected space X onto Y is universal.

(E.2) Let X, Y be metric spaces and $\varepsilon > 0$. A continuous $f : X \to Y$ is called an *ε-map* if $\delta(f^{-1}(y)) \leq \varepsilon$ for all $y \in Y$. Let X, Y be compact metric spaces and $f : X \to Y$ be a map. Assume that for each $\varepsilon > 0$ there exist a space Z_ε and an ε-map $f_\varepsilon : Y \to Z_\varepsilon$ such that $f_\varepsilon \circ f : X \to Z_\varepsilon$ is universal. Show: f is universal (Holsztyński [1969]).

(E.3) Let $f : K^{n+1} \to K^{n+1}$ be a map such that $f(S^n) \subset S^n$ and $f|S^n : S^n \to S^n$ is not homotopic to a constant. Show: f is universal (Holsztyński [1969]).

(E.4) Let $f : X \to K^{n+1}$ be a map of a normal space into the $(n+1)$-ball and $A = f^{-1}(S^n)$. Prove: f is universal if and only if the map $f|A : A \to S^n$ is not extendable over X (Lokutsievskiĭ [1957], Holsztyński [1967]).

F. Fixed point spaces

Given subsets A, B in a metric space (X, d) and $\varepsilon > 0$, an *ε-displacement* of A into B is any continuous map $f_\varepsilon : A \to B$ such that $d(a, f_\varepsilon(a)) \leq \varepsilon$ for all $a \in A$.

(F.1) Let (X, d) be a compact metric space and assume that for each $\varepsilon > 0$, there is an ε-displacement $p_\varepsilon : X \to X$ such that $p_\varepsilon(X)$ is a fixed point space. Show: X is a fixed point space (Borsuk [1932]).

(F.2) Let X be a compact metric space and assume that for each $\varepsilon > 0$ there is an ε-map $f : X \to K^{n+1}$ onto some $(n+1)$-ball such that $f|A : A \to S^n$, where $A = f^{-1}(S^n)$, is not extendable over X. Prove: X is a fixed point space (Lokutsievskiĭ [1957]).

(F.3) Show: The Hilbert cube I^∞ is a fixed point space.

[Use (F.1) and the Brouwer fixed point theorem.]

(F.4) Let (X, d) be a metric space and $\mathcal{K}(X)$ the set of all nonempty compact subsets of X. For $A \in \mathcal{K}(X)$ and $\varepsilon > 0$ we denote by $U_\varepsilon(A)$ the ε-neighborhood of the set A in X. We define the *Borsuk metric* D_{B} in $\mathcal{K}(X)$ by letting, for $A, B \in \mathcal{K}(X)$,

$$D_{\mathrm{B}}(A, B) = \inf_\varepsilon\{\text{there are } \varepsilon\text{-displacements } f_\varepsilon : A \to U_\varepsilon(B) \text{ and } g_\varepsilon : B \to U_\varepsilon(A)\}.$$

Prove: If $\{A_n\}$ is a sequence of fixed point spaces in $\mathcal{K}(X)$ and $D_{\mathrm{B}}(A_n, A) \to 0$, then A is also a fixed point space.

(F.5) Prove: The unit ball

$$K^\infty = \left\{x = \{x_i\} \in l^2 \,\middle|\, \|x\|^2 = \sum_{i=1}^{\infty} x_i^2 \le 1\right\}$$

in the Hilbert space l^2 is not a fixed point space (Kakutani [1943]).

[Show that $\varphi : K^\infty \to K^\infty$ given by $(x_1, x_2, \dots) \mapsto (\sqrt{1 - \|x\|^2}, x_1, x_2, \dots)$ is a continuous map without fixed points.]

(F.6) Let X be a normal space and L a closed subset of X homeomorphic to the half-line $[1, \infty)$. Show: X is not a fixed point space (Klee [1955]).

[Prove that L is a retract of X using the Tietze–Urysohn theorem.]

(F.7) Let E be an infinite-dimensional normed linear space and K its closed unit ball. Prove: K is not a fixed point space (Dugundji [1955]).

[Define by induction a sequence $\{e_n\}$ in ∂K such that $\operatorname{dist}(e_{n+1}, \operatorname{span}(e_1, \dots, e_n)) = 1$. Prove that $L = [e_1, e_2] \cup [e_2, e_3] \cup \cdots$ is closed in K and that L is homeomorphic to $[1, \infty)$; then apply (F.6). The argument in this hint was suggested by V. Klee and independently by C. Bowszyc.]

G. Vector fields

Let $A \subset \boldsymbol{R}^{n+1}$, and let $F : A \to \boldsymbol{R}^{n+1}$ be a map; from F we obtain a *vector field* $f : A \to \boldsymbol{R}^{n+1}$ by $f(x) = x - F(x)$. A zero of f is called also a *singularity* of f; clearly, the singularities of f are precisely the fixed points of F.

(G.1) Let $f, g \; S^n \to \boldsymbol{R}^{n+1} - \{0\}$ be two singularity free vector fields that are never opposite. Show: $f \simeq g$.

(G.2) Let $f : (K^{n+1}, \partial K^{n+1}) \to (\boldsymbol{R}^{n+1}, \boldsymbol{R}^{n+1} - \{0\})$ be a vector field on K^{n+1} without singular points on ∂K^{n+1} We say that f is *essential* if for any given $g : (K^{n+1}, \partial K^{n+1}) \to (\boldsymbol{R}^{n+1}, \boldsymbol{R}^{n+1} - \{0\})$ satisfying $g|\partial K^{n+1} = f|\partial K^{n+1}$ there is a singular point for g; f is said to be *inessential* if $f|S^n$ can be extended without singularities over K^{n+1} Show: A vector field $f \; (K^{n+1}, \partial K^{n+1}) \to (\boldsymbol{R}^{n+1}, \boldsymbol{R}^{n+1} - \{0\})$ is essential if and only if the map $\varphi : S^n \to S^n$ defined by $x \mapsto f(x)/\|f(x)\|$ is not nullhomotopic.

(G.3) Let $f, g : (K^{n+1}, \partial K^{n+1}) \to (\boldsymbol{R}^{n+1}, \boldsymbol{R}^{n+1} - \{0\})$ be two homotopic vector fields. Prove: If f is essential, then so is g.

(G.4) Let $f : (K^{n+1}, \partial K^{n+1}) \to (\boldsymbol{R}^{n+1}, \boldsymbol{R}^{n+1} - \{0\})$ be an essential vector field. Show: The vector field $f|\partial K^{n+1} : S^n \to \boldsymbol{R}^{n+1} - \{0\}$ points in every direction.

(G.5) Let $f : (K^{n+1}, \partial K^{n+1}) \to (\boldsymbol{R}^{n+1}, \boldsymbol{R}^{n+1} - \{0\})$ be a vector field such that the vectors at antipodal points never have the same direction. Show: f is essential.

(G.6) Let $f : K^{n+1} \to R^{n+1} - \{0\}$ be a nonsingular vector field on K^{n+1}. Show: There is a pair of antipodal points on $S^n = \partial K^{n+1}$ at which the vectors have the same direction.

(G.7) Show: There is no singularity free vector field $f : K^{n+1} \to R^{n+1} - \{0\}$ on K^{n+1} such that f is everywhere outward normal or everywhere inward normal on the boundary.

[If f $S^n \to R^{n+1} - \{0\}$ has no inward normal, then it can be deformed to a field that has outward normal everywhere.]

H. Topological theory of KKM-maps

In this subsection, C stands for a nonempty convex set in R^s and X for a subset of C; by $\langle X\rangle$ we denote the set of finite subsets of X, and for $A \in \langle X\rangle$ we let $[A] = \operatorname{conv} A$. We recall that $F : X \to 2^C$ is a KKM-map if $[A] \subset \bigcup\{Fx \mid x \in A\}$ for each $A \in \langle X\rangle$. We say that a map $T : X \to 2^C$ has the *matching property* if the condition

$$[A] \subset \bigcup\{Tx \mid x \in A\} \quad \text{for some } A \in \langle X\rangle$$

implies that

$$\bigcap\{Tx \cap [B] \mid x \in B\} \neq \emptyset \quad \text{for some } B \in \langle X\rangle.$$

(H.1) (*"Closed" and "open" versions of the KKM-property*) Prove: If $F : X \to 2^C$ is a closed-valued (or an open-valued) KKM-map, then

$$\bigcap\{Fx \cap [A] \mid x \in A\} \neq \emptyset \quad \text{for each } A \in \langle X\rangle.$$

(H.2) (*Matching theorems*) Prove: If $T \cdot X \to 2^C$ is an open-valued (or a closed-valued) KKM-map, then T has the matching property.

[Use the map $x \mapsto Fx = C - Tx$ and (H.1).]

(H.3) (*Fixed point theorems*) Let $T : C \to 2^C$ be a convex-valued map with open fibers (or with closed fibers) such that for some $A \in \langle C\rangle$, we have $Tx \cap A \neq \emptyset$ for all $x \in [A]$. Show: $\operatorname{Fix}(T) \neq \emptyset$.

[Use (H.2) and the following observation: if $T : C \to 2^C$ is convex-valued and $R = T^{-1}$, then, given any $A \in \langle C\rangle$, the following properties are equivalent: (i) $Tx \cap A \neq \emptyset$ for each $x \in [A]$, (ii) $[A] \subset \bigcup\{Rx \mid x \in A\}$.]

(The above results, except the "closed" version of (H.1), are due to Lassonde [1990].)

10. Notes and Comments

The antipodal theorem of Borsuk

This is one of the central results in fixed point theory. It was established by Borsuk [1933a] together with (4.4); the Borsuk- Ulam theorem was conjectured by Ulam and proved by Borsuk. Theorem (4.4) was discovered earlier by Lusternik Schnirelmann [1930] in their work on topological methods in analysis.

The combinatorial proof of (5.2) presented in the text is an adaptation of that in Granas's tract [1962]. The first proof of this type was given by Tucker [1945] for $n = 3$; Fan [1952a] extended Tucker's result to arbitrary n

and established some generalizations of the Borsuk–Ulam and Lusternik–Schnirelmann–Borsuk theorems. A combinatorial proof of the antipodal theorem was also found by Krasnosel'skiĭ–Krein [1949]. The first proof of the antipodal theorem using the analytical definition of the degree was given in the lecture notes by J.T. Schwartz [1969]. For yet another analytical proof see the lecture notes by Nirenberg [1973]. An elementary proof based on degree theory may also be found in Dugundji's book [1965]. A noteworthy algebraic proof of the Borsuk Ulam theorem is given by Arason–Pfister [1982].

K. Borsuk and P. Alexandroff, Radachówka, 1962

The Lusternik-Schnirelmann category

Let X be a topological space. A set $A \subset X$ has *Lusternik–Schnirelmann category* $\leq n$, written $\operatorname{Cat} A \leq n$, if A is the union of n closed sets each deformable to a point in X. It is easy to establish that:

(a) $\operatorname{Cat} B \leq \operatorname{Cat} A$ if $B \subset A$.
(b) $\operatorname{Cat}(A \cup B) \leq \operatorname{Cat} A + \operatorname{Cat} B$.
(c) $\operatorname{Cat} A \leq \dim A - 1$.
(d) If $f : A \to X$ is homotopic to the inclusion $i : A \to X$, then $\operatorname{Cat} A \leq \operatorname{Cat}(f(A))$.

The following theorem of Lusternik Schnirelmann [1930] is equivalent to Theorem (4.4), and hence to Borsuk's antipodal theorem: *If P^n is the n-dimensional real projective space, then* $\mathrm{Cat}(P^n) = n+1$.

This notion of category plays a basic role in the critical point theory developed by Lusternik Schnirelmann. Let f be a smooth real-valued function on a smooth manifold M. A point $p \in M$ is called a *critical point* of f provided there is a local coordinate system $(x_1, \ldots, x_n)$ in a nbd of p with $\partial f(p)/\partial x_i = 0$ for all $i = 1, \ldots, n$. The number of critical points is governed by the following fundamental result of Lusternik Schnirelmann: *If M is compact, then f has at least* $\mathrm{Cat}\, M$ *critical points on M.*

The following is a simple application. Let $f(x) = f(x_1, \ldots, x_n)$ be a smooth function defined on a nbd of the unit sphere S^{n-1} in R^n The critical points of f are determined by the equations

$$(*) \qquad d\Big[f(x) - \lambda \sum_{i=1}^{n} x_i^2\Big] = 0, \quad \sum_{i=1}^{n} x_i^2 = 1$$

(if f is a quadratic form, then the corresponding critical points coincide with eigenvectors of f, and to Lagrange multipliers correspond eigenvalues of f). Assume that the function f is even, i.e., $f(x) = f(-x)$ for all $x \in S^{n-1}$ Since by identifying x with $-x$ for all $x \in S^{n-1}$ we obtain the projective space P^{n-1}, it follows from the above theorem of Lusternik-Schnirelmann that $f : P^{n-1} \to R$ has at least $\mathrm{Cat}(P^{n-1}) = n$ different critical points and hence the equation $(*)$ has at least n different pairs of solutions.

Results related to the Borsuk–Ulam theorem

The Borsuk Ulam theorem suggested deriving more precise results for maps $f : S^n \to R$. In 1942 Kakutani ($n = 2$) and in 1950 Yamabe–Yujobô (for general n) showed that there exist $n+1$ points $\{x_i\}$ satisfying $(x_i, x_j) = 0$ for $i \neq j$ and such that $f(x_1) = \cdots = f(x_{n+1})$. This has a consequence that *any compact convex $K \subset R^{n+1}$ has a circumscribing $(n+1)$-cube C* (i.e., every face of C meets K): for each direction $\alpha \in S^n$ let $f(\alpha)$ be the distance between two parallel planes perpendicular to α that contain K between them, each of the planes meeting K. In 1950 Dyson ($n = 2$) and in 1954 Yang (arbitrary n) showed that any $f : S^n \to R$ maps the $2n$ endpoints of some n mutually orthogonal diameters to a single point. For more details see Yang [1954].

Theorem of Brouwer

This is one of the oldest and best known results in topology. It was proved for $n = 3$ by Brouwer [1909]; for differentiable maps an equivalent result was established earlier by Bohl [1904]. Hadamard [1910] (using the Kronecker

index) gave an analytic proof for arbitrary n; somewhat earlier, Brouwer gave a proof using the simplicial approximation technique and the notion of degree; that proof appeared in Brouwer [1912] (cf. comment on p. 277). Other proofs depending on various definitions of degree were also given by J.W. Alexander [1922] and Birkhoff–Kellogg [1922].

A simple combinatorial proof of the Brouwer theorem (based on Sperner's lemma [1928]) was given by Knaster Kuratowski–Mazurkiewicz [1929]; they noted also that for a map $f : \boldsymbol{K}^n \to \boldsymbol{R}^n$ the condition $f(\partial \boldsymbol{K}^n) \subset \boldsymbol{K}^n$ suffices for the existence of a fixed point. For a noteworthy analytical proof of the Brouwer theorem see Lax [1999]; for another proof due to Milnor [1978] see "Miscellaneous Results and Examples".

Equivalent formulations

The fact that the Brouwer fixed point theorem admits an equivalent formulation in terms of homotopy and another one in terms of retraction was observed by Borsuk [1931a], [1931b]; these simple but significant observations provided the justification for the study of nonextendability problems and were the starting point of many further important developments. The fact that there is no retraction $r : \boldsymbol{K}^{n+1} \to \boldsymbol{S}^n$ (besides being equivalent to Brouwer's theorem) provides a key to a number of other results in topology; it can be used, for example, to prove the domain invariance in $\boldsymbol{R}^n$ and also the "tiling" covering theorem of Lebesgue (see the book of Hurewicz–Wallman [1941]). Among the proofs of this nonretraction result we mention an analytic proof by Milnor (cf. Milnor's book [1965]) based on the approach of M. Hirsch [1963], the inductive proof by Sieklucki [1983], and that of Alexandroff–Pasynkoff [1957] based on the Knaster–Kuratowski–Mazurkiewicz theorem. For a discussion of the not entirely correct approach of Hirsch [1963] in the simplicial context see Joshi [2000].

Computing fixed points

Brouwer's theorem ensures that each self-map $f : \sigma \to \sigma$ of a simplex has at least one fixed point. Until the late sixties computer methods to find a fixed point of a given such map were severely limited: the techniques used were all based on iterative procedures that required additional restrictions on the map in order to guarantee convergence. Scarf [1967] developed a finite algorithm (based essentially on Sperner's lemma) for approximating a fixed point for any continuous $f : \sigma \to \sigma$; to improve accuracy, the early programs included a Newton method subroutine, depending on the function considered. Scarf's paper initiated considerable activity in computation of fixed points. By using homotopy techniques, Eaves [1972] gave an algorithm with improved accuracy over that of Scarf and avoiding the Newton method:

working on $\sigma \times I$ with the given function on $\sigma \times \{0\}$, a known function with a unique fixed point on $\sigma \times \{1\}$, and a suitable homotopy joining them, this approach relies on the fact that connected 1-manifolds (the fibers of the homotopy) are homeomorphic either to the circle or to the unit interval. An introduction to this currently active research area and to some of its immediate applications may be found in Scarf–Hansen [1973] and the book of Allgower Georg [1990].

Brouwer's theorem in the infinite-dimensional case

The fact that the Brouwer theorem does not hold for arbitrary continuous maps of the unit ball in infinite-dimensional Banach spaces was observed by several authors.

In 1935 (answering a question of Ulam), Tychonoff showed that the unit sphere in l^2 is a retract of the unit ball. Leray [1935] observed that the unit sphere in $C[0,1]$ is contractible. Kakutani [1943] gave an example of a fixed point free homeomorphism of the unit ball in l^2 into itself. Dugundji [1951] proved the following theorem: *A closed unit ball in a normed linear space is a fixed point space if and only if it is compact.* Klee [1955] generalized this result to arbitrary convex sets in metrizable locally convex spaces.

In their study of fixed points of Lipschitz maps, Lin Sternfeld [1985] established the following result: *A convex set in a Banach space, having the fixed point property for Lipschitz maps, must be compact.* From this one can deduce the following earlier result of Nowak [1979]: *In every infinite-dimensional Banach space there exists a Lipschitz retraction of the unit ball on its boundary.* For more details the reader is referred to Bessaga [1994] and to the book of Goebel-Kirk [1991].

Universal maps

A map $f : X \to Y$ is called *universal* if for each $g : X \to Y$ there is $x \in X$ such that $f(x) = g(x)$ (clearly X is a fixed point space if and only if the identity map 1_X is universal). This notion is due to Holsztyński [1964], who obtained the following generalizations of the theorems of Hurewicz and Brouwer:

(i) *Let X be a normal space and $J^n_k = \{(x_1, \dots, x_n) \in J^n \mid x_{|k|} = \operatorname{sgn} k\}$ for $k = \pm 1, \pm 2, \dots, \pm n$, where $J = [-1, +1]$. Let $f : X \to J^n$ be a map and $A_1, \dots, A_n$ be a sequence of closed sets in X such that A_k partitions X between $f^{-1}(J^n_{-k})$ and $f^{-1}(J^n_k)$ for $k = 1, \dots, n$. Then f is universal if and only if $\bigcap_{i=1}^n A_i \neq \emptyset$.*

(ii) *If X is a normal space, then the covering dimension* $\dim X \geq n$ *if and only if there is a universal map $f : X \to J^n$.*

Further results (and some applications to dimension theory) can be found in Holsztyński [1967], [1969].

Fixed point spaces

A space X has the fixed point property (or is a fixed point space) if every continuous $f : X \to X$ has a fixed point. Clearly, this property is topologically invariant. Borsuk [1931a] observed that if X is a fixed point space, so also is every retract of X. For Cartesian products of fixed point spaces, the result depends on the number of factors. The product of two compact fixed point spaces need not be a fixed point space. We mention the following examples:

(i) there is a finite polyhedron P that is a fixed point space while $P \times [0, 1]$ is not a fixed point space (see an example of Lopez [1967] given below and Bredon [1971]);

(ii) there is a finite polyhedron P with the fixed point property such that the suspension of P is not a fixed point space (Holsztyński [1970]);

(iii) there are two manifolds X, Y that are fixed point spaces while $X \times Y$ is not a fixed point space (Husseini [1977]).

In contrast with finite products, an infinite product of compact nonempty fixed point spaces is a fixed point space whenever every finite product of those spaces is a fixed point space (Dyer [1956]). Thus by the Brouwer theorem, the Hilbert cube I^∞ and in fact any Tychonoff cube are fixed point spaces.

Several important results are summarized in the following list of fixed point spaces:

(i) projective spaces of even dimension (J.W. Alexander [1922]),

(ii) compact convex sets in L^2 and in $C^n[0, 1]$ (Birkhoff–Kellogg [1922]),

(iii) compact convex sets in Banach spaces (Schauder [1927a], [1927b], [1930]),

(iv) weakly compact convex sets in separable Banach spaces (with respect to weakly continuous maps) (Schauder [1927a]),

(v) compact absolute retracts (Borsuk [1931a]),

(vi) compact convex sets in locally convex linear topological spaces (Tychonoff [1935]).

Two compact metric spaces X and Y are *quasi-homeomorphic* if for each $\varepsilon > 0$ there is an ε-map f $X \to Y$ and an ε-map $g : Y \to X$. The question of whether the fixed point property is invariant under quasi-homeomorphisms was treated by Borsuk [1938]: it is not invariant for arbitrary continua but is invariant if X and Y are compact ANRs.

Some examples

We give some noteworthy examples of continua lacking the fixed point property.

(a) (Knill [1967]) Let $S = \{x \in \boldsymbol{R}^2 \mid \|x\| = 1\}$ be the unit circle and A be a closed half-line spiraling to S. Let $X = A \cup S$ and regard X as a subset of $\boldsymbol{R}^2 \times \{0\} \subset \boldsymbol{R}^3$. Let $Z = CX \subset \boldsymbol{R}^3$ be the cone over X with vertex $p = (0, 0, -1)$ as shown in Figure 1. Clearly, Z is contractible (to p); Knill showed that Z is not a fixed point space but is a cell-like continuum (i.e., it is the intersection of a decreasing sequence of closed 3-cells in $\boldsymbol{R}^3$).

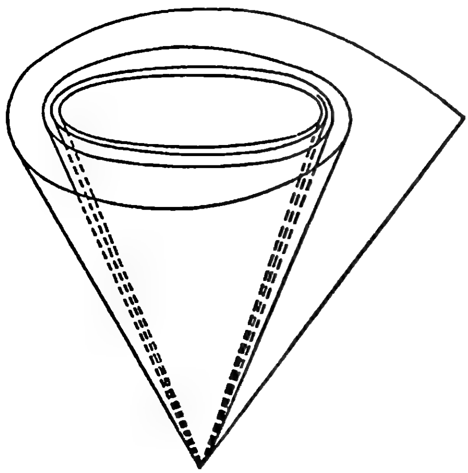


Figure 1

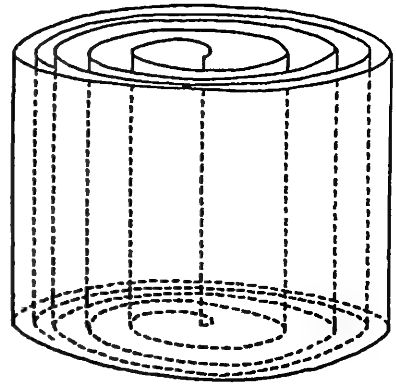


Figure 2

(b) (Kinoshita [1953]) Let X be the "can-with-a-roll-of-toilet-paper" shown in Figure 2. Clearly, X is a contractible continuum; Kinoshita showed that neither X nor the cone CX over X has the fixed point property. In connection with the above two examples see Sieklucki [1985].

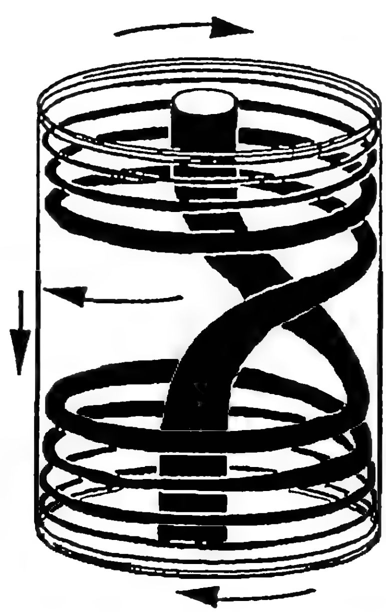


Figure 3

(c) (Borsuk [1935]) Let X be the continuum in $\boldsymbol{R}^3$ shown in Figure 3. It consists of a solid cylinder with two tunnels carved out; the width of each tunnel tends to zero as the tunnel approaches the limiting circle. Borsuk showed that X is a cell-like continuum that admits a fixed point free homeomorphism. Such a homeomorphism can be described as follows: the top and the bottom of the cylinder rotate about the axis of the cylinder, and each point lying between the top and the bottom is moved down below its original position. Borsuk's example can be used to construct a fixed point free flow with bounded orbits in $\boldsymbol{R}^3$ (Kuperberg–Reed [1981]). For related results, the reader is referred to Kuperberg et al. [1993]. We remark that there exists a 3-dimensional continuum in $\boldsymbol{R}^4$ that has the properties of the Borsuk example and in addition is simply connected (Verchenko [1940]).

(d) Let X be a plane continuum not separating the plane. It can be shown that X is the intersection of a decreasing sequence of closed 2-cells in $\boldsymbol{R}^2$. Since the twenties, the following problem has remained unsolved: does X have the fixed point property? The following related partial result was established by Cartwright–Littlewood [1951]: *Let h be an orientation-preserving homeomorphism of $\boldsymbol{R}^2$ onto itself, and let $X = h(X)$ be a continuum not separating the plane. Then $h|X$ has a fixed point.*

(e) (Lopez [1967]) We now describe a polyhedron Z with the fixed point property such that $Z \times [0,1]$ lacks this property. Let $P^n(\boldsymbol{C})$ denote the complex n-dimensional projective space (of real dimension $2n$). It is obtained from $\boldsymbol{C}^{n+1} - \{0\}$ by identifying all the points on each complex line through 0 to any particular point. Specific embeddings $P^n(\boldsymbol{C}) \subset P^{n+1}(\boldsymbol{C})$ can be obtained from the inclusions $\boldsymbol{C}^{n+1} \times \{0\} \subset \boldsymbol{C}^{n+1} \times \boldsymbol{C}$. The space $P^1(\boldsymbol{C})$ is readily seen to be homeomorphic to the 2-sphere S^2

Let $(a,b) \in S^2 \times S^2$ and form a quotient space Z of the disjoint union $P^2(\boldsymbol{C}) + S^2 \times S^2 + P^4(\boldsymbol{C}) + \Sigma P^8(\boldsymbol{C})$ by identifying

$$S^2 = P^1(\boldsymbol{C}) \subset P^2(\boldsymbol{C}) \quad \text{with} \quad S^2 \times \{b\} \text{ in } S^2 \times S^2,$$
$$S^2 = P^1(\boldsymbol{C}) \subset P^4(\boldsymbol{C}) \quad \text{with} \quad \{a\} \times S^2 \text{ in } S^2 \times S^2,$$

and (a,b) with any point of $\Sigma P^8(\boldsymbol{C})$, the suspension of $P^8(\boldsymbol{C})$.

The space Z is triangulable; Lopez showed that Z has the fixed point property but $Z \times [0,1]$ does not. This is another example showing that the fixed point property is not an invariant of homotopy type. Although not so easily visualized as example (a), it has the advantage of having no local pathology.

Further, more special examples of fixed point spaces and additional references can be found in the surveys by Bing [1969], Fadell [1970], and Brown [1974].

§6. Fixed Points for Compact Maps in Normed Linear Spaces

In this paragraph we obtain the basic fixed point theorems for compact operators in normed linear spaces. The approach is based on two general techniques: (1) The approximation of compact maps by finite-dimensional ones, which leads to the analogs of the Brouwer and Borsuk theorems for compact maps; (2) Topological transversality, which, applied to compact maps, leads to the Leray–Schauder principle, and when applied to compact fields, gives domain invariance. One feature of this approach is that the results are established for compact maps having ranges restricted to an arbitrary preassigned convex set.

1. Compact and Completely Continuous Operators

(1.1) DEFINITION. Let X, Y be topological spaces.

(i) A map $F : X \to Y$ is called *compact* if $F(X)$ is contained in a compact subset of Y.

(ii) If X is a metric space, $F : X \to Y$ is called *completely continuous* if the image of each bounded set in X is contained in a compact subset of Y

(iii) If Y is a linear space, a compact map $F : X \to Y$ is called *finite-dimensional* if $F(X)$ is contained in a finite-dimensional linear subspace of Y.

(iv) If Y is a metric space, $F : X \to Y$ is called *bounded* if $F(X)$ is a bounded subset of Y

The set of all compact maps from X to Y is denoted by $\mathscr{K}(X, Y)$. If (Y, ϱ) is a metric space, $\mathscr{K}(X, Y)$ is contained in the metric space $(\mathscr{B}(X, Y), d)$ of all bounded maps from X to Y with metric $d(F, G) = \sup_{x \in X} \varrho(F(x), G(x))$. If $(E, \| \ \|)$ is a normed linear space, $\mathscr{K}(X, E)$ is a linear subspace of the normed linear space $(\mathscr{B}(X, E), | \ |)$ with norm $|F| = \sup_{x\in X} \|F(x)\|$. For a complete Y the space $(\mathscr{B}(X, Y), d)$ is also complete; if E is complete, then $\mathscr{B}(X, E)$ is a Banach space.

Slightly less evident is

(1.2) PROPOSITION. *If Y is complete, then so is the space $\mathscr{K}(X, Y)$. In particular, if $Y = E$ is a Banach space, then $\mathscr{K}(X, E)$ is also a Banach space.*

PROOF Since $(\mathscr{B}(X, Y), d)$ is complete, it is sufficient to show that $\mathscr{K}(X,Y)$ is closed in $\mathscr{B}(X, Y)$. Let $\{F_n\}$ be a sequence in $\mathscr{K}(X, Y)$ with $d(F_n, F) \to 0$ for some $F \in \mathscr{B}(X, Y)$; we show that $F \in \mathscr{K}(X, Y)$.

According to Hausdorff's theorem (see Appendix) it is sufficient to establish that $F(X)$ is totally bounded. Let $\varepsilon > 0$ be given; by assumption there is an integer k such that $\sup_{x\in X} \varrho(F(x), F_k(x)) \le \varepsilon/2$. Let $N = \{y_1, \dots, y_n\}$ be an $\varepsilon/2$-net for the totally bounded set $F_k(X)$. Given a point $x \in X$ we have, for some $y_i \in N$,

$$\varrho(F(x), y_i) \le \varrho(F(x), F_k(x)) + \varrho(F_k(x), y_i) \le \varepsilon,$$

which shows that N is an ε-net for the set $F(X)$. □

We now give some examples of compact and completely continuous operators.

EXAMPLE 1. Let E be a Banach space, $U \subset E$ open, and $F : U \to E$ a compact map that has a derivative $G = F'(x_0)$ at a point $x_0 \in U$. We prove that G is a completely continuous linear map from E to E. For suppose not; then for some $\varepsilon > 0$ there is a sequence $\{h_n\}$ in E with $\|h_n\| \le 1$ such that

$$\text{(i)} \qquad \|G(h_n - h_m)\| \ge \varepsilon$$

for all $n, m = 1, 2, \dots$ $(n \ne m)$. By assumption

$$\text{(ii)} \qquad F(x_0 + h) - F(x_0) = G(h) + \omega(h)$$

in a neighborhood of x_0, where

$$\text{(iii)} \qquad \lim_{h\to 0} \frac{\|\omega(h)\|}{\|h\|} = 0.$$

By (iii), there is a $\delta > 0$ such that $x_0 + \delta h_n \in U$ for all $n = 1, 2, \dots$ and

$$\text{(iv)} \qquad \|h\| < \delta \Rightarrow \|\omega(h)\| \le \frac{\varepsilon}{4}\|h\|.$$

By (i), (ii), (iv), we have, for all n, m with $n \ne m$,

$$\begin{aligned}
&\|F(x_0 + \delta h_n) - F(x_0 + \delta h_m)\| \\
&\qquad = \|[F(x_0 + \delta h_n) - F(x_0)] - [F(x_0 + \delta h_m) - F(x_0)]\| \\
&\qquad \ge \delta\|G(h_n - h_m)\| - \|\omega(\delta h_n) - \omega(\delta h_m)\| \\
&\qquad \ge \delta\varepsilon - \delta\varepsilon/2 = \delta\varepsilon/2.
\end{aligned}$$

But this is a contradiction, because F is compact.

EXAMPLE 2 (Urysohn integral operators). Let Ω be a bounded domain in $\boldsymbol{R}^n$, and let $K : \overline{\Omega} \times \overline{\Omega} \times \boldsymbol{R} \to \boldsymbol{R}$ be continuous. Putting

$$\text{(1)} \qquad [Fu](x) = \int_{\Omega} K(x, y, u(y))\, dy \quad (x \in \overline{\Omega})$$

for each $u \in C(\overline{\Omega})$ we obtain a nonlinear map $F : C(\overline{\Omega}) \to C(\overline{\Omega})$ known as the *Urysohn integral operator*. We prove that F is completely continuous:

by Arzelà's theorem it suffices to show that given $\{u_n\}$ with $\|u_n\| \le r$, the sequence $v_n = F(u_n)$ is bounded and equicontinuous.

(a) $\{v_n\}$ is bounded: for $M = \sup\{|K(x,y,u)| \mid x,y \in \overline{\Omega},\ u \in [-r,r]\} < \infty$, we have

$$\|v_n\| = \sup_{x\in\overline{\Omega}} |v_n(x)| \le \sup_{x\in\overline{\Omega}} \int_\Omega |K(x,y,u_n(y))|\,dy \le M\mu(\Omega).$$

(b) $\{v_n\}$ is equicontinuous: let $\varepsilon > 0$ be given; because $K : \overline{\Omega} \times \overline{\Omega} \times [-r,r] \to \boldsymbol{R}$ is uniformly continuous there is a $\delta > 0$ such that

$$[\|x_1 - x_2\| < \delta] \Rightarrow |K(x_1,y,u) - K(x_2,y,u)| < \varepsilon$$

for all $y \in \overline{\Omega}, u \in [-r,r]$, and consequently,

$$|v_n(x_1) - v_n(x_2)| \le \int_\Omega |K(x_1,y,u_n(y)) - K(x_2,y,u_n(y))|\,dy < \varepsilon\mu(\Omega)$$

for all n, whenever $\|x_1 - x_2\| < \delta$. Thus $\{v_n\}$ is equicontinuous.

Important special cases are the *Hammerstein operators* of the form

$$[Fu](x) = \int_\Omega K(x,y) f(y,u(y))\,dy \tag{2}$$

for suitable $f : \overline{\Omega} \times \boldsymbol{R} \to \boldsymbol{R}$. One general procedure for studying (2) is based on the factorization

$$\begin{tikzcd} C(\overline{\Omega}) \arrow[rr, "F"] \arrow[dr, "\hat{f}"'] & & C(\overline{\Omega}) \\ & C(\overline{\Omega}) \arrow[ur, "K"'] & \end{tikzcd}$$

where the nonlinear map $\hat{f} : C(\overline{\Omega}) \to C(\overline{\Omega})$ (called the *Niemytzki operator*) is given by

$$[\hat{f}u](y) = f(y,u(y)), \quad y \in \overline{\Omega},$$

and K is a linear integral operator

$$[Ku](x) = \int_\Omega K(x,y)u(y)\,dy.$$

Many variations of these examples are possible by considering nonlinear integral operators in spaces other than $C(\overline{\Omega})$. For instance, to prove complete continuity of the Hammerstein operator $F : L^p(\Omega) \to L^p(\Omega)$ one may factorize F through L^q $(1/p + 1/q = 1)$ and impose conditions that guarantee the complete continuity of the linear operator $K : L^q(\Omega) \to L^p(\Omega)$ and boundedness on bounded sets of the Niemytzki operator $\hat{f} : L^p(\Omega) \to L^q(\Omega)$.

EXAMPLE 3 (Carathéodory operators). Denote by $(C^k[a,b], \|\ \|_k)$ the Banach space of C^k functions $v : [a,b] \to \boldsymbol{R}$ with

$$\|v\|_k = \max\{\|v\|_0, \|v'\|_0, \ldots, \|v^{(k)}\|_0\}$$

($\|\ \|_0$ is the sup norm in $C[a,b]$) and by $(L^p[a,b], \|\ \|_p)$ the Banach space of pth power integrable functions with $\|v\|_p = (\int_a^b |v(s)|^p\, ds)^{1/p}$ for $p \geq 1$.

Given $p \geq 1$, we call $f : [a,b] \times \boldsymbol{R}^k \to \boldsymbol{R}$ an *L^p-Carathéodory function* if:

(i) $y \mapsto f(s,y)$ is continuous for almost all $s \in [a,b]$,

(ii) $s \mapsto f(s,y)$ is measurable for all $y \in \boldsymbol{R}^k$,

(iii) for each $r > 0$ there exists a nonnegative $\varphi_r \in L^p[a,b]$ such that $\|y\| \leq r$ implies $|f(s,y)| \leq \varphi_r(s)$ for almost all $s \in [a,b]$.

Putting

$$[Fv](t) = \int_a^t f(s, v(s), \ldots, v^{(k-1)}(s))\, ds$$

for each $v \in C^{k-1}[a,b]$, we obtain a nonlinear map $F : C^{k-1}[a,b] \to C[a,b]$ known as the *L^p-Carathéodory operator* associated with the L^p-Carathéodory function f.

Leaving to the reader the verification that F is well defined and continuous, we show that F is completely continuous: by Arzelà's theorem, it suffices to prove that given $\{u_n\}$ in $C^{k-1}[a,b]$ with $\|u_n\|_{k-1} \leq r$, the sequence $v_n = Fu_n$ is bounded and equicontinuous in $C[a,b]$.

(a) $\{v_n\}$ is bounded: since by assumption, $\max\{\|u_n\|_0, \ldots, \|u_n^{(k-1)}\|_0\} \leq r$, there exists, in view of (iii), a function $\varphi_r \in L^p[a,b]$ such that

$$|f(s, u_n(s), \ldots, u_n^{(k-1)}(s))| \leq \varphi_r(s)$$

for almost all $s \in [a,b]$, and therefore

$$\|v_n\|_0 = \|Fu_n\|_0 \leq \int_a^b \varphi_r(s)\, ds.$$

(b) $\{v_n\}$ is equicontinuous: let $\varepsilon > 0$ be given; by the absolute continuity of the Lebesgue integral, there exists a $\delta > 0$ such that $\int_t^{t'} \varphi_r(s)\, ds \leq \varepsilon$ whenever $|t - t'| < \delta$; consequently,

$$|v_n(t) - v_n(t')| = |Fu_n(t) - Fu_n(t')| \leq \int_t^{t'} \varphi_r(s)\, ds \leq \varepsilon$$

for all n whenever $|t - t'| < \delta$. Thus $\{v_n\}$ is equicontinuous.

EXAMPLE 4 (Boundary value problems). Let Ω be a bounded domain in $\boldsymbol{R}^n$ with a smooth boundary $\partial\Omega$. Let $f : \overline{\Omega} \times \boldsymbol{R} \to \boldsymbol{R}$ be Hölder continuous and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ the Laplace operator. Consider the nonlinear elliptic

boundary value problem

$$\text{(a)} \qquad \begin{cases} -\Delta u = f(x,u), \\ u|_{\partial\Omega} = 0. \end{cases}$$

One seeks a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying (a). The above problem is a prototype of more general problems that arise in numerous applications. One method of treating (a) is to apply the Green function G for the operator $-\Delta$ subject to the Dirichlet boundary conditions, and so to reduce it to the nonlinear integral equation of the Hammerstein type

$$\text{(b)} \qquad u(x) = \int_\Omega G(x,y) f(y,u(y))\,dy, \qquad x \in \overline{\Omega}.$$

The equation (b) may then be considered in various spaces, e.g. in $L^p(\Omega)$ $(1 < p < \infty)$, $C(\overline{\Omega})$, or $C^1(\Omega)$. In each case, however, due to the regularity of Green's function, it turns out to be equivalent to the problem (a). Moreover, in each of the above cases the nonlinear Hammerstein operator

$$[Fu](x) = \int_\Omega G(x,y) f(y,u(y))\,dy$$

is a completely continuous self-map of the corresponding function space, and thus the boundary value problem (a) reduces to the fixed point problem for the map F.

2. Schauder Projection and Approximation Theorem

The approximation of compact maps into normed linear spaces by finite-dimensional maps will be based on a simple projection of any finite union of ε-balls into the convex hull of their centers:

(2.1) DEFINITION. Let $N = \{c_1, \dots, c_n\}$ be a finite subset of a normed linear space, and for any fixed $\varepsilon > 0$, let

$$(N,\varepsilon) = \bigcup\{B(c_i,\varepsilon) \mid i \in [n]\}.$$

For $i \in [n]$ let $\mu_i : (N,\varepsilon) \to \boldsymbol{R}$ be the map $x \mapsto \max[0, \varepsilon - \|x - c_i\|]$. The *Schauder projection* $p_\varepsilon : (N,\varepsilon) \to \operatorname{conv} N$ is given by

$$p_\varepsilon(x) = \frac{1}{\sum_{i=1}^n \mu_i(x)} \sum_{i=1}^n \mu_i(x) c_i.$$

Note that p_ε is well defined, because each $x \in (N,\varepsilon)$ belongs to some $B(c_i,\varepsilon)$, so that $\sum_{i=1}^n \mu_i(x) \neq 0$; moreover, $p_\varepsilon[(N,\varepsilon)] \subset \operatorname{conv} N$, since each $p_\varepsilon(x)$ is a convex combination of the points $c_1, \dots, c_n$. The main properties we require are summarized in:

(2.2) PROPOSITION. *Let $c_1, \ldots, c_n$ belong to some convex $C \subset E$ and p_ε be the Schauder projection. Then:*
(a) *p_ε is a compact map of (N, ε) into* $\operatorname{conv} N \subset C$,
(b) *$\|x - p_\varepsilon(x)\| < \varepsilon$ for all $x \in (N, \varepsilon)$,*
(c) *if $N \subset C$ is symmetric with respect to* 0, *i.e., if*

$$N = \{c_1, \ldots, c_k, -c_1, \ldots, -c_k\},$$

then $(N, \varepsilon) = -(N, \varepsilon)$ and $p_\varepsilon(-x) = -p_\varepsilon(x)$ for all $x \in (N, \varepsilon)$.

PROOF. (a) is immediate; (b): we have

$$\begin{aligned}
\|x - p_\varepsilon(x)\| &= \left\| \frac{\sum_{i=1}^n \mu_i(x)}{\sum_{i=1}^n \mu_i(x)} x - \frac{\sum_{i=1}^n \mu_i(x) c_i}{\sum_{i=1}^n \mu_i(x)} \right\| \\
&= \frac{1}{\sum_{i=1}^n \mu_i(x)} \left\| \sum_{i=1}^n \mu_i(x)[x - c_i] \right\| \\
&\le \frac{\sum_{i=1}^n \mu_i(x) \|x - c_i\|}{\sum_{i=1}^n \mu_i(x)} < \varepsilon.
\end{aligned}$$

(c) Write $-c_i = c_{-i}$, and note that from the definition of the μ_i we get

$$\mu_i(x) = \mu_{-i}(-x),$$

and our assertion follows. □

This leads to the basic approximation result:

(2.3) THEOREM (Schauder approximation theorem). *Let X be a topological space, let C be a convex subset of a normed linear space E, and let $F : X \to C$ be a compact map. Then for each $\varepsilon > 0$, there is a finite set $N = \{c_1, \ldots, c_n\} \subset F(X) \subset C$ and a finite-dimensional map $F_\varepsilon : X \to C$ such that:*
(a) *$\|F_\varepsilon(x) - F(x)\| < \varepsilon$ for all $x \in X$,*
(b) *$F_\varepsilon(X) \subset$* $\operatorname{conv} N \subset C$.

PROOF. Since $F(X)$ is relatively compact, there exists a finite set $N = \{c_1, \ldots, c_n\} \subset F(X)$ with $F(X) \subset (N, \varepsilon)$. Define $F_\varepsilon : X \to C$ by $x \mapsto p_\varepsilon F(x)$, where $p_\varepsilon : (N, \varepsilon) \to \operatorname{conv} N$ is the Schauder projection; according to (2.2), the map F_ε has the required properties. □

As an application, we will establish a general extension property for compact maps.

(2.4) LEMMA. *Let X be a compact metric space, and let $A \subset X$ be closed. Let E be a normed linear space. Then each continuous $f : A \to E$ is extendable to a continuous $F : X \to E$.*

PROOF. For each $\varepsilon = 1/n^2$, $n = 1, 2, \ldots$, let $f_n : A \to E$ be a Schauder approximation to f with

$$\|f_n(a) - f(a)\| \leq 1/n^2$$

for $a \in A$. Since each f_n is a continuous map of A into a finite-dimensional linear space E^{k_n} (which is homeomorphic to $\boldsymbol{R}^{k_n}$), Tietze's theorem yields an extension $f_n^* : X \to E^{k_n} \subset E$ for $n = 1, 2, \ldots$. Let $g_n(x) = f_{n+1}^*(x) - f_n^*(x)$; then

$$\|g_n(a)\| \leq \|f_{n+1}(a) - f(a)\| + \|f(a) - f_n(a)\| \leq 2/n^2$$

for $a \in A$. Now for each $n = 1, 2, \ldots$, let

$$U_n = g_n^{-1} B(0, 3/n^2) \cap \{x \mid d(x, A) < 1/n\},$$

which is an open set in X containing A; replacing each U_n by $U_1 \cap \cdots \cap U_n$, we can assume that $U_1 \supset U_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} U_n = A$. Finally, for each $n = 1, 2, \ldots$ choose an Urysohn function λ_n that is 1 on A and zero off U_n, and let $h_n(x) = \lambda_n(x) \cdot g_n(x)$; clearly, $\|h_n(x)\| \leq 3/n^2$. Consider now the series $\sum_{n=1}^{\infty} h_n(x)$. This converges for each x: in fact, if $x \notin A$, then because x belongs to at most finitely many U_n, the series reduces to a finite sum; and if $a \in A$, then the nth partial sum is $f_{n+1}(a) - f_1(a)$, which converges to $f(a) - f_1(a)$. Since the convergence is uniform on X, the function $h(x) = \sum_{n=1}^{\infty} h_n(x)$ is therefore continuous, and $f_1^*(x) + h(x)$ is the required extension of f. □

(2.5) THEOREM. *Let X be normal, $A \subset X$ closed, and $F_0 : A \to E$ a compact map into a normed linear space E. Then F_0 is extendable to a compact map $F : X \to E$.*

PROOF. Because $\overline{F_0(A)}$, as a compact subset of E, is a compact metric space, it is embeddable as a closed subset Q of the Hilbert cube $\boldsymbol{I}^{\infty}$ Letting $h : Q \to E$ be the inverse of this embedding map, we have the commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "F_0"] \arrow[dr, "g"'] & & E \\
& Q \arrow[ur, "h"'] \arrow[d, phantom, "\cap"] & \\
& \boldsymbol{I}^{\infty} &
\end{tikzcd}$$

where g is the map $a \mapsto h^{-1} F_0(a)$. Because X is normal, g is extendable to a map $G : X \to \boldsymbol{I}^{\infty}$; and by (2.4), h is extendable to a map $H : \boldsymbol{I}^{\infty} \to E$. The map $H \circ G : X \to E$ is clearly compact, and is the required extension of F_0. □

3. Extension of the Brouwer and Borsuk Theorems

Let A be a subset of a metric space (X, d) and $F : A \to X$. Given an $\varepsilon > 0$, any point $a \in A$ with $d(a, F(a)) < \varepsilon$ is called an *ε-fixed point* for F. To relate this concept to that of fixed point, we have

(3.1) PROPOSITION. *Let A be a closed subset of a metric space (X, d). and $F : A \to X$ a compact map. Then F has a fixed point if and only if it has an ε-fixed point for each $\varepsilon > 0$.*

PROOF. "Only if" being trivial, we prove "if". For each $n = 1, 2, \dots$, let a_n be a $1/n$-fixed point for F, so that $d(a_n, F(a_n)) < 1/n$. Since F is compact, we may assume that $F(a_n) \to x \in \overline{F(A)}$. It follows that $a_n \to x$, and since A is closed, that $x \in A$. By the continuity of F, we find $F(a_n) \to F(x)$; consequently, $x = F(x)$, and F has a fixed point. □

We can now formulate Brouwer's theorem in a manner valid for all normed linear spaces.

(3.2) THEOREM (Schauder fixed point theorem). *Let C be a convex (not necessarily closed) subset of a normed linear space E. Then each compact map $F : C \to C$ has at least one fixed point.*

PROOF. According to (3.1), it is enough to show that F has an ε-fixed point for each $\varepsilon > 0$. Fix $\varepsilon > 0$; by the approximation theorem (2.3), there is an $F_\varepsilon : C \to C$ such that

$$\|F_\varepsilon(x) - F(x)\| < \varepsilon, \qquad x \in C,$$
$$F_\varepsilon(C) \subset \operatorname{conv} N \subset C \qquad \text{for some finite set } N \subset C.$$

Since $F_\varepsilon[\operatorname{conv} N] \subset \operatorname{conv} N$, and $\operatorname{conv} N$ is homeomorphic to a finite-dimensional ball, it follows from Brouwer's fixed point theorem that F_ε has a fixed point x_0; and because $\|x_0 - F(x_0)\| = \|F_\varepsilon(x_0) - F(x_0)\| \le \varepsilon$, this is the required ε-fixed point for F. □

With the same approximation technique, we extend Borsuk's fixed point theorem to all normed linear spaces.

(3.3) THEOREM (Borsuk). *Let U be a bounded convex open symmetric neighborhood of the origin in a normed linear space E. Then each compact map $F : \overline{U} \to E$ that is antipode-preserving on the boundary, i.e., $-F(a) = F(-a)$ on ∂U, has at least one fixed point.*

PROOF. Let $\varepsilon > 0$ be given; it suffices to show that F has an ε-fixed point. First, choose a finite subset $N \subset E$ symmetric with respect to the origin and such that $F(\overline{U}) \subset (N, \varepsilon)$. It follows, using (2.2)(c), that the finite-dimensional ε-approximation $p_\varepsilon \circ F$ is still antipode-preserving on ∂U. Let

E^k be a finite-dimensional linear subspace of E with $p_\varepsilon \circ F(\overline{U}) \subset E^k$, and set $F^* = p_\varepsilon \circ F|(\overline{U} \cap E^k)$. Now, $U^k = U \cap E^k$ is an open bounded symmetric neighborhood of 0 in E^k, and $F^* : \overline{U}^k \to E^k$ is antipode-preserving on $\partial(U^k)$, so by Borsuk's theorem, F^* has a fixed point x_0. Since

$$\|x_0 - F(x_0)\| = \|F^*(x_0) - F(x_0)\| = \|p_\varepsilon F(x_0) - F(x_0)\| < \varepsilon,$$

this is the required ε-fixed point for F. □

4. Topological Transversality. Existence of Essential Maps

Let $F : X \to E$ be a compact operator defined on some $X \subset E$ and satisfying a "boundary condition" on a closed $A \subset X$. One technique for determining whether or not the equation $x = F(x)$ has solutions starts by deforming F, and possibly also the boundary values $F|A$, to a simpler operator G, and attempts to reduce the problem to that for the equation $x = G(x)$. In geometric terms, one deforms the graph of F to that of G and seeks to conclude, from the nature of the deformation, that if the graph of G meets the diagonal $\Delta \subset X \times E \subset E \times E$, then the graph of F must also do so. The topological transversality theorem gives conditions under which such a conclusion is valid.

To formulate the theorem in broad generality, we work entirely within a convex set $C \subset E$, a normed linear space. By a *pair* (X, A) in C is meant an arbitrary subset X of C and an $A \subset X$ closed in X. We call a homotopy $H : X \times I \to Y$ *compact* if it is a compact map. If $X \subset Y$, the homotopy H is called *fixed point free on* $A \subset X$ if for each $t \in I$, the map $H|A \times \{t\} : A \to Y$ has no fixed point.

We denote by $\mathscr{K}_A(X, C)$ the set of all compact maps $F : X \to C$ such that the restriction $F|A : A \to C$ is fixed point free.

(4.1) DEFINITION. Two maps $F, G \in \mathscr{K}_A(X, C)$ are called *admissibly homotopic*, written $F \simeq G$ in $\mathscr{K}_A(X, C)$, provided there is a compact homotopy $H_t : X \to C$ $(0 \leq t \leq 1)$ that is fixed point free on A and such that $H_0 = F$ and $H_1 = G$. Clearly, $\simeq$ is an equivalence relation in $\mathscr{K}_A(X, C)$.

(4.2) PROPOSITION. *Let $F, G \in \mathscr{K}_A(X, C)$ be two maps and assume that one of the following conditions holds:*
(i) *$tG(a) + (1 - t)F(a) \neq a$ for each $(a, t) \in A \times [0, 1]$,*
(ii) *$\sup_{a \in A} \|F(a) - G(a)\| \leq \inf_{a \in A} \|a - F(a)\|$.*
Then $F \simeq G$ in $\mathscr{K}_A(X, C)$.

PROOF. Condition (ii) implies that given $a \in A$ the line segment $[F(a), G(a)]$ joining $F(a)$ to $G(a)$ does not contain a, which is precisely condition (i). Thus it is sufficient to show that (i) implies $F \simeq G$ in $\mathscr{K}_A(X, C)$.

Write

$$H_t(x) = tG(x) + (1-t)F(x) \quad \text{for } (x, t) \in X \times [0, 1]$$

and note that $\{H_t\}_{0 \le t \le 1}$ is a compact homotopy that is fixed point free on A and such that $H_0 = F$ and $H_1 = G$. □

(4.3) DEFINITION. Let (X, A) be a pair in a convex $C \subset E$. A map $F \in \mathscr{K}_A(X, C)$ is called *essential* provided every $G \in \mathscr{K}_A(X, C)$ such that $F|A = G|A$ has a fixed point. A map that is not essential is called *inessential*.

In geometric terms, a compact map $F : X \to C$ is essential if the graph of $F|A$ does not meet the diagonal $\Delta \subset X \times C$, but the graph of every compact $G : X \to C$ that coincides with F on A must cross the diagonal Δ.

(4.4) PROPOSITION. *Let (X, A) be a pair in a normed linear space E, let L be a closed linear subspace meeting $A \subset X$, and let $A_L = A \cap L$ and $X_L = X \cap L$. Let $F \in \mathscr{K}_A(X, E)$ be an essential map such that $F(X) \subset L$. Then $F|X_L \in \mathscr{K}_{A_L}(X_L, L)$ is also essential.*

PROOF. We need to show that a compact extension $F_0 : X_L \to L$ of $F|A_L$ over X_L has a fixed point. Consider the map $G : A \cup X_L \to L$ given by

$$G(x) = \begin{cases} F_0(x), & x \in X_L, \\ F(x), & x \in A. \end{cases}$$

This is continuous, since the two definitions agree on $A \cap X_L = A_L$; moreover $A \cup X_L$ is closed in X. By (2.5), G can be extended to a compact map $\widetilde{G} : X \to L$, and $\widetilde{G}$, being an extension of $F|A$, must have a fixed point x. Since $x = \widetilde{G}(x)$ shows $x \in X_L$, and $\widetilde{G}|X_L = F_0$, we find $F_0(x) = x$. □

(4.5) PROPOSITION. *Let $B_r = \{x \in E \mid \|x - x_0\| < r\}$ and $F \in \mathscr{K}_{\partial B_r}(\overline{B}_r, E)$ be an essential map such that x_0 is the only fixed point for F. Then for every $0 < r_0 < r$ the restriction $F|\overline{B}_{r_0}$ is an essential map in $\mathscr{K}_{\partial B_{r_0}}(\overline{B}_{r_0}, E)$.*

PROOF. Assume $F|\partial B_{r_0}$ has a fixed point free extension F_0 over $\overline{B}_{r_0}$. Then

$$F(x) = \begin{cases} F_0(x), & x \in \overline{B}_{r_0}, \\ F(x), & x \in \overline{B}_r - B_{r_0}, \end{cases}$$

would be a fixed point free extension of $F|\partial B_r$. □

We now characterize the inessential maps in terms of homotopy.

(4.6) LEMMA. *Let (X, A) be a pair in a convex $C \subset E$. The following conditions on $F \in \mathscr{K}_A(X, C)$ are equivalent:*

(i) *F is inessential,*
(ii) *there is a fixed point free $G \in \mathscr{K}_A(X,C)$ such that $F \simeq G$ in $\mathscr{K}_A(X,C)$.*
(iii) *F is homotopic in $\mathscr{K}_A(X,C)$ to a fixed point free $F^* \in \mathscr{K}_A(X,C)$ by a homotopy keeping $F|A$ pointwise fixed.*

PROOF. (i)$\Rightarrow$(ii). Let $G \in \mathscr{K}_A(X,C)$ be a fixed point free map such that $F|A = G|A$. The compact homotopy $(x,t) \mapsto tG(X) + (1-t)F(x)$ joins F to G and is fixed point free on A.

(ii)$\Rightarrow$(iii). Let $H : X \times \boldsymbol{I} \to C$ be a compact homotopy from G to F such that $H|A \times \{t\}$ is fixed point free for each $t \in \boldsymbol{I}$. Let

$$B = \{x \mid x = H(x,t) \text{ for some } t \in \boldsymbol{I}\}.$$

There is no loss of generality in assuming that B is nonempty; then B is a closed subset of the compact set $\overline{H(X \times \boldsymbol{I})}$, so is compact, and therefore closed in X. Since $A \cap B = \emptyset$ because H is fixed point free on A, there is an Urysohn function $\lambda : X \to \boldsymbol{I}$ with $\lambda|A = 1$ and $\lambda|B = 0$. Define

$$F^*(x) = H(x, \lambda(x)).$$

Clearly, F^* is compact; it is also fixed point free: for if $F^*(x) = H(x,\lambda(x)) = x$ then $x \in B$ so $\lambda(x) = 0$ and $x = H(x,0) = G(x)$, which contradicts the assumption that G is fixed point free. To show F^* homotopic to F keeping $F|A$ pointwise fixed, consider the compact homotopy

$$H^*(x,t) = H(x, (1-t) + t\lambda(x)).$$

Then $H^*(x,0) = H(x,1) = F(x)$ and $H^*(x,1) = H(x,\lambda(x)) = F^*(x)$; moreover $\lambda(a) = 1$ for all $a \in A$; therefore, $H^*(a,t) = H(a,1) = F(a)$ for all $t \in \boldsymbol{I}$. For each t in $\boldsymbol{I}$, $H^*(x,t)$ is clearly fixed point free on A.

(iii)$\Rightarrow$(i) is obvious. □

As an immediate consequence we obtain

(4.7) THEOREM (Topological transversality). *Let (X,A) be a pair in a convex $C \subset E$, and let F, G be maps in $\mathscr{K}_A(X,C)$ such that $F \simeq G$ in $\mathscr{K}_A(X,C)$. Then F is essential if and only if G is essential.* □

The concept of topological transversality, which is invariant under fixed point free deformations on A, is also invariant under small modifications of F on A:

(4.8) COROLLARY. *Let (X,A) be a pair in the convex $C \subset E$, and let $F \in \mathscr{K}_A(X,C)$ be essential. Then there exists an $\varepsilon > 0$ with the properties:* (i) *any compact $G : X \to C$ satisfying $\|G(a) - F(a)\| < \varepsilon$ for all $a \in A$ is in $\mathscr{K}_A(X,C)$, and* (ii) *G is essential.*

PROOF. Because F is compact and fixed point free on A, there exists an $\varepsilon > 0$ such that $\|a - F(a)\| \geq \varepsilon$ for all $a \in A$. If $G : X \to C$ satisfies $\|G(a) - F(a)\| < \varepsilon$ for all $a \in A$, it is fixed point free on A; by (4.2) we have $F \simeq G$ in $\mathscr{K}_A(X, C)$, so by (4.6), the desired conclusion follows. □

To prepare for applications of (4.7), we will show that some simple maps are essential.

(4.9) THEOREM. *Let U be an open subset of a convex set $C \subset E$, and let $(\overline{U}, \partial U)$ be the pair consisting of the closure of U in C and the boundary of U in C. Then for any $u_0 \in U$, the constant map $F|\overline{U} = u_0$ is essential in $\mathscr{K}_{\partial U}(\overline{U}, C)$.*

PROOF. We must show that any compact map $G : \overline{U} \to C$ with $G|\partial U = u_0$ has a fixed point. For this purpose, extend G to a compact map $G^* : C \to C$ by setting $G^*|C - \overline{U} = u_0$. By Schauder's theorem, G^* must have a fixed point; and since no point in $C - U$ is fixed, the fixed point x must be in U. Thus, $x = G(x)$. □

As another basic example, we have

(4.10) THEOREM. *Let U be a convex open bounded symmetric neighborhood of zero in a normed linear space E. Then any compact map $F \in \mathscr{K}_{\partial U}(\overline{U}, E)$ that is antipode-preserving on $\partial\overline{U}$ is essential.*

PROOF. This is an immediate consequence of (3.3). □

5. Equation $x = F(x)$. The Leray–Schauder Principle

In this section we apply the topological transversality theorem to the equation $x = F(x)$, where F is a compact or completely continuous operator; first we establish the following fundamental result:

(5.1) THEOREM (Leray–Schauder principle). *Let $C \subset E$ be a convex set, and let U be open in C. Let $\{H_t : \overline{U} \to C\}$ be an admissible compact homotopy such that $H_0 = F$ and $H_1 = G$, where G is the constant map sending $\overline{U}$ to a point $u_0 \in U$ Then F has a fixed point.*

PROOF. By (4.9), G is an essential map; from (4.7) it follows that so is F, and the conclusion follows. □

As an immediate consequence we obtain an important result that yields many of the standard fixed point theorems.

(5.2) THEOREM (Nonlinear alternative). *Let $C \subset E$ be a convex set, and let U be open in C and such that $0 \in U$. Then each compact map $F : \overline{U} \to C$ has at least one of the following two properties:*

(a) *F has a fixed point,*
(b) *there exist* $x \in \partial U$ *and* $\lambda \in (0,1)$ *such that* $x = \lambda F(x)$.

PROOF. We can assume $F|\partial U$ is fixed point free, else we have property (a). Let $G : \overline{U} \to C$ be the constant map $u \mapsto 0$, and consider the compact homotopy $H_t : \overline{U} \to C$ given by $H(u,t) = tF(u)$ joining G to F; either this homotopy is fixed point free on ∂U or it is not. If it is fixed point free, then by (5.1) we find that F must have a fixed point. If the homotopy is not fixed point free on ∂U, then there is an $x \in \partial U$ with $x = \lambda F(x)$; since $\lambda \neq 0$ because $0 \notin \partial U$, and $\lambda \neq 1$ because $F|\partial U$ has been assumed to be fixed point free, the property (b) is satisfied. □

Many of the customary fixed point theorems can be derived from the nonlinear alternative by imposing conditions that prevent occurrence of the second property. As an illustration of such conditions, let $p : E \to \boldsymbol{R}^+$ be any (not necessarily continuous) function such that $p^{-1}(0) = 0$ and $p(\lambda x) = \lambda p(x)$ for all $\lambda > 0$; any norm, not necessarily equivalent to the given one in E, is an example of such a function. Then we have

(5.3) COROLLARY. *Let* $C \subset E$ *be convex, and* $U \subset C$ *an open subset that contains* 0. *Let* $F : \overline{U} \to C$ *be a compact map. If either*
(1) (Rothe type condition)

$$p[F(x)] \leq p(x) \quad \textit{for all } x \in \partial U,$$

or
(2) (Altman type condition)

$$[pF(x)]^2 \leq [p(F(x) - x)]^2 + [p(x)]^2 \quad \textit{for all } x \in \partial U,$$

then F has a fixed point.

PROOF. The routine calculation, to show that the second property in (5.2) cannot occur, is left to the reader. □

The nonlinear alternative, applied to completely continuous operators, yields

(5.4) THEOREM (Leray–Schauder alternative). *Let C be a convex subset of E, and assume* $0 \in C$. *Let* $F : C \to C$ *be a completely continuous operator, and let*

$$\mathcal{E}(F) = \{x \in C \mid x = \lambda F(x) \textit{ for some } 0 < \lambda < 1\}.$$

Then either $\mathcal{E}(F)$ *is unbounded or F has a fixed point.*

PROOF. Assume that $\mathcal{E}(F)$ is bounded, and let $B(0,r)$ be an open ball containing $\mathcal{E}(F)$. Then $F|C \cap \overline{B(0,r)} : C \cap \overline{B(0,r)} \to C$ is a compact map, and no $x \in \partial[C \cap B(0,r)]$ can satisfy the second property in (5.2). □

As an immediate consequence of the topological transversality theorem (4.7) and Borsuk's theorem, we obtain

(5.5) THEOREM. *Let $F : E \times \boldsymbol{I} \to E$ be a completely continuous operator such that for some $\varrho > 0$, $F(x, 0) = -F(-x, 0)$ for all $x \in E$ with $\|x\| \geq \varrho$. Let*

$$\mathcal{E}(F) = \{x \in E \mid x = F(x, t) \text{ for some } t \in (0, 1)\}.$$

Then either $\mathcal{E}(F)$ is unbounded or $x \mapsto F(x, 1)$ has a fixed point.

PROOF. Assume that $\mathcal{E}(F)$ is bounded, and let $B = B(0, r)$ be an open ball in E containing $\mathcal{E}(F)$ such that $r \geq \varrho$.

Consider the compact homotopy $F_t(x) = F(x, t)$ for $(x, t) \in \overline{B} \times \boldsymbol{I}$; we may assume that $\{F_t\}$ is fixed point free on ∂B, so that $F_0 \simeq F_1$ in $\mathscr{K}_{\partial B}(\overline{B}, E)$. Then since F_0 is antipode-preserving on ∂B, F_0 is essential by (4.10), so by (4.7), the desired conclusion follows. □

As another application, recall that an operator $F : E \to E$ is called *quasi-bounded* whenever

$$|F| = \overline{\lim_{\|x\| \to \infty}} \frac{\|F(x)\|}{\|x\|} = \inf_{\varrho > 0} \sup_{\|x\| \geq \varrho} \frac{\|F(x)\|}{\|x\|} < \infty;$$

for example, any bounded linear operator is quasi-bounded and satisfies $|F| = \|F\|$.

(5.6) THEOREM. *Let $F : E \to E$ be a quasi-bounded completely continuous operator. Then for each real $|\lambda| < 1/|F|$ (and for all real λ whenever $|F| = 0$) the operator λF has at least one fixed point. More generally: for each $y \in E$ and $|\lambda| < 1/|F|$ the equation*

$$y = x - \lambda F(x)$$

has at least one solution.

PROOF. Given $y \in E$, consider the completely continuous operator $G(x) = y + \lambda F(x)$; it is easy to see that $|G| = |\lambda| \cdot |F|$, so that if $|\lambda| < 1/|F|$, we have $|G| < 1$, and there exists an $r > 0$ with $\|G(x)\|/\|x\| < 1$ for all $\|x\| \geq r$ By the nonlinear alternative for G on $\{x \mid \|x\| \leq r\}$ we find that G has a fixed point x, so $x = G(x) = y + \lambda F(x)$. □

6. Equation $x = \lambda F(x)$. Birkhoff–Kellogg Theorem

In this section we apply the topological transversality theorem to the equation $x = \lambda F(x)$, where λ is a real parameter, i.e., to the problem of the existence of an invariant direction for a compact map. We prove that under certain conditions an invariant direction always exists.

(6.1) THEOREM (Birkhoff Kellogg). *Let U be a bounded open neighborhood of 0 in an infinite-dimensional normed linear space, and $F : \partial U \to E$ a compact map satisfying $\|F(x)\| \geq \alpha > 0$ for all $x \in \partial U$. Then F has an invariant direction, i.e., there is an $x \in \partial U$ and a $\mu > 0$ such that $x = \mu F(x)$.*

PROOF. By (2.5) we may assume that F is in fact defined on $\overline{U}$. We first show that there is a compact map $G : \overline{U} \to E$, coinciding with F on ∂U, such that $0 \notin \overline{G(U)}$. For the set $\overline{F(\overline{U})}$, being compact, cannot cover the noncompact ball $B(0, \alpha/2)$, so there is some $v_0 \in B(0, \alpha/2) - \overline{F(\overline{U})}$. Define a homeomorphism $h : (E, B(0, \alpha/2)) \to (E, B(0, \alpha/2))$ that is the identity on $E - B(0, \alpha/2)$ and maps v_0 to 0 as follows: each $z \in B(0, \alpha/2) - \{v_0\}$ is uniquely expressible in the form

$$z = (1-t)v_0 + t\dot{z}, \qquad \dot{z} \in \partial B(0, \alpha/2),\ 0 < t < 1.$$

Set

$$h(z) = \begin{cases} 0, & z = v_0, \\ h((1-t)v_0 + t\dot{z}) = t\dot{z}, & z \in B(0, \alpha/2) - \{v_0\}, \\ z, & z \in E - B(0, \alpha/2). \end{cases}$$

Then $G = h \circ F$ is the required map, and $G|\partial U = F|\partial U$ because $\|F(x)\| \geq \alpha$ for $x \in \partial U$.

Now choose an integer N so large that $\overline{U}$ is contained in $B(0, N)$, and consider the compact map $H : \overline{U} \to E$ given by

$$H(x) = N \frac{G(x)}{\|G(x)\|}.$$

This map cannot have a fixed point, because $\overline{U}$ is in $B(0, N)$ whereas $H(\overline{U}) \subset \partial B(0, N)$.

Applying the nonlinear alternative to H we get

$$\frac{\lambda N}{\|G(x)\|} G(x) = x \qquad \text{for some } x \in \partial U \text{ and } \lambda \in (0, 1),$$

and since $G|\partial U = F|\partial U$, the proof is complete. □

7. Compact Fields

Further applications of the topological transversality theorem (in particular to the equation $y = x - F(x)$) will be expressed in terms of compact fields. Let X be a subset of a normed linear space E, and $F : X \to E$ a compact map. Functions of the form $f(x) = x - F(x)$ are called compact fields. In an arbitrary normed space, the class of compact fields has various features that the class of compact maps does not have: for example, the identity map

of an infinite-dimensional normed space is a compact field, but it is not a compact map. In this section we obtain some of the more immediate general properties of this class of functions.

(7.1) DEFINITION. Let (X, A), (Y, B) be two pairs in the normed space E. A map $f : (X, A) \to (Y, B)$ is called a *compact field* if $x \mapsto x - f(x)$ is a compact map of X into E.

Whenever the compact field is denoted by a lowercase letter, e.g., $f : X \to Y$, the compact map $x \mapsto x - f(x)$ will be denoted by the corresponding capital letter, as in $F(x) = x - f(x)$; we call F the (uniquely determined) compact map *associated* with the given field. Compact fields are described by the terminology used for the associated maps: thus, the field f is called finite-dimensional if the associated compact map is finite-dimensional.

It is obvious that if $A \subset X \subset E$, where E is a normed space, then (a) the inclusion map $i : A \to X$ is a compact field (in fact, finite-dimensional, since the associated map is $F(x) = 0$) and (b) the restriction $f|A$ of a compact field $f : X \to E$ is also a compact field. Moreover, (c) the composition of compact fields is also a compact field: for if $f : X \to Y$ and $g : Y \to Z$ have the associated compact maps F and G, respectively, then

$$g[f(x)] = f(x) - G[f(x)] = x - [F(x) + G(f(x))],$$

and $F + G \circ f$ is clearly a compact map. Slightly less evident is

(7.2) THEOREM. *Let $X \subset E$ be closed. Then:*
(a) *a compact field $f : X \to Y$ is a closed proper map (i.e., the image of each closed set in X is closed in Y, and the preimage of each compact subset of Y is compact),*
(b) *a bijective compact field $f : X \to Y$ is bicontinuous, and its inverse is also a compact field.*

PROOF. (a) We first show that f is proper. Let $K \subset Y$ be compact; then

$$f^{-1}(K) = \{x \mid x - F(x) \in K\} \subset \{x \mid x \in K + F(x)\} \subset K + \overline{F(X)}.$$

Because f is continuous, the set $f^{-1}(K)$ is closed in the closed X, so it is closed in E, and being contained in the compact $K + \overline{F(X)}$, it is therefore compact. To show that f is a closed map, it is enough to prove (since Y is a metric space) that $K \cap f(A)$ is closed in K for each closed $A \subset X$ and compact $K \subset Y$. To establish this, note that $f^{-1}(K)$ is compact by what we have just shown, so that $A \cap f^{-1}(K)$ is also compact, and therefore $f[A \cap f^{-1}(K)] = f(A) \cap K$, being compact, is closed.

(b) The bicontinuity follows immediately from (a). Since

$$y = f[f^{-1}(y)] = f^{-1}(y) - F[f^{-1}(y)]$$

we find that f^{-1} is associated with the compact map $-(F \circ f^{-1})$, so f^{-1} is a compact field. □

We now introduce a suitable notion of homotopy for compact fields:

(7.3) DEFINITION. Let (X, A), (Y, B) be two pairs in E. Two compact fields $f, g : (X, A) \to (Y, B)$ are *homotopic* if there is a continuous map $h : (X, A) \times \boldsymbol{I} \to (Y, B)$ such that the map $(x, t) \mapsto x - h(x, t)$ is a compact map of $X \times \boldsymbol{I}$ into E, $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.

It is clear that the homotopy $H(x, t) = x - h(x, t)$ associated with h is a compact homotopy of the associated compact maps.

To describe the notion of topological transversality directly in terms of compact fields, we consider maps into the fixed pair $(E, E - \{0\})$. Given any pair (X, A), observe that by making each compact field correspond to its associated compact map, the set of all compact fields $f : (X, A) \to (E, E - \{0\})$ is in one-to-one correspondence with the set $\mathscr{K}_A(X, E)$ of all compact maps $F : X \to E$ that are fixed point free on A; and moreover, that a homotopy $f \simeq g : (X, A) \to (E, E - \{0\})$ of compact fields corresponds to a compact homotopy of the associated compact maps that is fixed point free on A. To facilitate the discussion, we make

(7.4) DEFINITION. Let (X, A) be a pair in E, and $f : (X, A) \to (E, E - \{0\})$ a compact field. The field f is called *inessential* on X if there is a compact field $g : X \to E - \{0\}$ such that $g|A = f|A$. A field is called *essential* on X if it is not inessential on X.

It follows from the preceding discussion that a compact field $f : (X, A) \to (E, E - \{0\})$ is essential on X if and only if the associated map F is essential in $\mathscr{K}_A(X, E)$. In these terms, we can now restate a part of Theorem (4.7) as

(7.5) THEOREM. *Let $f, g : (X, A) \to (E, E - \{0\})$ be two homotopic compact fields. Then f is essential if and only if g is essential.* □

We can express the result of (4.9) as

(7.6) THEOREM. *Let U be an open subset of E, and let $x_0 \in U$. Then the compact field $f : (\overline{U}, \partial U) \to (E, E - \{0\})$ given by $x \mapsto x - x_0$ is essential on $\overline{U}$.* □

8. Equation $y = x - F(x)$. Invariance of Domain

In this section we apply the topological transversality to the equation $y = x - F(x)$. Our results are expressed in terms of compact and completely continuous fields.

To formulate the main results, we need the general

(8.1) DEFINITION. Let $f : X \to Y$ be a map of metric spaces. If there are $\varepsilon > 0$ and $\delta \geq 0$ such that $\operatorname{diam} f^{-1}(B(y,\delta)) < \varepsilon$ for all $y \in Y$, we say that f is a *δ-based ε-map* ([1]). If $\delta = 0$ then f is called simply an *ε-map* and for $\delta > 0$ it is called an *ε-map in the narrow sense*.

The main results will appear as consequences of

(8.2) LEMMA. *Let $B(x_0, \varepsilon)$ be an open ball in the normed linear space E, and let $f : \overline{B(x_0, \varepsilon)} \to E$ be a compact field. If f is an ε-map, then there is an $\eta > 0$ such that*

$$f(B(x_0, \varepsilon)) \supset B(f(x_0), \eta),$$

and if f is a δ-based ε-map with $\delta > 0$, then

$$f(B(x_0, \varepsilon)) \supset B(f(x_0), \delta).$$

PROOF. Because f is a δ-based ε-map, the associated compact map F has the property that $\|x - y\| < \varepsilon$ whenever $\|F(x) - F(y) - (x - y)\| < \delta$.

There is no loss of generality to assume $x_0 = 0$, so that $F(0) = -f(0)$. We consider the compact map $G(x) = F(x) - F(0)$ on the ball $\overline{B} = \overline{B(0, \varepsilon)}$ and show first that if f is a δ-based ε-map for some $\delta \geq 0$, then G is essential in $\mathscr{K}_{\partial B}(\overline{B}, E)$. For this purpose, define a homotopy $H : \overline{B} \times I \to E$ by

$$H(x,t) = F\left(\frac{x}{1+t}\right) - F\left(\frac{-tx}{1+t}\right).$$

This map is fixed point free on ∂B: for if

$$F\left(\frac{x}{1+t}\right) - F\left(\frac{-tx}{1+t}\right) = x = \frac{x}{1+t} - \left(\frac{-tx}{1+t}\right)$$

for some $x \in \partial B$ and $0 \leq t \leq 1$, this would imply that $\|x\| < \varepsilon$ because f is a δ-based ε-map. Since $H(x,1) = F(x/2) - F(-x/2)$ is clearly antipode-preserving on ∂B, we find from (4.10) and the topological transversality theorem that $H(x, 0) = G(x)$ is essential.

Now consider the case where f is a 0-based ε-map. According to (4.8), there is $\eta > 0$ such that any compact operator G_1 with $\|G_1(x) - G(x)\| < \eta$ on ∂B is in $\mathscr{K}_{\partial B}(\overline{B}, E)$ and is also essential; in particular, the operator $G_1(x) = G(x) + y$ satisfies this requirement for each $\|y\| < \eta$ so the equation $x = G(x) + y$ has a solution in B. This says that $x - F(x) = -F(0) + y$, i.e., $f(x) = f(0) + y$ has a solution for each $\|y\| < \eta$, so $f(B(0, \varepsilon)) \supset B(f(0), \eta)$.

To see that we can take $\eta = \delta$ whenever f is a δ-based ε-map with $\delta > 0$, note that for each $\|y\| < \delta$ and $0 \leq t \leq 1$ the compact homotopy $(x, t) \mapsto F(x) - F(0) + ty$ is fixed point free on ∂B: for if $F(x) - F(0) + ty = x$

([1]) If $\delta = 0$, then $B(y, \delta)$ means $\{y\}$.

for some $x \in \partial B$ and $0 \le t \le 1$, then

$$\|F(x) - F(0) - (x - 0)\| \le t\|y\| < \delta,$$

which would require that $\|x\| < \varepsilon$. Thus, $F(x) - F(0) + y$ is essential in $\mathscr{K}_{\partial B}(\overline{B}, E)$ for all $\|y\| < \delta$, so as before, $f(B(0,\varepsilon)) \supset B(f(0), \delta)$. □

This has immediate corollaries:

(8.3) THEOREM. *Let $f : E \to E$ be a completely continuous field in a normed space E.*
(a) *If f is an ε-map, then $f(E)$ is open in E.*
(b) *If f is an ε-map in the narrow sense, then f is surjective, i.e., $f(E) = E$.*

PROOF. (a) is immediate from Lemma (8.2). For (b): according to the lemma, the ball $B(y, \delta)$ of fixed radius $\delta > 0$ is contained in $f(E)$ for each $y \in f(E)$, so f is surjective. □

(8.4) THEOREM (Schauder domain invariance). *Let U be open in the normed space E, and let $f : U \to E$ be an injective completely continuous field. Then* (a) *f is an open map,* (b) *$f(U)$ is open in E, and* (c) *f is a homeomorphism of U onto $f(U)$.*

PROOF. Since f is injective, it is an ε-map for each $\varepsilon > 0$; so the result follows from (8.2). □

As an immediate consequence we obtain

(8.5) COROLLARY (Fredholm alternative). *Let E be an arbitrary normed space, and let $F : E \to E$ be a completely continuous linear operator. Then either*
(a) *the equation $0 = x - F(x)$ has a nontrivial solution, or*
(b) *the equation $y = x - F(x)$ has a unique solution for each $y \in E$.*

PROOF. The completely continuous field $f(x) = x - F(x)$ is either not injective, or it is injective. In the first event, property (a) holds. In the second event, the image of E under f is a linear subspace which, by domain invariance, must therefore be the entire space E. Thus, the injective field $f : E \to E$ is also surjective, and therefore bijective. □

For the nonlinear case, the following invertibility condition is useful.

(8.6) COROLLARY. *Let $f : E \to E$ be a completely continuous field in a normed space E. If $\|f(x) - f(y)\| \ge M\|x - y\|$, then f is invertible.*

PROOF. Given $\varepsilon > 0$, the hypothesis on f shows that f is an ε-map in the narrow sense, so by (8.3), f is surjective. Since f is also injective, it is bijective, and so by (8.2) a homeomorphism, therefore invertible. □

9. Miscellaneous Results and Examples

A. Compact maps and compact fields

(A.1) Let $E = A + B$ be a direct sum decomposition of a normed linear space E, let $P_A : E \to A$ and $P_B : E \to B$ be the corresponding linear projections, and let $F : X \to E$ be a compact map. Show: For each $\varepsilon > 0$ there exists a compact map $F_\varepsilon : X \to E$ such that:

(i) $\|F(x) - F_\varepsilon(x)\| \leq \varepsilon$ for all $x \in X$,
(ii) $P_B \circ F_\varepsilon = P_B \circ F$,
(iii) $P_A \circ F_\varepsilon$ is finite-dimensional.

[If G_ε is the Schauder approximation for $P_A \circ F$, define $F_\varepsilon = G_\varepsilon + P_B \circ F$.]

(A.2) Let E be a normed linear space, U an open convex symmetric neighborhood of zero in E, and C a nonempty closed convex symmetric (with respect to 0) subset of E. Let $F : \overline{U} \cap C \to C$ be a compact map such that $F(x) = -F(-x)$ for each $x \in \partial U \cap C$. Show: F has a fixed point (Kaczynski [1987]).

[Use (2.5) and Borsuk's theorem (3.3).]

(A.3) Let $(E^n, \|\ \|)$ be a normed space of dimension n, and U an open symmetric convex nbd of 0 in E^n; let $f : (\overline{U}, \partial U) \to (E^n, E^n - \{0\})$ be a compact field satisfying

$$f(x) \neq tf(-x) \qquad \text{for all } t > 0 \text{ and } x \in \partial U.$$

(a) Denote by p the Minkowski functional of U, and let $V = \{y \in E^n \mid \|y\| < \varrho\}$ be an open ball with $F(\overline{U}) \subset \frac{1}{2}V$. Define $G : \overline{V} \to E^n$ by

$$G(x) = \begin{cases} F(x) & \text{for } x \in \overline{U} \cap \overline{V}, \\ p(x)F[x/p(x)] & \text{for } x \in \overline{V} - U. \end{cases}$$

Show: $G(x) \neq x$ for each $x \in \partial V \cup \overline{V} - U$.

(b) Let $g : (\overline{V}, \partial V) \to (E^n, E^n - \{0\})$ be given by $g(x) = x - G(x)$. Prove: $g(x) \neq tg(-x)$ for each $t > 0$ and $x \in \partial V$.

[Show that assuming $g(x_0) = t_0 g(-x_0)$ for some $t_0 > 0$ and $x_0 \in \partial V$ leads to a contradiction; consider separately $x_0 \in \partial V \cap \overline{U}$ and $x_0 \in \partial V - U$.]

(c) Show: g is essential and $g \simeq g^*$, where g^* is antipode-preserving on ∂V

[Consider the homotopy $h_t(x) = [g(x) - tg(-x)]$.]

(d) Prove: f is an essential compact field.

(A.4) Let U be an open convex symmetric neighborhood of 0 in $(E, \|\ \|)$, and f $(\overline{U}, \partial U) \to (E, E - \{0\})$ a compact field such that $f(x) \neq tf(-x)$ for each $(x, t) \in \partial U \times I$. Show: f is essential.

[First consider the special case $f(x) = -f(-x)$ for all $x \in \partial U$ and use (A.3); for the general case consider the compact homotopy $h_t(x) = \frac{1}{1+t}[f(x) - tf(-x)]$.]

(A.5) Let U be an open bounded neighborhood of 0 in $(E, \|\ \|)$, and let f $(\overline{U}, \partial U) \to (E, E - \{0\})$ be a compact field. Given $x \in \partial U$ let $L_x = \{y \in E \mid y = \lambda x,\ 0 \leq \lambda < \infty\}$ be the ray joining 0 to x. Show: If f is essential, then $f(\partial U) \cap L_x \neq \emptyset$ for each $x \in \partial U$.

(A.6) Let $E^{\infty - n}$ be a subspace of $E = E^\infty$ of codimension n, $S^{\infty - n} = \{x \in E^{\infty - n} \mid \|x\| = 1\}$, and let $f : S^\infty \to E^{\infty - 1}$ be a compact field. Show: There is a point $x \in S^\infty$ such that $f(x) = f(-x)$ (Granas [1962]).

[Consider on S^∞ the compact field $\varphi(x) = (f(x) - f(-x))/2$; use (A.4) and (A.5).]

(A.7) Prove: There is no bijective compact field $f : S^{\infty-k} \to S^{\infty-l}$ for $k \neq l$.

(A.8) Let U be an open set in a Banach space E, and $f : U \to E$ a map of the form $f(x) = x - [F(x) + G(x)]$, where F is completely continuous and G is contractive. Prove: If f is injective then (i) f is open, in particular $f(U)$ is open in E, and (ii) $f : U \to f(U)$ is a homeomorphism (Schauder [1932]).

(A.9) (*Separation criterion*) Let $X \subset E$ be closed; given $x_1, x_2 \in E - X$, we let $f_i(x) = x - x_i$ $(i = 1, 2)$. Show: X does not separate E between x_1 and x_2 if and only if the fields $f_1, f_2 : X \to E - \{0\}$ are homotopic (Borsuk [1933], Granas [1962]).

(A.10) Given $X \subset Y \subset E$, a homotopy $d_t : X \to Y$ $(0 \leq t \leq 1)$ is called a *compact deformation* provided $d_0(x) = x$ for all $x \in X$ and $d_t(x) = x - D(x, t)$, where $D : X \times I \to E$ is compact. Let $X \subset E$ be closed, $x_1, x_2 \in E - X$, and $d_t : X \to E - \{x_1\} - \{x_2\}$ a compact deformation. Show: If X separates E between x_1 and x_2 then so does $d_1(X)$.

[Assuming the contrary take an arc $\varphi : I \to E - d_1(X)$ joining x_1 and x_2 and consider on X compact homotopies into $E - \{0\}$ given by $(x, t) \mapsto d_1(x) - \varphi(t)$, $(x, t) \mapsto d_t(x) - x_1$, $(x, t) \mapsto d_t(x) - x_2$. Obtain a contradiction using (A.9).]

(A.11) (*Sweeping theorem*) Given a deformation $d_t : X \to E$ $(0 \leq t < 1)$ a point $y_0 \in E - X$ is said to be *swept* by $\{d_t\}$ if $y_0 = d_t(x)$ for some $(x, t) \in X \times I$. Let $X \subset E$ be closed, and $d_t : X \to E$ be a compact deformation. Show: If $x_1, x_2 \in E - X$ belong to different components of $E - X$ and to the same component of $E - d_1(X)$, then either x_1 or x_2 must be swept by $\{d_t\}$ (Borsuk [1931], Gęba–Granas–Jankowski [1959]).

B. Surjectivity of maps. Quasi-bounded operators

Recall that a map $T : E \to F$ between normed linear spaces is *quasi-bounded* if

$$|T| = \limsup_{\|x\| \to \infty} \|x\|^{-1} \|T(x)\| < \infty;$$

$|T|$ is called the *quasi-norm* of T. A map $T : E \to F$ is *asymptotically linear* if there exists a bounded linear operator $S : E \to F$ such that

$$\lim_{\|x\| \to \infty} \frac{\|T(x) - S(x)\|}{\|x\|} = 0;$$

then $S = T'(\infty)$ is uniquely determined and called the *asymptotic derivative* of T.

(B.1) Let E be a Banach space, $F : E \to E$ an asymptotically linear completely continuous operator, and $f(x) = x - F(x)$ the corresponding field. Prove: (a) $F'(\infty)$ is completely continuous; (b) if $\|F'(\infty)\| < 1$ then f is surjective (Krasnosel'skiĭ).

[Show that F is quasi-bounded and $|F| \leq \|F'(\infty)\|$; then apply (5.5).]

(B.2) Let $f, g : E \to E$ be completely continuous fields such that (a) $\|f(x) - g(x)\| \leq \alpha\|x\| + \beta$ for some $\alpha, \beta > 0$ and all $x \in E$, and (b) g is an invertible linear vector field with $\|g^{-1}\| < 1/\alpha$. Show: f is surjective.

[Given $y_0 \in E$, apply (5.5) to get $y_0 = f(x)$ for some $x \in E$.]

(B.3) (*Intersection theorem*) Let $E = A + B$ be a normed linear space represented as a direct sum of two linear subspaces A and B with linear projections $P_A : E \to A$ and $P_B : E \to B$. Let $f(a) = a - F(a)$, $a \in A$, and $g(b) = b - G(b)$, $b \in B$, be completely continuous vector fields such that F and G are quasi-bounded, and $\|P_A\| |F| + \|P_B\| |G| < 1$. Show: $f(A) \cap g(B) \neq \emptyset$ (Granas [1962]).

[Set $x = a - b$ and consider the completely continuous field $h(x) = x - H(x)$ given by $h(x) = a - b - [G(b) - F(a)]$; prove that H is quasi-bounded and $|H| < 1$, then apply (5.5).]

C. Set contractions and condensing maps

Let E be a Banach space and $\mathscr{B}_E$ the class of bounded subsets of E. The *Kuratowski measure of noncompactness* $\alpha : \mathscr{B}_E \to R$ is defined by $\alpha(X) = \inf\{\varepsilon > 0 \mid X = \bigcup_{i=1}^n X_i$ with $\delta(X_i) < \varepsilon$ for $i = 1, \dots, n\}$. Let $X \subset E$ and $F : X \to E$ be continuous. Then F is a *set contraction* if for some constant $0 \le \lambda < 1, \alpha[F(A)] \le \lambda\alpha(A)$ for all $A \subset X$: and F is a *condensing map* if $\alpha[F(A)] < \alpha(A)$ for all $A \subset X$ with $\alpha(A) \neq 0$.

(C.1) Establish the following properties of α:
 (i) $\alpha(X) = 0 \Leftrightarrow \overline{X}$ is compact.
 (ii) $\alpha(\lambda X) = |\lambda|\alpha(X)$, $\alpha(X_1 + X_2) \le \alpha(X_1) + \alpha(X_2)$.
 (iii) $X \subset Y \Rightarrow \alpha(X) \le \alpha(Y)$; $\alpha(X \cup Y) = \max\{\alpha(X), \alpha(Y)\}$.
 (iv) $\alpha(\operatorname{conv} X) = \alpha(X)$ (Darbo [1955]).
 (v) $\alpha(X) = \alpha(\overline{X})$.
[Hint for (iv): for convex C_1, C_2 show $\alpha[\operatorname{conv}(C_1 \cup C_2)] \le \max\{\alpha(C_1), \alpha(C_2)\}$.]

(C.2) Let $\{X_n\}$ be a decreasing sequence of closed subsets of E such that $\alpha(X_n) \to 0$. Show: The intersection $X = \bigcap_{n=1}^\infty X_n$ is nonempty and compact (Kuratowski [1930]).

[Letting $x_n \in X_n$ for $n = 1, 2, \dots$, show $\{x_n\}$ has a subsequence converging to $x \in X$; observe that $\alpha(X) \le \alpha(X_k)$.]

(C.3) (*Darbo theorem*) Let C be a closed convex bounded subset of E and let $F : C \to C$ be a set contraction.

(a) Define a sequence $\{C_n\}$ of convex sets by: $C_0 = C$ and $C_n = \operatorname{Conv} F(C_{n-1})$ for $n \ge 1$. Show: $C_\infty = \bigcap_n C_n$ is a compact convex set.

(b) Prove: $F : C \to C$ has a fixed point (Darbo [1955]).

(C.4) Let $C \subset E$ be a closed convex bounded set and let $H : C \to C$ be a continuous map of the form $H = F + G$, where $F : C \to E$ is compact and $G : C \to E$ is contractive. Show: H has a fixed point (Krasnosel'skiĭ [1955]).

(C.5) Let $X \subset E$ be closed, and let $F : X \to E$ be a continuous map such that (i) for each $\varepsilon > 0$ there is an ε-fixed point for F; (ii) the field $I - F$ is proper. Show: F has a fixed point.

(C.6) (*Sadovskiĭ theorem*) Let C be a closed convex bounded subset of E. Prove: Each condensing map $F : C \to C$ has a fixed point (Sadovskiĭ [1967]).

[Assuming $0 \in C$ define $F_n = \frac{1}{n}F$ for $n = 1, 2, \dots$; show that a condensing field is proper and apply (C.5) and (C.3).]

(C.7) (*Nonlinear alternative*) Let $C \subset E$ be closed and convex, U an open subset of C containing 0, and $F : \overline{U} \to C$ a condensing map. Prove: Either (i) F has a fixed point, or (ii) there are $x \in \partial U$ and $\lambda \in (0, 1)$ with $x = \lambda F(x)$.

[See the proof of (0.3.3).]

(C.8) Let $C \subset E$ be closed and convex, $U \subset C$ an open subset containing 0, and $F : \overline{U} \to C$ a condensing map. Assume that for all $x \in \partial U$ one of the following conditions holds:
 (a) $\|F(x)\| \le \|x\|$,
 (b) $\|F(x)\| \le \|x - F(x)\|$,
 (c) $\|F(x)\| \le \sqrt{\|x\|^2 + \|x - F(x)\|^2}$.
Show: F has a fixed point.

(C.9) Let E be a Banach space, $G: E \to E$ a linear condensing map, and $F(x) = Gx + z_0$, where z_0 is a fixed element of E. Prove: If F admits a bounded orbit $\{F^k(x_0) \mid k \in N\}$, then F has a fixed point (Chow Hale [1974]).

D. Fixed point spaces

Let X be a space and $K \subset X$ a subset; by $\mathrm{Cov}_X(K)$ we denote the set of all open coverings of K in X. Given two coverings $\alpha, \beta \in \mathrm{Cov}_X(K)$, we write $\alpha \leq \beta$ provided β refines α (i.e., each member of β is contained in some member of α); clearly, $\leq$ is a preorder relation converting $\mathrm{Cov}_X(K)$ into a directed set. We let $\mathrm{Cov}_X(X) = \mathrm{Cov}(X)$.

(D.1) Let Y be a space and $\alpha \in \mathrm{Cov}(Y)$; two maps $f, g: X \to Y$ are *α-close* (written $f =_\alpha g$) provided $f(x)$ and $g(x)$ belong to a common $U_x \in \alpha$ for each $x \in X$. Let $f: Y \to Y$ be a map and $\alpha \in \mathrm{Cov}(Y)$ A point $y \in Y$ is said to be an *α-fixed point* if y and $f(y)$ belong to a common $U \in \alpha$.

(a) Let $f: Y \to Y$ be a map and $K = f(Y)$: assume that there is a cofinal family $\mathscr{D} \subset \mathrm{Cov}_Y(K)$ such that f has an α-fixed point for every $\alpha \in \mathscr{D}$. Show: f has a fixed point.

(b) Let f $Y \to Y$ be a map and $K = f(Y)$; assume that there is a cofinal family $\mathscr{D} \subset \mathrm{Cov}_Y(K)$ satisfying: for each $\alpha \in \mathscr{D}$ there is a map $f_\alpha: Y \to Y$ such that (i) f_α is α-close to f and (ii) f_α has a fixed point. Show: f has a fixed point.

(D.2) Let X be a space and $\alpha \in \mathrm{Cov}(X)$; we say that X is *α-dominated* by Y provided there are maps $s: X \to Y$ and $r: Y \to X$ such that $rs =_\alpha 1_X$. Given a class Ω of spaces, we define a new class $\mathscr{D}(\Omega)$ as follows: $X \in \mathscr{D}(\Omega)$ if and only if for each $\alpha \in \mathrm{Cov}(X)$ there is a $Y_\alpha \in \Omega$ that α-dominates X.

(a) Let $\Omega = \{X_i\}_{i \in I}$ be a family of compact spaces. Denote by $J = \{j\}$ the family of all finite subsets of I, and let $X_j = \prod_{i \in j} X_i$ for each $j \in J$ and $\Omega^* = \{X_j\}_{j \in J}$. Show: The product $X = \prod_{i \in I} X_i$ belongs to $\mathscr{D}(\Omega^*)$.

(b) Let Ω be a class of fixed point spaces. Prove: Every $X \in \mathscr{D}(\Omega)$ is a fixed point space.

(c) Let $\{X_i\}_{i \in I}$ be an infinite family of compact spaces. Show: $X = \prod_{i \in I} X_i$ is a fixed point space if and only if every finite product $\prod_{i \in j} X_i$ ($j \in J$) is a fixed point space (Dyer [1956]).

E. Fixed points in locally convex spaces

In the following results E stands for an arbitrary but fixed locally convex space.

(E.1) Let $N = \{c_1, \ldots, c_n\}$ be a finite subset of E, let p_U be a seminorm corresponding to a convex symmetric nbd U of the origin, and let $N_U = \bigcup\{c_i + U \mid i = 1, \ldots, n\}$. For each $i = 1, \ldots, n$ define $\mu_i: N_U \to \boldsymbol{R}$ by $x \mapsto \max\{0, 1 - p_U(x - c_i)\}$. The *Schauder projection* $\pi_U: N_U \to \operatorname{conv} N$ is given by

$$\pi_U(x) = \frac{1}{\sum_{i=1}^n \mu_i(x)} \sum_{i=1}^n \mu_i(x) c_i.$$

Show:

(i) $x - \pi_U(x) \in U$ for all $x \in N_U$.

(ii) If $N = \{c_1, \ldots, c_k, -c_1, \ldots, -c_k\}$ is symmetric with respect to 0, then $\pi_U(x) = -\pi_U(-x)$ for all $x \in N_U$.

(E.2) (*Approximation theorem*) Let X be a topological space, $C \subset E$ convex, and let $F\ X \to C$ be a compact map. Show: For each nbd V of the origin there exists a finite-dimensional map $F_V : X \to C$ such that $F(x) - F_V(x) \in V$ for all $x \in X$ (Leray [1950]).

(E.3) Let C be a convex subset of E, and $F\ C \to C$ a compact map. Show: F has a fixed point (Mazur [1938], Hukuhara [1950]).

(E.4) (*Antipodal theorem*) Let U be a convex symmetric nbd of the origin in E, and let $F : \overline{U} \to E$ be a compact map such that $F(-x) = -F(x)$ for all $x \in \partial U$. Show: F is an essential map (Altman [1958]).

[Use (E.2) and (A.3).]

(E.5) Let U be open in a convex $C \subset E$, and let $F : \overline{U} \to C$ be a constant map $F(x) \equiv p$ with $p \in U$. Show: F is essential.

(E.6) (*Leray-Schauder principle*) Let U be open in a convex set $C \subset E$, and let $\{H_t : \overline{U} \to C\}$ be an admissible compact homotopy such that $H_0 = F$ and $H_1 = G$, where G is the constant map sending $\overline{U}$ to a point $u_0 \in U$. Show: F has a fixed point.

[Follow the proof of (0.3.3) and use the fact that if A and B are two closed disjoint subsets of a completely regular space X such that at least one of them is compact, then there is an Urysohn function $\lambda : X \to [0,1]$ with $\lambda|A = 0$, $\lambda|B = 1$.]

(E.7) (*Topological transversality*) Let U be open in a convex subset $C \subset E$, and let $F, G \in \mathscr{K}_{\partial U}(\overline{U}, C)$ be two compact operators such that $F \simeq G$. Show: If F is essential then so is G (Granas [1976]).

(E.8) (*Nonlinear alternative*) Let $C \subset E$ be convex, U an open subset of C containing the origin 0 of E, and $F : \overline{U} \to C$ a compact map. Prove: Either

(i) F has a fixed point, or

(ii) there are $x \in \partial U$ and $\lambda \in (0,1)$ with $x = \lambda F(x)$ (Granas [1976]).

(E.9) Let $p : E \to \boldsymbol{R}^+$ be any (not necessarily continuous) function such that $p^{-1}(0) = 0$ and $p(\lambda x) = \lambda p(x)$ for all $\lambda > 0$. Let $C \subset E$ be convex, $U \subset C$ an open subset containing 0, and $F : \overline{U} \to C$ a compact map. Assume that for all $x \in \partial U$ one of the following conditions holds:

(a) $p[F(x)] \le p(x)$,

(b) $p[F(x)] \le p[F(x) - x]$,

(c) $p[F(x)] \le \{[p(x)]^k + [p(F(x) - x)]^k\}^{1/k}$ for some $k > 1$.

Show: F has a fixed point (Granas [1976]).

(E.10) (*Domain invariance*) Let $U \subset E$ be open and $f\ U \to E$ an injective compact vector field. Show:

(a) $f(U)$ is open in E,

(b) f is a homeomorphism of U onto $f(U)$

(Leray [1950], Altman [1958]).

(E.11) (*Fredholm alternative*) Let E be a locally convex space and $F : E \to E$ a completely continuous linear operator. Show: Either

(a) the equation $y = x - F(x)$ has a unique solution for each $y \in E$, or

(b) the equation $x = F(x)$ has a nontrivial solution

(Leray [1950]).

F. Topological transversality for coincidences

In this subsection X and Y are fixed Banach spaces. We let $L \in \mathcal{L}(X, Y)$ be a Fredholm operator of index zero, i.e., $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$.

We use the following notation:

(i) Y_1 is a closed complement to $Y_2 = \operatorname{Im} L$ in Y, and $Q \in \mathcal{L}(Y, Y_1)$ is a linear projection of Y onto Y_1 along Y_2;

(ii) X_2 is a closed complement to $X_1 = \operatorname{Ker} L$ in X, and $P \in \mathcal{L}(X, X_1)$ is a linear projection of X onto X_1 along X_2.

We have topological direct sums $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$, and because L is of index zero, $\dim X_1 = \dim Y_1$; we let $J : X_1 \to Y_1$ be a linear isomorphism.

(F.1) Let $N : X \to Y$ be a continuous (possibly nonlinear) operator. Show:

(a) The linear operator $L + JP : X \to Y$ is invertible and its inverse is $(L + JP)^{-1} = L_2^{-1}(I - Q) + J^{-1}Q$, where $L_2 \in \mathcal{L}(X_2, Y)$ is the restriction of L to X_2.

(b) The coincidence problem $Lx = Nx$ is equivalent to the fixed point problem $Mx = x$, where $M = (L + JP)^{-1}(N + JP)$.

(F.2) Let $U \subset X$ be open, and let $\mathscr{K}_{\partial U}(\overline{U}, Y; L)$ be the set of all maps $f : (\overline{U}, \partial U) \to (Y, Y - \{0\})$ of the form $f = L - F$, where $F : \overline{U} \to Y$ is a compact operator. Two maps $f, g \in \mathscr{K}_{\partial U}(\overline{U}, Y; L)$ are called *L-homotopic* if there is a compact homotopy $H_t : \overline{U} \to Y$ such that $h_t = L - H_t \in \mathscr{K}_{\partial U}(\overline{U}, Y; L)$ for each $t \in [0, 1]$ and $h_0 = f$, $h_1 = g$. Assume $f, g \in \mathscr{K}_{\partial U}(\overline{U}, Y; L)$ are two maps $f = L - F$, $g = L - G$ such that $F(\overline{U}), G(\overline{U}) \subset Y_1$. Let $U_1 = U \cap X_1$ and $F_1 = F|\overline{U}_1$, $G_1 = G|\overline{U}_1$. Show: If the maps $F_1, G_1 : (\overline{U}_1, \partial U_1) \to (Y_1, Y_1 - \{0\})$ are homotopic, then f and g are L-homotopic.

(F.3) (*Topological transversality for coincidences*) We say that a map $f \in \mathscr{K}_{\partial U}(\overline{U}, Y; L)$ is *L-essential* if any $g \in \mathscr{K}_{\partial U}(\overline{U}, Y; L)$ with $f|\partial U = g|\partial U$ has a zero in U. Show: If $f, g \in \mathscr{K}_{\partial U}(\overline{U}, Y; L)$ are L-homotopic, then f is L-essential if and only if g is L-essential.

(F.4) (*Antipodal theorem for coincidences*) Let U be an open bounded convex subset of X and $c \in X$. Call U *symmetric with respect to c* if $x \in U \Leftrightarrow 2c - u \in U$. Let U be such a set, and let $k : \overline{U} \to Y$ We say that k is *odd* on $\overline{U}$ (respectively on ∂U) if $k(2c - x) = -k(x)$ for all $x \in \overline{U}$ (respectively $x \in \partial U$).

Let U as above be symmetric with respect to some $c \in \operatorname{Ker} L$. Let $f = L - F \in \mathscr{K}_{\partial U}(\overline{U}, Y; L)$ be odd on ∂U. Show: f is L-essential.

[Consider the compact map $G : \overline{U} - c \to Y$ given by $G(x - c) = F(x)$ for $x \in \overline{U}$ and the compact map $M = (L + JP)^{-1}(G + JP) : \overline{U} - c \to X$. Use (F.1) and Borsuk's fixed point theorem.]

(F.5) Let U be as in (F.4), and let $g = L - G \in \mathscr{K}_{\partial U}(\overline{U}, Y; L)$ be such that $G(\overline{U}) \subset Y_1$ and $G|\overline{U}_1 : (\overline{U}_1, \partial U_1) \to (Y_1, Y_1 - \{0\})$ is homotopic to an odd map, where $U_1 = U \cap X_1$. Show: g is L-essential.

[Prove that g is L-homotopic to an odd map and use (F.4) and (F.3).]

(F.6) (*Nonlinear alternative for coincidences*) Let $N : X \to Y$ be completely continuous. Let U be as in (F.4). Assume furthermore that

(a) $QNx \neq 0$ for all $x \in U_1 = U \cap X_1$,

(b) the map $QN : (\overline{U}_1, \partial U_1) \to (Y_1, Y_1 - \{0\})$ is homotopic to an odd map.

Show: Either (i) $Lx = Nx$ for some $x \in U$, or (ii) there are $y \in \partial U$ and $\lambda \in (0, 1)$ such that $Ly = \lambda Ny$.

(The results (F.1)–(F.6) are from Granas Guenther Lee [1991].)

10. Notes and Comments

Compact operators

Compact and completely continuous operators occur in many problems of classical analysis. In the nonlinear case, the first comprehensive research on compact operators with numerous applications to partial differential equations (both linear and nonlinear) was due to Schauder. In particular, the approximation technique for compact operators in Banach spaces goes back to Schauder; the proof of (2.3) given in the text is, with slight modifications, that of Leray–Schauder [1934]. Theorem (2.3) extends to locally convex spaces (see the survey of Leray [1950] and Nagumo [1951]) and also to some linear topological spaces that are not locally convex (Klee [1960]). The proof of (2.4) was communicated to the authors by L. Waelbroeck; we remark that (2.4) is valid in any linear metric space for which the Schauder approximation theorem holds. Theorem (2.5) on the extension of compact mappings was first proved (for Banach spaces) in Krasnosel'skiĭ's book [1964a]; the proof in the text is taken from an unpublished paper of Fournier and Granas.

Extension of the theorems of Brouwer and Borsuk

Theorem (2.3), due to Schauder [1930], evolved from a number of special results concerned with extensions of the Brouwer theorem to the case of function spaces. The first important result in this direction was due to Birkhoff–Kellogg [1922], who generalized the Brouwer theorem to compact convex subsets of $L^2(0,1)$ and $C^n(0,1)$. Schauder [1927a], [1927b] extended first the above result to Banach spaces of type (S) with a basis and then in [1930] to an arbitrary Banach space. (A Banach space E is called an *S-space* provided each bounded sequence in E has a weakly convergent subsequence. This type of spaces, introduced by Schauder [1927a], turned out to be especially useful in applications; we remark that several years later Eberlein proved that Banach spaces of type (S) coincide with reflexive spaces.) In [1930] Schauder also obtained the following theorem: *If C is a convex weakly compact subset of a separable Banach space, then every weakly continuous map $f : C \to C$ has a fixed point* (in connection with this result see also Arino–Gautier–Penot [1984]). Mazur [1930] proved that the convex closure of a compact set in a Banach space is compact; this implies a more general version of the Schauder theorem that is especially convenient in applications: *If C is a convex closed subset of a Banach space, then every compact map $f : C \to C$ has a fixed point.*

Theorem (3.3) was established (in a somewhat different form) with the aid of the Leray–Schauder degree by Krasnosel'skiĭ [1951].

Fixed points of discontinuous maps

Let X, Y be two spaces, let $f : X \to Y$ be a function, and let G_f stand for the graph of f. The function f is called a *connectivity function* if for each connected subset $A \subset X$ the graph $G_{f|A}$ is connected. Stallings [1959] obtained a generalization of the Brouwer theorem by showing that the unit ball in $\boldsymbol{R}^n$ is a fixed point space for connectivity functions. The following extension of the Schauder theorem was established by Girolo [1981]: *If C is a closed convex subset of a Banach space and $f : C \to C$ is a connectivity function such that $\overline{f(C)}$ is compact, then f has a fixed point.*

Topological transversality

Homotopy arguments for compact operators were introduced by Leray and Schauder in the context of their theory of the degree for compact fields in Banach spaces. Using homotopy invariance of the degree, Leray–Schauder [1934] developed an important theoretical principle that abstracts (in the nonlinear context) the classical method of continuation of solutions along a parameter. The approach presented in the text does not use the degree theory and is based on the notion of the essential map and on the topological transversality theorem (Granas [1976]); its essential feature is that it extends to other classes of operators and gives results that cannot be obtained by using directly the Leray–Schauder degree. An alternative but less general method based on the homotopy extension theorem was developed earlier for Banach spaces in Granas [1959a], [1962]; this method was suggested by prior finite-dimensional results of Borsuk [1931b], [1937]. We remark that (as shown by Simon Volkmann [1988]) the notions of topological transversality theory also permit one to establish localization and multiplicity results for nonlinear problems (see "Miscellaneous Results and Examples" in §7).

For an extension of topological transversality to the class of condensing operators the reader is referred to Krawcewicz [1988]. For topological transversality in the coincidence setting (for various classes of operators) see Volkmann [1984], Granas–Guenther Lee [1991], Gęba–Granas–Kaczynski–Krawcewicz [1985]; for closely related results see Furi–Martelli Vignoli [1978], [1980], Precup [1995].

The Leray Schauder principle and nonlinear alternative

For open subsets in Banach spaces Theorem (5.1) was established by the degree-theoretic method in Leray Schauder [1934]; for arbitrary convex sets see Granas [1976]. The nonlinear alternative, which follows at once from (5.1), quickly yields a number of fixed point results. In connection with (5.3) see Rothe [1938], Altman [1955], and Granas [1976]. Theorem (5.4)

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V. Klee, NN, J. Leray, K. Fan, Baton Rouge, 1967

was proved first for Banach spaces in the context of degree theory by Leray Schauder [1934]; a direct elementary proof of (5.4) was given by Schaefer [1955] (see also Potter [1972], Reich [1976], [1979] and Granas [1993]).

Birkhoff–Kellogg theorem

Theorem (6.1) was established in Birkhoff and Kellogg [1922]; for another proof of (6.1) (based on the sweeping theorem (A.11)) see Gęba Granas-Jankowski [1959]. Topological transversality and (2.5) can be used to get the following theorem (Krasnosel'skiĭ Ladyzhenskiĭ [1954] and Granas [1962]): *Let C be a positive cone in a normed space E, $U \subset C$ an open set containing* 0, *and let $F : \overline{U} \subset C$ be a compact map such that $\|F(x)\| \geq a > 0$ for all $x \in \partial U$. Then there are $\lambda > 0$ and $x \in \partial U$ such that $x = \lambda F(x)$.*

Compact fields

This notion was introduced by Schauder and Leray (Schauder [1929], Leray Schauder [1934], Leray [1935]); the terms "completely continuous field" and "compact field" appeared in Rothe [1937a] and Granas's tract [1962]. J. Schauder and J. Leray discovered that many facts of finite-dimensional topology can be carried over to infinite dimensions provided attention is restricted to the class of compact fields. In particular, for compact fields a

generalization of Brouwer's degree theory was developed and with its aid various applications were obtained. Among these the following generalization of the Jordan Alexandroff theorem was established by Leray [1935]: *Let X and Y be two closed bounded subsets of a Banach space E, and $f : X \to Y$ a homeomorphism. If f is a compact field, then the complements $E - X$ and $E - Y$ have the same number of components.* For more details see Rothe [1937], [1938], Granas's tract [1962], and Krasnosel'skiĭ's book [1964b].

Invariance of domain

The invariance of domain theorem in $\boldsymbol{R}^n$ is due to Brouwer [1912]. Schauder [1929] extended the Brouwer theorem to compact fields in Banach spaces with a basis and of type (S). A general result (for arbitrary Banach spaces) was established (using the degree theory for compact fields) by Leray [1935]. The proof of (8.4) presented in the text (and based on Borsuk's theorem) is found in Granas [1958]. Using invariance of domain as a tool, Schauder [1929], [1932] developed an important theoretical method for solving nonlinear equations, applicable to problems in which the uniqueness implies the existence of a solution. Schauder [1932] also gave significant applications of the method to nonlinear elliptic equations. For more recent uses of Schauder's method, see Chow–Lasota [1973], where some further references can be found.

Surjectivity results

The first result of this type was established by Borsuk [1933c], who proved that: *if $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is an ε-map in the narrow sense, then f is surjective*; Theorem (8.3)(b), extending the above result to normed linear spaces, and also (8.6) are contained in Granas [1958]. For Theorem (5.5), see Granas [1957]; a similar argument to that in the proof of (5.5) can be applied to obtain the following theorem: *Let E be a reflexive Banach space and let $F : E \to E$ be a quasi-bounded map with $|F| < 1$. Assume that one of the following conditions is satisfied:*

(i) *E has normal structure and F is nonexpansive,*

(ii) *F is weakly continuous.*

Then $x \mapsto x - Fx$ is surjective.

Theorem (8.3)(b) combined with the Banach Mazur theorem gives the following result: *Let E be a normed linear space with* $\dim E \geq 3$. *Let $f : E \to E$ be a completely continuous field such that:*

(i) *f is locally invertible for all $x \in E$ with $\|x\| > M$, where $M > 0$ is a constant,*

(ii) $\|f(x)\| \to \infty$ *as* $\|x\| \to \infty$.

Then f is surjective.

Fixed points in locally convex spaces

The main results in the text extend to locally convex spaces. A number of such results (Tychonoff and Schauder–Tychonoff theorems) are given in §7.

Borsuk's antipodal theorem was generalized to locally convex spaces by Altman [1958a]. Leray [1950] extended the invariance of domain theorem to locally convex spaces and applied it to establish the Fredholm alternative in such spaces; another proof of the invariance of domain similar to that given in the text and based on Borsuk's theorem can be found in Altman [1958b].

Fixed points in arbitrary linear topological spaces

Some results in the text extend also to spaces that are not locally convex. Fan [1964], using his coincidence theorem (7.1.3), proved that the Tychonoff fixed point theorem (cf. Tychonoff [1935]) is valid in linear topological spaces with sufficiently many linear functionals. A few other fixed point results of this type are given in §7.

Following Klee [1960a], we call a linear topological space E *admissible* if for any $X \subset E$ every compact map $\varphi : X \to E$ is approachable by finite-dimensional maps (meaning that for any nbd V of the origin in E, there is a finite-dimensional map $\varphi_V : X \to E$ with $\varphi_V(x) \in \varphi(X) + V$ for $x \in X$). It was shown by Klee [1960a], using arguments similar to those given in the text, that the antipodal theorem (3.3) and the invariance of domain (8.4) can be established in arbitrary admissible linear topological spaces.

For more details about fixed point results in nonlocally convex spaces the reader is referred to Klee [1960a], Granas [1976], S. Hahn [1978], to the lecture notes by Riedrich [1975] and Granas [1980], and also to Hadžić's book [1984], where further references may be found.

Schauder–Tychonoff theorem in linear topological spaces

The study of fixed points in spaces that are not locally convex evolved from the problem, posed by Schauder, whether a compact convex subset of an arbitrary linear topological space has the fixed point property (problem 54 in the Scottish Book). This problem proved to be very difficult and for over 65 years defied the efforts of many mathematicians. An affirmative answer was given only recently by Cauty [2001], who proved the following: *Let E be a linear topological space, and let $C \subset E$ be convex. Then any compact map $F : C \to C$ has a fixed point.* This result together with Theorem (0.3.3) implies that the Leray–Schauder principle (5.1) and the nonlinear alternative (5.2) remain valid in arbitrary linear topological spaces.

§7. Further Results and Applications

This paragraph is primarily devoted to illustrating the use of the results established in this chapter; among the applications given, we obtain a number of significant results from various areas of mathematics. The first section contains extensions and applications of the Brouwer theorem; the main feature is the systematic use of a very versatile technique introduced by Fan based on the topological KKM-principle. In the following sections, we give applications of the Borsuk and Schauder theorems, as well as of topological transversality. The paragraph ends with a generalized Schauder theorem, fixed point results for Kakutani maps and the Ryll-Nardzewski theorem.

1. Applications of the Topological KKM-Principle

a. Fixed points and coincidences for set-valued maps of Fan

We first establish two geometric results for some classes of set-valued maps that have proved of importance in convex analysis, game theory and mathematical economics.

For the convenience of the reader we recall some notation and terminology. If $S : X \to 2^Y$ is a set-valued map, its inverse $S^{-1} : Y \to 2^X$ and its dual $S^* : Y \to 2^X$ are the maps $y \mapsto S^{-1}y = \{x \in X \mid y \in Sx\}$ and $y \mapsto S^*y = X - S^{-1}y$. The values of S^{-1} (respectively of S^*) are the fibers (respectively the cofibers) of the map S. The sets $G_S = \{(x, y) \in X \times Y \mid y \in Sx\}$ and $S(X) = \bigcup\{Sx \mid x \in X\}$ are the graph and image of S, respectively. Note that S is surjective [i.e., $S(X) = Y$] if and only if all its fibers $S^{-1}y$ are nonempty. By a fixed point of a set-valued map $S : X \to 2^X$ is meant a point $x_0 \in X$ for which $x_0 \in Sx_0$. Clearly, if S has a fixed point, then so does S^{-1}

(1.1) DEFINITION. Let X and Y be two subsets of linear topological spaces and $T, S : X \to 2^Y$ be two set-valued maps.

(i) If Y is convex, then $T : X \to 2^Y$ is called a *Fan map* (or simply an $\mathbb{F}$*-map*) provided T has nonempty convex values and open fibers.

(ii) If X is convex, then $S : X \to 2^Y$ is called an $\mathbb{F}^*$*-map* provided S has open values and nonempty convex fibers.

The set of all $\mathbb{F}$-maps (respectively $\mathbb{F}^*$-maps) from X to 2^Y is denoted by $\mathbb{F}(X, Y)$ (respectively $\mathbb{F}^*(X, Y)$). We note that $S \in \mathbb{F}^*(X, Y)$ if and only if $S^{-1} \in \mathbb{F}(Y, X)$.

Throughout this section, unless explicitly stated otherwise, by a "compact convex set" we mean a nonempty compact convex subset of some linear topological space.

(1.2) THEOREM (Fan Browder). *Let X be a compact convex set, and let $T : X \to 2^X$ be such that either $T \in \mathbb{F}(X, X)$ or $T \in \mathbb{F}^*(X, X)$. Then T has a fixed point.*

PROOF. It suffices to consider the case of an $\mathbb{F}$-map. Note first that the values $T^*(y) = X - T^{-1}(y)$ of T^* are compact. From the surjectivity of T^{-1} we get

$$\bigcap\{T^*(y) \mid y \in X\} = \bigcap\{X - T^{-1}(y) \mid y \in X\}$$
$$= X - \bigcup\{T^{-1}(y) \mid y \in X\} = \emptyset,$$

and thus T^* cannot be a KKM-map. Because the values of T are convex and T^* is not a KKM-map, we conclude, in view of (3.1.2), that T has a fixed point. □

Given two set-valued maps $S, T : X \to 2^Y$ we say that S and T have a *coincidence* provided there exists $(x_0, y_0) \in X \times Y$ such that $y_0 \in Sx_0 \cap Tx_0$.

(1.3) THEOREM (Fan coincidence theorem). *Let X and Y be compact convex sets. Let $S, T : X \to 2^Y$ be such that $S \in \mathbb{F}^*(X, Y)$ and $T \in \mathbb{F}(X, Y)$. Then S and T have a coincidence.*

PROOF. Let $Z = X \times Y$ and define $H : Z \to 2^Z$ by $z = (x, y) \mapsto T^{-1}y \times Sx$. Because the values of H are open and the fibers $H^{-1}(x', y') = S^{-1}y' \times Tx'$ of H are nonempty and convex, it follows that H is an $\mathbb{F}^*$-map. By (1.2), we find a fixed point $(x_0, y_0) \in T^{-1}y_0 \times Sx_0$. Thus $y_0 \in Sx_0 \cap Tx_0$. □

We give an immediate application of the coincidence theorem to game theory by establishing a general version of the von Neumann minimax principle due to M. Sion.

Recall that a real-valued function $f : X \to \boldsymbol{R}$ on a topological space is lower [respectively upper] semicontinuous if $\{x \in X \mid f(x) > r\}$ [respectively $\{x \in X \mid f(x) < r\}$] is open for each $r \in \boldsymbol{R}$; if X is a convex set in a linear space, then f is quasi-concave [respectively quasi-convex] if $\{x \in X \mid f(x) > r\}$ [respectively $\{x \in X \mid f(x) < r\}$] is convex for each $r \in \boldsymbol{R}$.

(1.4) THEOREM (von Neumann–Sion minimax principle). *Let X and Y be compact convex sets, and let $f : X \times Y \to \boldsymbol{R}$ satisfy:*
(i) $y \mapsto f(x, y)$ is l.s.c. and quasi-convex on Y for each $x \in X$,
(ii) $x \mapsto f(x, y)$ is u.s.c. and quasi-concave on X for fixed $y \in Y$.
Then

$$\max_x \min_y f(x, y) = \min_y \max_x f(x, y).$$

PROOF. Because of upper semicontinuity, $\max_x f(x, y)$ exists for each y and is a lower semicontinuous function of y, so $\min_y \max_x f(x, y)$ exists; similarly,

$\max_x \min_y f(x,y)$ exists. Since $f(x,y) \le \max_x f(x,y)$, we have

$$\min_y f(x,y) \le \min_y \max_x f(x,y);$$

therefore

$$\max_x \min_y f(x,y) \le \min_y \max_x f(x,y).$$

We shall show that strict inequality cannot hold. For assume it did; then there would be some r with

$$\max_x \min_y f(x,y) < r < \min_y \max_x f(x,y).$$

Define $S, T : X \to 2^Y$ by

$$Sx = \{y \mid f(x,y) > r\} \quad \text{and} \quad Tx = \{y \mid f(x,y) < r\}.$$

These set-valued maps would then satisfy the conditions of Theorem (1.3): Each Sx is open by the lower semicontinuity of $y \mapsto f(x,y)$, each Tx is convex by the quasi-convexity of $y \mapsto f(x,y)$, and is nonempty because $\max_x \min_y f(x,y) < r$ Since

$$S^{-1}y = \{x \mid f(x,y) > r\} \quad \text{and} \quad T^{-1}y = \{x \mid f(x,y) < r\},$$

we find in the same way that each $S^{-1}y$ is nonempty and convex, and each $T^{-1}y$ is open. Thus, $S \in \mathbb{F}^*(X,Y)$ and $T \in \mathbb{F}(X,Y)$; then by (1.3), there would be some (x_0, y_0) with $y_0 \in Sx_0 \cap Tx_0$, which gives the contradiction $r < f(x_0, y_0) < r$. □

b. Analytic formulations of the geometric results and minimax inequalities

We now give analytic formulations of the geometric results (1.2) and (1.3).

(1.5) THEOREM. *Let X be a compact convex set, and let $f, g : X \times X \to \mathbf{R}$ satisfy:*
(i) *$y \mapsto f(x,y)$ is l.s.c. for each $x \in X$,*
(ii) *$x \mapsto f(x,y)$ is quasi-concave for each $y \in X$.*
Then, for any $\lambda \in \mathbf{R}$, one of the following properties holds:
(a) *there exists a $y_0 \in X$ such that $f(x, y_0) \le \lambda$ for all $x \in X$,*
(b) *there exists a $w \in X$ such that $f(w,w) > \lambda$.*

PROOF. Let $\lambda \in \mathbf{R}$ and define $S : X \to 2^X$ by $Sx = \{y \in X \mid f(x,y) > \lambda\}$. According to (i) and (ii), S has open values and convex fibers. The map S is either not surjective, or surjective. In the first event, we have $S^{-1}y_0 = \emptyset$ for some $y_0 \in X$, i.e., $f(x, y_0) \le \lambda$ for all $x \in X$, and thus property (a) holds. In the second event, by the Fan–Browder theorem (1.2), S has a fixed point $w \in Sw$, and therefore $f(w, w) > \lambda$. □

(1.6) THEOREM. *Let X and Y be two compact convex sets, and let $f, g : X \times Y \to \boldsymbol{R}$ satisfy:*

(i) $f(x,y) \leq g(x,y)$ *for all* $(x,y) \in X \times Y$,
(ii) $y \mapsto f(x,y)$ *is l.s.c. on* Y *for each* $x \in X$,
(iii) $x \mapsto f(x,y)$ *is quasi-concave on* X *for each* $y \in Y$,
(iv) $y \mapsto g(x,y)$ *is quasi-convex on* Y *for each* $x \in X$,
(v) $x \mapsto g(x,y)$ *is u.s.c. on* X *for each* $y \in Y$

Then, for any $\lambda \in \boldsymbol{R}$, *one of the following properties holds:*

(a) *there exists a* $y_0 \in Y$ *such that* $f(x, y_0) \leq \lambda$ *for all* $x \in X$,
(b) *there exists an* $x_0 \in X$ *such that* $g(x_0, y) \geq \lambda$ *for all* $y \in Y$.

PROOF. Let $\lambda \in \boldsymbol{R}$; we define $S : X \to 2^Y$ and $T : Y \to 2^X$ by

$$Sx = \{y \in Y \mid f(x,y) > \lambda\} \quad \text{and} \quad Ty = \{x \in X \mid g(x,y) < \lambda\}.$$

According to (ii)–(v), both S and T have open values and convex (possibly empty) fibers. Observe now that either

(*) one among S and T is not surjective, or
(**) both S and T are surjective.

Consider first the event (*): if S is not surjective, then for some y_0, we have $S^{-1}y_0 = \emptyset$, i.e., $f(x, y_0) \leq \lambda$ for all $x \in X$, and thus the property (a) holds. Similarly, if T is not surjective, then (b) is satisfied. In the event (**), both S and T would be $\mathbb{F}^*$-maps, and therefore, by Theorem (1.3), there would be some (x_0, y_0) with $y_0 \in Sx_0 \cap T^{-1}x_0$, which gives the contradiction $\lambda < f(x_0, y_0) \leq g(x_0, y_0) < \lambda$. Thus, the case (**) is excluded, and the proof is complete. □

Theorems (1.5) and (1.6) imply at once two general principles in the topological KKM theory.

(1.7) THEOREM (Fan minimax inequality). *Let C be a compact convex set, and let $f : C \times C \to \boldsymbol{R}$ satisfy:*

(i) $y \mapsto f(x,y)$ *is l.s.c. for each* $x \in C$,
(ii) $x \mapsto f(x,y)$ *is quasi-concave for each* $y \in C$.

Then the following minimax inequality holds:

$$\inf_{y \in C} \sup_{x \in C} f(x,y) \leq \sup_{x \in C} f(x,x).$$

PROOF. We may assume that $\lambda = \sup_{x \in C} f(x,x) < \infty$. By (1.5), there is a $y_0 \in C$ such that $f(x, y_0) \leq \lambda$ for all $x \in C$. Then $\sup_{x \in C} f(x, y_0) \leq \lambda$ and

$$\inf_{y \in C} \sup_{x \in C} f(x,y) \leq \sup_{x \in C} f(x, y_0) \leq \lambda. \qquad \square$$

(1.8) THEOREM (F.C. Liu minimax inequality). *Let X and Y be compact convex sets, and let $f, g : X \times Y \to \boldsymbol{R}$ satisfy:*

(i) $f(x,y) \leq g(x,y)$ *for all* $(x,y) \in X \times Y$,
(ii) $y \mapsto f(x,y)$ *is l.s.c. on* Y *for each* $x \in X$,
(iii) $x \mapsto f(x,y)$ *is quasi-concave on* X *for each* $y \in Y$,
(iv) $y \mapsto g(x,y)$ *is quasi-convex on* Y *for each* $x \in X$,
(v) $x \mapsto g(x,y)$ *is u.s.c. on* X *for each* $y \in Y$

Then
$$\alpha = \inf_{y \in Y} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} \inf_{y \in Y} g(x,y) = \beta.$$

PROOF. Suppose that the inequality does not hold. Then for some $\lambda \in \boldsymbol{R}$ we have $\beta < \lambda < \alpha$. We now apply (1.6): assume there is a $y_0 \in Y$ such that $f(x,y_0) \leq \lambda$ for all $x \in X$; then $\alpha = \inf_{y \in Y} \sup_{x \in X} f(x,y) \leq \lambda < \alpha$, a contradiction.

In a similar way, the assumption that for some $x_0 \in X$, $g(x_0, y) \geq \lambda$ for all $y \in Y$ leads to a contradiction. □

We observe that by taking $f = g$ in (1.8), we obtain as a special case the von Neumann–Sion theorem (1.4).

c. Applications to fixed point theory. Theorems of Tychonoff and Schauder–Tychonoff

We now give the simplest applications of KKM-maps to fixed point theory. First we extend Theorem (5.8.4) to normed spaces.

(1.9) THEOREM (Fan). *Let* C *be a nonempty compact convex set in a normed space* E. *Let* $f : C \to E$ *be continuous and such that for each* $x \in C$ *with* $x \neq f(x)$ *the line segment* $[x, f(x)]$ *contains at least two points of* C. *Then* f *has a fixed point.*

PROOF. Define $G : C \to 2^C$ by
$$Gx = \{y \in C \mid \|y - fy\| \leq \|x - fy\|\}.$$

Because f is continuous, the sets Gx are closed, therefore compact. Clearly, G is strongly KKM, and hence because C is convex, it is a KKM-map. By the topological KKM-principle, we find a point y_0 such that $y_0 \in \bigcap\{Gx \mid x \in C\}$, and hence $\|y_0 - fy_0\| \leq \|x - fy_0\|$ for all $x \in C$. We claim that y_0 is a fixed point: if not, the segment $[y_0, fy_0]$ must contain a point of C other than y_0, say $x = ty_0 + (1-t)fy_0$ for some $0 < t < 1$; then $\|y_0 - fy_0\| \leq t\|y_0 - fy_0\|$, and since $t < 1$, we must have $\|y_0 - fy_0\| = 0$. □

The theorem just proved implies that any continuous self-map of a compact convex set in a normed space has a fixed point. We extend this result to arbitrary locally convex spaces.

(1.10) THEOREM (Tychonoff). *Let C be a nonempty compact convex set in a locally convex linear topological space E. Then every continuous $F : C \to C$ has a fixed point.*

PROOF. Let $\{p_i\}_{i\in I}$ be the family of all continuous seminorms in E. For each $i \in I$ set

$$A_i = \{y \in C \mid p_i(y - Fy) = 0\}.$$

A point $y_0 \in C$ is a fixed point for F if and only if $y_0 \in \bigcap_{i\in I} A_i$. By compactness of C we need to show only that each finite intersection $A_{i_1} \cap \cdots \cap A_{i_n}$ is nonempty. Given $(i_1, \ldots, i_n)$ define $G : C \to 2^E$ by

$$Gx = \Big\{ y \in C \,\Big|\, \sum_{j=1}^{n} p_{i_j}(y - Fy) \le \sum_{j=1}^{n} p_{i_j}(x - Fy) \Big\}.$$

It is easily seen that G is strongly KKM, and so KKM since C is convex. By the topological KKM-principle there is a point $y_0 \in C$ such that

$$\sum_{j=1}^{n} p_{i_j}(y_0 - Fy_0) \le \sum_{j=1}^{n} p_{i_j}(x - Fy_0) \quad \text{for all } x \in C.$$

Therefore, $p_{i_j}(y_0 - Fy_0) = 0$ $(1 \le j \le n)$, and thus $y_0 \in A_{i_1} \cap \cdots \cap A_{i_n}$. □

Using Theorem (1.5) we now extend the Schauder fixed point theorem to locally convex spaces. We first establish two preliminary results.

(1.11) LEMMA. *Let E be a locally convex linear topological space, $X \subset E$, and let $f : X \to X$ be a compact map. Letting $\mathscr{V}$ be the system of all open symmetric convex neighborhoods of zero in E, assume that for each $U \in \mathscr{V}$, f has a U-fixed point (i.e., a point $x_0 \in X$ such that $f(x_0) - x_0 \in U$). Then f has a fixed point.*

PROOF. Suppose to the contrary that f has no fixed points. Then for each $x \in X$ we can find $V_x, W_x \in \mathscr{V}$ such that:

(i) $(x + V_x) \cap (f(x) + W_x) = \emptyset$,

(ii) $f((x + V_x) \cap X) \subset f(x) + W_x$.

Since $\overline{f(X)}$ is compact, there is a finite set $\{x_1, \ldots, x_k\} \subset \overline{f(X)} \subset X$ such that

$$f(X) \subset \bigcup_{i=1}^{k} (x_i + \tfrac{1}{2}V_{x_i}).$$

Define $U = \bigcap\{\frac{1}{2}V_{x_i} \mid i = 1, \ldots, k\}$ and let x be any point in X. For some i, we have $f(x) \in x_i + \frac{1}{2}V_{x_i}$. We claim that $x \notin x_i + V_{x_i}$: if not, we would have, because of (ii), $f(x) \in f(x_i) + W_{x_i}$, and therefore, by (i), $f(x) \notin x_i + V_{x_i}$, which is a contradiction. Thus $x \notin x_i + V_{x_i}$; since $f(x) + \frac{1}{2}V_{x_i} \subset x_i + V_{x_i}$,

we infer that $x \notin f(x) + \frac{1}{2}V_{x_i}$, and therefore $x - f(x) \notin U$. This implies that f has no U-fixed points; we have thus obtained a contradiction. □

(1.12) LEMMA. *Let C be a nonempty compact convex subset of a linear topological space E, U an open symmetric convex neighborhood of* 0, *and $f : C \to E$ a continuous map such that $f(C) \subset C + U$. Then f has a U-fixed point.*

PROOF. Define $T : C \to 2^C$ by $Tx = \{y \in C \mid y \in f(x)+U\} = C \cap (f(x)+U)$ for $x \in C$ and note that the values of T are nonempty and convex and its fibers $T^{-1}y = \{x \in C \mid f(x) \in y - U\}$ are open (by the continuity of f); thus $T \in \mathbb{F}(C, C)$. Now (1.2) gives a point $x_0 \in f(x_0) + U$ □

(1.13) THEOREM (Schauder–Tychonoff). *Let C be a nonempty convex subset of a locally convex linear topological space E, and let $F : C \to C$ be a compact map. Then F has a fixed point.*

PROOF. Let V be a convex symmetric neighborhood of zero. By Lemma (1.11), because E is locally convex, it is enough to show that F has a V-fixed point, i.e., a point x_0 such that $F(x_0) \in x_0 + V$ Let $\{x_i + V\}_{i=1}^k$ be a finite covering of the compact set $\overline{F(C)}$, and let $K = \operatorname{conv}\{x_1, \dots, x_k\}$. Since $F(K) \subset K + V$, there exists by Lemma (1.12) a point $x_0 \in K \subset C$ such that $F(x_0) \in x_0 + V$ □

d. Fixed points in spaces with sufficiently many linear functionals

We now establish some general fixed point results in linear topological spaces with sufficiently many linear functionals. We recall that given such a space E, its adjoint E^* separates the elements of E. By the Hahn–Banach theorem, every locally convex space has sufficiently many linear functionals.

In this subsection E stands for a linear topological space with sufficiently many linear functionals.

We first give some extensions of Theorem (1.9).

(1.14) THEOREM. *Let C be a nonempty compact convex subset of E. Let $f : C \to E$ be continuous such that for each $y \in C$ with $y \neq f(y)$ there exists a λ (real or complex depending on whether E is real or complex) with $|\lambda| < 1$ and $\lambda y + (1 - \lambda)f(y) \in C$. Then f has a fixed point.*

PROOF. Assume $f(y) \neq y$ for all $y \in C$ and consider the compact set $K = \{y - f(y) \mid y \in C\}$; clearly, $0 \notin K$. For each $x \in K$ there is a linear functional $l_x \in E^*$ with $l_x(x) \neq 0$; by continuity of l_x there is a neighborhood U_x of x such that $l_x(y) \neq 0$ for all $y \in U_x$. Let $\{U_{x_1}, \dots, U_{x_k}\}$ be a finite subcovering of the covering $\{U_x\}_{x \in K}$ of the compact set K and

put $p(y) = \sum_{i=1}^{k} |l_{x_i}(y)|$ for each $y \in E$. Then p is a continuous seminorm on E such that $p(y) > 0$ for all $y \in K$. For this seminorm we apply, as in (1.10), the topological KKM-principle to get $y_0 \in C$ such that

$$(*) \qquad 0 < p(y_0 - f(y_0)) \leq p(x - f(y_0)) \quad \text{for all } x \in C.$$

By the hypothesis applied to y_0, there exists a λ with $|\lambda| < 1$ such that $x = \lambda y_0 + (1-\lambda) f(y_0) \in C$. Then from $(*)$ we get

$$0 < p(y_0 - f(y_0)) \leq |\lambda| p(y_0 - f(y_0)),$$

and since $|\lambda| < 1$, this is a contradiction. □

As a special case of Theorem (1.14) we obtain

(1.15) THEOREM. *Let C be a nonempty compact convex subset of E. Let $f : C \to E$ be continuous and such that for each $x \in C$ with $x \neq f(x)$, the line segment $[x, f(x)]$ contains at least two points of C. Then f has a fixed point.* □

As our last result we derive a version of a fixed point theorem due to F. Browder and B. Halpern.

Let C be a convex subset of a vector space E; for each $x \in C$, let

$$I_C(x) = \{y \in E \mid y = x + \lambda(y_0 - x) \text{ for some } y_0 \in C \text{ and } \lambda > 0\},$$
$$O_C(x) = \{y \in E \mid y = x - \lambda(y_0 - x) \text{ for some } y_0 \in C \text{ and } \lambda > 0\}.$$

A map $f : C \to E$ is said to be *inward* (respectively *outward*) if $f(x) \in I_C(x)$ (respectively $f(x) \in O_C(x)$) for each $x \in C$.

(1.16) THEOREM. *Let C be a nonempty convex compact subset of a linear topological vector space E with sufficiently many linear functionals. Then every continuous inward (or outward) map $f : C \to E$ has a fixed point.*

PROOF. The case of an inward map follows directly from (1.15); if f is outward then $g : C \to E$ given by $x \mapsto 2x - f(x)$ is inward with the same set of fixed points as f, and the conclusion follows. □

e. Fan intersection theorem and Nash equilibria

In this subsection we give another application to game theory by establishing a general version of the Nash equilibrium theorem.

We begin by introducing some special notation and terminology that will be used in our discussion. Let $X_1, \ldots, X_n$ be topological spaces and $X = \prod_{j=1}^{n} X_j$. For each $i \in [n]$, we let $X^i = \prod_{j \neq i} X_j$ and denote by $p_i : X \to X_i$, $p^i : X \to X^i$ the corresponding projections. Given $x, y \in X$

and letting $p_i(x) = x_i$ and $p^i(y) = y^i$ we write

$$(y_i, x^i) = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) \in X = X_i \times X^i$$

Observe that given any map $S : X^i \to 2^{X_i}$, its graph G_S is contained in X.

Assume for each $i \in [n]$ we are given a function $g_i : X \to \boldsymbol{R}$. A point $\widehat{x} \in X$ is said to be a *Nash equilibrium* for the system $\{g_1, \dots, g_n\}$ provided for each $i \in [n]$ we have

$$g_i(\widehat{x}) = \max\{g_i(z_i, \widehat{x}^i) \mid z_i \in X_i\}.$$

The proof of the theorem of Nash relies on the following geometric result:

(1.17) THEOREM (Fan intersection theorem). *Let $X_1, \dots, X_n$ be compact convex sets, let $X = \prod_{j=1}^n X_j$, and suppose we are given a set-valued map $S_i \in \mathbb{F}(X^i, X_i)$ for $i \in [n]$ (or equivalently $S_i^{-1} \in \mathbb{F}^*(X_i, X^i)$ for $i \in [n]$). Then each of the following equivalent properties holds:*
(a) *the intersection $\bigcap_{i=1}^n G_{S_i}$ is nonempty,*
(b) *there is an $\widehat{x} \in X$ such that $\widehat{x}_i \in S_i(\widehat{x}^i)$ for each $i \in [n]$.*

PROOF. Define $T : X \to 2^X$ by

$$\text{(i)} \qquad Tx = \{y \in X \mid y_i \in S_i(x^i) \text{ for each } i \in [n]\} = \prod_{i=1}^n S_i(x^i).$$

For $y \in X$ we have

$$\text{(ii)} \quad T^{-1}y = \{x \in X \mid y_i \in S_i(x^i) \text{ for each } i \in [n]\} = \bigcap_{i=1}^n X_i \times S_i^{-1}(y_i).$$

According to (i) and (ii), the values of T are convex and nonempty, and the fibers of T are open. Thus T is an $\mathbb{F}$-map, and therefore by Theorem (1.2) it has a fixed point $\widehat{x}$; this means that $\widehat{x} \in \bigcap_{i=1}^n G_{S_i}$. □

We now give an analytic formulation of the intersection theorem.

(1.18) THEOREM. *Let $X_1, \dots, X_n$ be compact convex sets, let $X = \prod_{j=1}^n X_j$, and let $f_1, \dots, f_n : X \to \boldsymbol{R}$ satisfy:*
(i) *$x^i \mapsto f_i(y_i, x^i)$ is l.s.c. on X^i for each $y_i \in X_i$,*
(ii) *$y_i \mapsto f_i(y_i, x^i)$ is quasi-concave on X_i for each $x^i \in X^i$,*
(iii) *for each $i \in [n]$ and $x^i \in X^i$ there exists a $y_i \in X_i$ such that $f_i(y_i, x^i) > 0$.*

Then there exists an $\widehat{x} \in X$ such that $f_i(\widehat{x}) > 0$ for each $i \in [n]$.

PROOF. For each $i \in [n]$ define $S_i : X^i \to 2^{X_i}$ by $S_i(x^i) = \{y_i \in X_i \mid f_i(y_i, x^i) > 0\}$. We note that S_i is in fact an $\mathbb{F}$-map: each set $S_i(x^i)$ is convex by the quasi-concavity of $y_i \mapsto f_i(y_i, x^i)$ and is nonempty because of (iii).

Since $S_i^{-1}(y_i) = \{x^i \in X^i \mid f_i(y_i, x^i) > 0\}$, each fiber of S_i is open by lower semicontinuity of $x^i \mapsto f_i(y_i, x^i)$ on X^i. Thus $S_1, \ldots, S_n$ satisfy the conditions of the intersection theorem (1.17), and hence we get a point $\widehat{x} \in X$ such that $\widehat{x}_i \in S_i(\widehat{x}^i)$ for each $i \in [n]$. By the definition of S_i, this means that $f_i(\widehat{x}) = f_i(\widehat{x}_i, \widehat{x}^i) > 0$ for each $i \in [n]$. □

We are now in a position to establish the fundamental result:

(1.19) THEOREM (Nash equilibrium theorem). *Let $X_1, \ldots, X_n$ be compact convex sets, let $X = \prod_{j=1}^n X_j$, and let $g_1, \ldots, g_n : X \to \boldsymbol{R}$ be continuous. Assume that for each $y \in X$ and each $i \in [n]$ the function $x_i \mapsto g_i(x_i, y^i)$ is quasi-concave on X_i. Then the system $\{g_1, \ldots, g_n\}$ admits a Nash equilibrium point.*

PROOF. For $\varepsilon > 0$ and $i \in [n]$, we first define $h_i, f_i : X \to \boldsymbol{R}$ by

$$x \mapsto h_i(x) = h_i(x_i, x^i) = \max\{g_i(z_i, x^i) \mid z_i \in X_i\},$$
$$x \mapsto f_i(x) = g_i(x) - h_i(x) + \varepsilon,$$

and we let $Q_\varepsilon = \{x \in X \mid f_i(x) \geq 0 \text{ for each } i \in [n]\}$; since for each $i \in [n]$, the uniform continuity of g_i implies that h_i is continuous, it follows that the set Q_ε is compact.

We claim that Q_ε is not empty. To see this, observe that the conditions of (1.18) are satisfied for the continuous functions $f_1, \ldots, f_n$: indeed, for each $i \in [n]$ we have:

(a) $x_i \mapsto f_i(x_i, y^i) = g_i(x_i, y^i) - \max\{g_i(z_i, y^i) \mid z_i \in X_i\} + \varepsilon$ is quasi-concave on X_i for each $y^i \in X^i$,

(b) for any $y^i \in X^i$ there exists an $x_i \in X_i$ such that $f_i(x_i, y^i) > 0$.

Applying (1.18), we get a point $\widehat{x} \in X$ such that $f_i(\widehat{x}) > 0$ for all $i \in [n]$, and thus Q_ε is not empty.

Consider now the decreasing family $\{Q_\varepsilon \mid \varepsilon > 0\}$ of nonempty compact sets and take a point $\widehat{x} = \{\widehat{x}_i\}$ in the intersection $Q = \bigcap\{Q_\varepsilon \mid \varepsilon > 0\}$. By the definition of Q, for any $i \in [n]$ we have

$$g_i(\widehat{x}) \geq \max\{g_i(z_i, \widehat{x}^i) \mid z_i \in X_i\} - \varepsilon \qquad \text{for all } \varepsilon > 0,$$

and therefore $g_i(\widehat{x}) = \max\{g_i(z_i, \widehat{x}^i) \mid z_i \in X_i\}$. Thus $\widehat{x} = \{\widehat{x}_i\}$ is a Nash equilibrium for the system $\{g_1, \ldots, g_n\}$. □

2. Some Applications of the Antipodal Theorem

a. Measure theory

Given a Lebesgue measurable set $A \subset \boldsymbol{R}^n$, we denote its measure by $\mu(A)$, and for any $x \in \boldsymbol{R}^n$, its translation $\{a + x \mid a \in A\}$ by A^x.

(2.1) THEOREM (Kuratowski-Steinhaus). *Let $\Delta = [p_0, \dots, p_n]$ be an n-simplex in $\boldsymbol{R}^n$ that contains the origin $0 \in \boldsymbol{R}^n$ in its interior, and for each $i = 0, \dots, n$, let M_i be the cone consisting of the union of all the rays joining 0 to the points of the $(n-1)$-dimensional face $[p_0, \dots, p_{i-1}, p_{i+1}, \dots, p_n]$ of Δ. Then given any bounded Lebesgue measurable set A with $\mu(A) = 1$ and any $n+1$ nonnegative numbers $\mu_0, \dots, \mu_n$ with $\sum \mu_i = 1$, there exists at least one $y \in \boldsymbol{R}^n$ with $\mu(A^y \cap M_i) = \mu_i$ for each $i = 0, \dots, n$.*

PROOF. Let $K_r = \{x \in \boldsymbol{R}^n \mid \|x\| \le r\}$ and $S_r = \partial K_r$. Choose r so large that for each $x \in S_r$, the set A^x does not meet at least one M_i, and for each $i = 0, \dots, n$, either $A^x \cap M_i$ or $A^{-x} \cap M_i$ is empty. For $y \in K_r$ and $i = 0, \dots, n$, let $\lambda_i(y) = \mu(A^y \cap M_i)$; using the properties of the Lebesgue measure, the functions λ_i can be shown to be continuous, and by the usual additivity properties of measure, $\sum_{i=0}^n \lambda_i(y) = 1$ for each $y \in K_r$. The rule $y \mapsto \sum \lambda_i(y) p_i$ therefore defines a continuous map $f : K_r \to \Delta$. By the choice of r, for each $x \in S_r$ at least one $\lambda_i(x) = 0$ so $f|S_r : S_r \to \partial\Delta$; and since at least one of $\lambda_i(x), \lambda_i(-x)$ is zero, it follows that $f(x) \neq f(-x)$, and hence $f|S_r$ is not nullhomotopic. Thus $f : K_r \to \Delta$ is surjective, and any $y \in f^{-1}(\sum \mu_i p_i)$ satisfies the required conditions. □

As another application, we have

(2.2) THEOREM. *Let $M_1, \dots, M_n$ be n bounded measurable subsets of $\boldsymbol{R}^n$. Then there exists an $(n-1)$-dimensional flat that simultaneously bisects each one of $M_1, \dots, M_n$.*

PROOF. Identify $\boldsymbol{R}^n$ with the subspace $\{x_1, \dots, x_n, 0\} \subset \boldsymbol{R}^{n+1}$; for each $x \in S^n \subset \boldsymbol{R}^{n+1}$, let L_x be the n-dimensional flat of $\boldsymbol{R}^{n+1}$ that passes through $(0, \dots, 0, 1)$ and is perpendicular to the vector x. For each $r = 1, \dots, n$ let

$f_r(x) =$ measure of the part of M_r that lies on the same side of L_x as does $x + (0, \dots, 0, 1)$.

It can be verified that f_r is a continuous real-valued function on S^n, so that $x \mapsto (f_1(x), \dots, f_n(x))$ defines a continuous map $f : S^n \to \boldsymbol{R}^n$. By the Borsuk–Ulam theorem, there is at least one point $x \in S^n$ with $f(x) = f(-x)$ so that $f_r(x) = f_r(-x)$ for each $r = 1, \dots, n$. Since $L_x = L_{-x}$ and $x + (0, \dots, 0, 1), -x + (0, \dots, 0, 1)$ are on opposite sides of L_x, this means that L_x bisects each M_i. Thus, $L_x \cap \boldsymbol{R}^n$ is an $(n-1)$-dimensional flat in $\boldsymbol{R}^n$ that bisects each of $M_1, \dots, M_n$. □

b. Periodic transformations

(2.3) THEOREM. *Let $T : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be any homeomorphism. If $TT(a) = a$ for all $a \in \boldsymbol{R}^n$, then T has a fixed point.*

PROOF. Let $a \in \mathbf{R}^n$; we can assume $T(a) \neq a$. The sphere S^n contains in natural fashion the sequence $S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n$, each sphere being the equator of the next; we shall define, by induction, an $f : S^n \to \mathbf{R}^n$ satisfying $f(-x) = Tf(x)$. For S^0, the map $f(+1) = a$, $f(-1) = Ta$ clearly satisfies the requirement. Assuming f defined on S^k, extend f in any way over the northern hemisphere $S_+^{k+1} \subset S^{k+1}$, and then extend over S_-^{k+1} by the formula $f(-x) = Tf(x)$. This completes the induction. By the Borsuk Ulam theorem, the map $f : S^n \to \mathbf{R}^n$ must have some $x_0 \in S^n$ with $f(-x_0) = f(x_0)$; therefore, $f(x_0) = Tf(x_0)$, and T has a fixed point. □

c. Geometry of Banach spaces

The following result of Krein–Krasnosel'skiĭ–Milman plays an important role in perturbation theory for linear (unbounded) operators:

(2.4) THEOREM. *Let M and N be linear subspaces of a finite-dimensional Banach space $(E, \|\ \|)$. If $\dim M > \dim N$, then there is an $x_0 \in M$ such that*

$$\operatorname{dist}(x_0, N) = \|x_0\| > 0.$$

PROOF. Suppose first that the norm $\|\ \|$ in E is strictly convex: $\|x + y\| < \|x\| + \|y\|$ whenever x, y are linearly independent. It is then easily seen that each $x \in E$ has a unique nearest point $y = f(x)$ in N and that $x \mapsto f(x)$ is continuous. Furthermore, the map $f : E \to N$ has the property $f(-x) = -f(x)$ for all $x \in E$. Consequently, applying the Borsuk theorem (5.5.2) to $f|S_M : S_M \to N$ we obtain a point $x_0 \in S_M$ such that $f(x_0) = 0$. Clearly, x_0 is the required point.

In the general case, choose a basis $\varphi^1, \varphi^2, \ldots, \varphi^n$ in E^* and define

$$\|x\|_m = \left\{\|x\|^2 + \frac{1}{m}[(\varphi^1, x)^2 + \cdots + (\varphi^n, x)^2]\right\}^{1/2};$$

$\|x\|_m$ is a new and strictly convex norm in E. For each $m = 1, 2, \ldots$ there is an $x_m \in M$ with $\operatorname{dist}_m(x_m, N) = \|x_m\|_m = 1$. Since $\|x_m\| \leq \|x_m\|_m = 1$, the sequence $\{x_m\}$ contains a convergent subsequence relative to $\|\ \|$; the limit x_0 of this subsequence satisfies the requirements of the theorem. □

As an immediate consequence we get

(2.5) THEOREM. *Let M and N be linear subspaces of a finite-dimensional Banach space $(E, \|\ \|)$. Suppose*

$$\sup_{\|x\| \leq 1,\, x \in M} \operatorname{dist}(x, N) < 1 \quad \textit{and} \quad \sup_{\|y\| \leq 1,\, y \in N} \operatorname{dist}(y, M) < 1.$$

Then $\dim M = \dim N$. □

3. The Schauder Theorem and Differential Equations

In this section we give two applications of the Schauder fixed point theorem to differential equations. We first describe the simplest scheme that is used for these applications.

Let U, B, E be normed spaces. In the following we shall be given a linear operator $L : U \to B$, a completely continuous embedding $j : U \to E$ and a continuous operator $G : E \to B$ that is bounded on bounded sets; we seek to place conditions on G to ensure that the nonlinear equation $L(u) = Gj(u)$ will have a solution. The approach used here relies on the invertibility of the linear operator L: for if L is invertible, the question is equivalent to the fixed point problem $u = L^{-1}Gj(u)$ for the completely continuous nonlinear operator $L^{-1}Gj : U \to U$; we can then place restrictions on G to ensure that some fixed point principle (or topological transversality) can be applied.

Thus, showing the operator L to be invertible is crucial in this approach to existence theorems, and in our applications we will encounter two methods for doing this. In the first, L is bijective, and its invertibility follows from the Banach inverse mapping theorem; in the second, the Schauder invertibility theorem and appropriate a priori bounds are required.

Observe that with an invertible L, the problem of solving $L(u) = Gj(u)$ is in fact equivalent to the fixed point problem for any one of the nonlinear completely continuous operators $L^{-1}Gj : U \to U$, $jL^{-1}G : E \to E$, $GjL^{-1} : B \to B$; because the expression for the norm may be especially simple in one of the spaces U, B, E, it may be more convenient to verify that for a given G, some one rather than another of these operators has a fixed point.

a. Initial value problem (Peano's theorem)

On the interval $[0, T]$, consider the initial value problem

$$\begin{cases} \dfrac{du}{dt} = g(t, u), & 0 \le t \le T, \\ u(0) = 0. \end{cases}$$

If we let

$$C_0^1 = \{u \in C^1[0, T] \mid u(0) = 0\},$$

then the linear operator $d/dt \equiv L : C_0^1 \to C[0, T]$ is clearly bijective with inverse

$$L^{-1}(f)(t) = \int_0^t f(x)\,dx \equiv u(t).$$

We verify that L^{-1} is continuous by an a priori estimate: Recall that the norm in C_0^1 is $\|u\|_1 = \max\{\|u\|, \|u'\|\}$, where $\|\ \|$ denotes the supremum norm on $[0, T]$; since $\|u\| \le T\|Lu\|$ from the above formula, and $\|u'\| = \|Lu\|$, we

find $\|u\|_1 \le (T+1)\|Lu\|$. So, if $u = L^{-1}(f)$, we have $\|L^{-1}(f)\|_1 \le (T+1)\|f\|$, and $L^{-1} : C[0,T] \to C_0^1$ is therefore continuous. We now prove

(3.1) THEOREM (Peano). *Let $g : [0,T] \times \boldsymbol{R} \to \boldsymbol{R}$ be a bounded continuous function. Then the initial value problem above has at least one solution $u \in C_0^1$.*

PROOF. Let $C = C[0,T]$, let $G : C \to C$ be the operator $G(u)(t) = g(t, u(t))$, and let $j : C_0^1 \to C$ be the natural embedding. The problem is to show that $L(u) = Gj(u)$ has a solution or equivalently, because L is invertible, that the operator $jL^{-1}G : C \to C$ has a fixed point. Since j is completely continuous and G is bounded (because g is), the operator $jL^{-1}G$ is compact. By Schauder's theorem $jL^{-1}G$ has a fixed point. □

b. Two-point boundary value problem

On the interval $[0,T]$, consider the two-point boundary value problem

$$\begin{cases} \dfrac{d^2u}{dt^2} = g(t, u(t), u'(t)), & 0 \le t \le T, \\ u(0) = u(T) = 0. \end{cases}$$

Letting

$$C_0^2 = \{u \in C^2 \mid u(0) = u(T) = 0\}$$

and $L = d^2/dt^2$, we begin by studying the linear operator $L : C_0^2 \to C[0,T]$. This operator is in fact bijective: if

$$K(s,t) = 2T \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} \sin \frac{n\pi s}{T} \sin \frac{n\pi t}{T}$$

is the associated Green's function, then the inverse is given by

$$u(s) \equiv L^{-1}(f)(s) = -\int_0^T K(s,t) f(t)\, dt.$$

We verify (directly) the continuity of L^{-1}: Recalling that the norm in C^2 is $\|u\|_2 = \sup\{\|u\|, \|u'\|, \|u''\|\}$ and observing from the formula for the inverse that each of $\|u\|, \|u'\|, \|u''\|$ is bounded by some multiple of $\|Lu\|$, we find that $\|u\|_2 \le K\|Lu\|$, and therefore the "smoothing operator" $L^{-1} : C[0,T] \to C_0^2$ is continuous.

We now establish

(3.2) THEOREM. *Let $g : [0,T] \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ be a bounded continuous function. Then the two-point boundary value problem above has at least one solution $u \in C_0^2$.*

PROOF. Defining $G : C^1[0,T] \to C[0,T]$ by $G(u)(t) = g(t, u(t), u'(t))$ and letting $j : C_0^2 \to C^1[0,T]$ be the embedding, we have the problem $L(u) = Gj(u)$. Since $C[0,T]$ has the simplest expression for its norm, we reduce this problem to the equivalent problem for $GjL^{-1} : C \to C$. Since jL^{-1} is completely continuous, because j is, and since G is a bounded operator because g is bounded, we conclude that $jL^{-1}G$ is compact, and therefore, by Schauder's theorem, it has a fixed point. This implies that $\mathrm{Fix}(GjL^{-1}) \neq \emptyset$, and thus the desired conclusion follows. □

4. Topological Transversality and Differential Equations

In this section we give an application of topological transversality to differential equations. The approach presented here was first introduced by Leray and Schauder in the context of degree theory for compact fields, and has proved to be a powerful tool in the treatment of various nonlinear problems.

We will treat in full detail a problem for ordinary differential equations, by establishing a version of a classical result of S. Bernstein.

Consider the two-point boundary value problem

$$(\mathscr{P}) \qquad \begin{cases} y'' = f(t, y, y'), \\ y(0) = 0 = y(1), \end{cases}$$

where $f = f(t, y, p)$ is defined and continuous in $[0,1] \times \boldsymbol{R}^2$. We shall impose conditions on f that are milder than those in the previous section and indicate the "a priori bounds" technique that permits applying topological transversality to obtain the existence of a C^2-solution to the above problem.

(4.1) LEMMA. *Assume that $f : [0,1] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ satisfies:*
 (i) *there is a constant $M_0 > 0$ such that $yf(t, y, 0) > 0$ for $|y| \geq M_0$,*
 (ii) *there are constants $A, B > 0$ such that for all $(t, y, p) \in [0,1] \times [-M_0, M_0] \times \boldsymbol{R}$,*
$$|f(t, y, p)| \leq Ap^2 + B.$$

Then there is a constant M_1 such that if y is a solution to $(\mathscr{P})$, then $|y(t)| \leq M_0$ and $|y'(t)| \leq M_1$ for all $t \in [0,1]$.

PROOF. We first show that (i) implies an a priori bound on a solution. Let $y \in C^2[0,1]$ be a solution to $(\mathscr{P})$; assume that $y \not\equiv 0$. Then the function $y \mapsto |y|$ must attain a positive maximum at $t_0 \in (0,1)$; supposing $y(t_0) > 0$, we have

$$0 \geq y''(t_0) = f(t_0, y(t_0), 0), \qquad y(t_0) f(t_0, y(t_0), 0) \leq 0,$$

and the last inequality holds also if $y(t_0) < 0$. By assumption (i) it follows that $|y(t_0)| \leq M_0$, and hence $|y(t)| \leq M_0$ for all $t \in [0,1]$.

Now assuming that y is a solution to $(\mathscr{P})$, we establish an a priori bound on y' Since y' vanishes at least once in $[0,1]$, each $t \in [0,1]$ for which $y'(t) \neq 0$ belongs to an interval $[a,b]$ such that y' has a fixed sign on $[a,b]$ and $y'(a) = 0$ and/or $y'(b) = 0$. Let us take such an interval $[a,b]$ and to be definite assume $y'(a) = 0$, $y'(t) \geq 0$ for $t \in [a,b]$. Then from (ii),

$$\frac{d}{dt}[\log(Ay'^2 + B)] = \frac{2Ay'y''}{Ay'^2 + B} \leq 2Ay' \quad \text{on } [a,b],$$

and integrating from a to t yields

$$\log \frac{Ay'^2(t) + B}{B} \leq 4AM_0,$$

and hence

$$|y'(t)| \leq \left\{ \frac{B}{A}(e^{4AM_0} - 1) \right\}^{1/2} = M_1 \quad \text{on } [a,b].$$

The other possibilities that might occur are treated similarly, and the same bound $|y'(t)| \leq M_1$ is obtained on the entire interval $[0,1]$. □

(4.2) THEOREM. *Assume that $f = f(t,y,p)$ is continuous and satisfies:*
(i) *there is a constant $M_0 > 0$ such that $yf(t,y,0) > 0$ for $|y| \geq M_0$,*
(ii) *$|f(t,y,p)| \leq A(t,y)p^2 + B(t,y)$, where A, B are functions bounded on each compact subset of $[0,1] \times \boldsymbol{R}$.*
Then there exists a solution $y \in C^2[0,1]$ to problem $(\mathscr{P})$.

PROOF. Consider the family of problems

$$(\mathscr{P}_\lambda) \qquad \begin{cases} y'' = \lambda f(t,y,y'), \\ y(0) = 0 = y(1), \end{cases}$$

depending on the parameter λ $(0 \leq \lambda \leq 1)$, joining the problem $(\mathscr{P})$ to the corresponding problem $y'' = 0$, $y(0) = y(1) = 0$, whose (unique) solution is $y \equiv 0$. We claim that there are constants M_0, M_1, M_2 such that for each solution y to $(\mathscr{P}_\lambda)$,

$$(*) \qquad |y(t)| \leq M_0, \quad |y'(t)| \leq M_1, \quad |y''(t)| \leq M_2$$

for $t \in [0,1]$. Clearly, the bounds for y and y' follow from (4.1); the bound on y'' follows from the continuity of f on $[0,1] \times [-M_0, M_0] \times [-M_1, M_1]$.

Let $C_0^2 = \{u \in C^2 \mid u(0) = u(1) = 0\}$. Consider the linear operator $L : C_0^2 \to C$ given by $u \mapsto d^2u/dt^2$, the family of maps $T_\lambda : C^1 \to C$ $(0 \leq \lambda \leq 1)$ defined by

$$(T_\lambda v)(t) = \lambda f(t, v(t), v'(t)),$$

and the completely continuous embedding $j : C_0^2 \to C^1$ Let

$$r = 1 + \max\{M_0, M_1, M_2\}.$$

where M_0, M_1, M_2 are the constants in (*), and $K_r = \{u \in C_0^2 \mid \|u\|_2 \le r\}$. Since $L : C_0^2 \to C$ is invertible, we can define a homotopy $H_\lambda : K_r \to C_0^2$ by $H_\lambda = L^{-1}T_\lambda j$. It is easily seen that the fixed points of H_λ are precisely the solutions of $(\mathscr{P}_\lambda)$; therefore, by the choice of r and (*) the homotopy H_λ is fixed point free on the boundary of K_r. Moreover (because of the complete continuity of j), the homotopy H is compact. Since H_0 is a constant map $(H_0(K_r) = \{0\})$, it is essential. Because $H_0 \simeq H_1$, topological transversality shows that H_1 is also essential; in particular, H_1 has a fixed point which is a solution to problem $(\mathscr{P})$. □

As an immediate consequence, we have

(4.3) THEOREM (Bernstein). *Assume $f = f(t, y, p)$ has continuous partial derivatives f_y and f_p and satisfies:*

(i) $f_y \ge k > 0$,

(ii) $|f(t, y, p)| \le A(t, y)p^2 + B(t, y)$,

where k is a constant and A, B are functions bounded on compact subsets of $[0, 1] \times \mathbf{R}$. Then there exists a solution $y \in C^2[0, 1]$ to problem $(\mathscr{P})$. □

5. Application to the Galerkin Approximation Theory

Let E be a normed space, $U \subset E$ open, and $F : \overline{U} \to E$ a compact operator. We seek numerical solutions (assumed to exist) of the equation $x = F(x)$ in $\overline{U}$. One method for finding at least approximate solutions consists in "approximating" the given equation on $\overline{U}$ by an equation $x = F_\varepsilon(x)$ for which an exact solution x_ε can be found; it is hoped that the "approximating" equation $x = F_\varepsilon(x)$ can be selected so that x_ε will in fact be an approximate solution of the given equation $x = F(x)$. This theory underlies various specific methods for the numerical solution of operator, integral, differential, etc., equations. One scheme for this program, called the *perturbed Galerkin method*, is the following: Let L_n be a sequence of closed linear subspaces of E, and let $P_n : E \to L_n$ be projections. Choose a compact operator $F_n : \overline{U} \cap L_n \to L_n$ (the "approximation" to F) and measure the degree of approximation of F_n to F by the smallness of the operators

$$T_n = F_n - P_n \circ F : \overline{U} \cap L_n \to L_n, \qquad S_n = F - P_n \circ F : \overline{U} \to E.$$

The equation $x = P_n F(x)$ in L_n is called the *associated Galerkin equation*; and $x_n = F_n(x_n)$ in L_n is called the *perturbed Galerkin equation*.

We give conditions under which the perturbed Galerkin method provides approximate numerical solutions for the given equation. For a set $X \subset E$ we write $X_n = X \cap L_n$.

(5.1) THEOREM. *Let $F \in \mathscr{K}_{\partial U}(\overline{U}, E)$ be an essential compact operator. Assume that subspaces L_n and compact operators $F_n : \overline{U}_n \to L_n$ are chosen so that*

$$\sup_{x \in \overline{U}} \|F(x) - P_n F(x)\| \to 0$$

and

$$\sup_{x \in \overline{U}_n} \|F_n(x) - P_n F(x)\| \to 0.$$

Then there is an N such that for all $n \geq N$, the set $\Sigma_n = \{x \in U_n \mid x = F_n(x)\}$ of solutions of the perturbed Galerkin equation is not empty; moreover, $\sup_{x_n \in \Sigma_n} d(x_n, \Sigma_0) \to 0$, where Σ_0 is the set of solutions of $x = F(x)$.

PROOF. Since F is fixed point free on ∂U, we have

$$\inf_{\partial U} \|x - F(x)\| = \alpha > 0.$$

Choose N_1 so large that

$$\sup_{\partial U} \|F(x) - P_n F(x)\| < \alpha/2 \quad \text{for all } n \geq N_1.$$

Then, by the proof of (6.4.8), $P_n F \in \mathscr{K}_{\partial U}(\overline{U}, E)$ is also essential, and therefore, by (6.4.4), so is the restriction $P_n F|\overline{U}_n \in \mathscr{K}_{\partial U_n}(\overline{U}_n, L_n)$. Since

$$\begin{aligned}\inf_{\partial U_n} \|x - P_n F(x)\| &\geq \inf_{\partial U} \|x - P_n F(x)\| \\ &\geq \inf_{\partial U} \|x - F(x)\| - \sup_{\partial U} \|F(x) - P_n F(x)\| \\ &\geq \alpha - \alpha/2 = \alpha/2,\end{aligned}$$

we choose $N \geq N_1$ so that $\sup_{\partial U_n} \|F_n(x) - P_n F(x)\| \leq \alpha/4$, and applying (6.4.8) again, we conclude that $F_n \in \mathscr{K}_{\partial U_n}(\overline{U}_n, L_n)$ is essential. Therefore, the set Σ_n of solutions is nonempty for all $n \geq N$.

To prove the remaining part, we must know the sups on $\overline{U}$ and $\overline{U}_n$ (not just on ∂U and ∂U_n, as has been enough up to now). Choose any $\varepsilon > 0$ and draw a ball $B(x, \varepsilon_x) \subset U$ with $\varepsilon_x \leq \varepsilon$ centered at each $x \in \Sigma_0$; the union of these open balls is denoted by Σ_0^ε. Since F is fixed point free on $\overline{U} - \Sigma_0^\varepsilon$, we have $\inf_{\overline{U} - \Sigma_0^\varepsilon} \|x - F(x)\| = \alpha_\varepsilon > 0$.

Now, if $x \in L_n \cap (\overline{U} - \Sigma_0^\varepsilon)$, then

$$\|x - F_n(x)\| \geq \alpha_\varepsilon - \|F(x) - P_n F(x)\| - \|P_n F(x) - F_n(x)\|,$$

so that $\|x - F_n(x)\| > \alpha_\varepsilon/2$ on $\overline{U} \cap L_n - \Sigma^\varepsilon$ for all n larger than some n_ε. But $x - F_n(x) = 0$ for all $x \in \Sigma_n$, so for all $n > n_\varepsilon$ the set $\Sigma_n \subset \overline{U}_n$ is disjoint from $\overline{U}_n - \Sigma_0^\varepsilon$; that is, $\Sigma_n \subset \Sigma_0^\varepsilon$. Since ε is arbitrary, the proof is complete. □

(5.2) COROLLARY. *Let* $U = \{x \mid \|x - x_0\| < \delta\}$ *and* $F \in \mathscr{K}_{\partial U}(\overline{U}, E)$ *be an essential operator with the unique solution* $x_0 = F(x_0)$. *Let* F_n *and* L_n *be the compact operators and closed subspaces satisfying the requirements of the theorem. Then for all sufficiently large* n *the equation* $x = F_n(x)$ *has at least one solution* x_n *in* U, *and any sequence* $\{x_n\}$ *of these solutions converges in norm to* x_0. □

6. The Invariant Subspace Problem

Let E be a topological vector space. A closed linear subspace $L \subset E$ is called *nontrivial* if $L \neq \{0\}$ and $L \neq E$; it is called an *invariant subspace* for a linear operator $T : E \to E$ if $T(L) \subset L$.

In Hilbert space, it is well known that every completely continuous self-adjoint operator has a nontrivial closed invariant subspace; indeed, its spectral representation is directly based on the existence of such subspaces. Seeking to determine the structure of non-self-adjoint operators therefore raises the question: what types of operators have nontrivial closed invariant subspaces? The following result shows that the existence of such subspaces for an operator does not require the existence of an inner product: in any infinite-dimensional normed linear space, commutativity with some nontrivial completely continuous operator suffices to ensure their existence.

(6.1) THEOREM (Lomonosov). *Let* T *be a continuous linear operator on an infinite-dimensional normed linear space* E. *If there is a completely continuous linear operator* $K \neq 0$ *such that* $T \circ K = K \circ T$, *then* T *has a nontrivial closed invariant subspace.*

PROOF. Let $\mathrm{Comm}(T)$ be the set of all continuous linear operators that commute with T; this set is not empty, since $T \in \mathrm{Comm}(T)$. For each $y \neq 0$, let $\mathscr{L}_y = \{Ay \mid A \in \mathrm{Comm}(T)\}$. Each $\mathscr{L}_y$ is a linear subspace: if A, B commute with T, then so also does $A + B$; therefore, $Ay + By \in \mathscr{L}_y$. Moreover, $A(\mathscr{L}_y) \subset \mathscr{L}_y$ for every $A \in \mathrm{Comm}(T)$, since $AB \in \mathrm{Comm}(T)$.

If some $\mathscr{L}_{y_0}$ is not dense in E, then $0 \neq \overline{\mathscr{L}}_{y_0} \neq E$, so $\overline{\mathscr{L}}_{y_0}$ is a nontrivial closed linear subspace; and since $A(\overline{\mathscr{L}}_{y_0}) \subset \overline{A(\mathscr{L}_{y_0})} \subset \overline{\mathscr{L}}_{y_0}$ for every $A \in \mathrm{Comm}(T)$, and in particular for T, to complete the proof we need only consider the case where every $\mathscr{L}_y$ is dense in E.

The subspace $K^{-1}(0)$ is a closed linear subspace, and since K is nontrivial, it is not E. Choose $x_0 \in E$ with $K(x_0) \neq 0$ and consider the open ball $B(K(x_0), \|K(x_0)\|/2)$. It is clear that we can find a closed ball $Q = \overline{B}(x_0, r) \subset K^{-1}(B(K(x_0), \|K(x_0)\|/2))$ such that $0 \notin \overline{K(Q)}$; moreover, $\overline{K(Q)}$ is compact, as K is a completely continuous operator. Because $\mathscr{L}_y$ is dense in E for each $y \neq 0$, for each $c \in \overline{K(Q)}$ we can find a $D_c \in \mathrm{Comm}(T)$ such that $D_c(c)$ lies in the interior of Q, so by continuity, each $c \in \overline{K(Q)}$

has a ball $Q_c(\varepsilon_c) = \overline{B(c, \varepsilon_c)}$ such that $D_c[Q_c(\varepsilon_c)] \subset Q$. Because $\overline{K(Q)}$ is compact, finitely many such balls cover $\overline{K(Q)}$, say $Q_{c_1}(\varepsilon_1), \ldots, Q_{c_n}(\varepsilon_n)$; let $D_{c_1}, \ldots, D_{c_n}$ be the corresponding operators.

On $\overline{K(Q)}$ let

$$\alpha_i(c) = \max[0, \varepsilon_i - \|c - c_i\|], \quad \beta_i(c) = \frac{\alpha_i(c)}{\sum_{i=1}^n \alpha_i(c)}, \qquad i = 1, \ldots, n,$$

and consider the map $F : Q \to E$ given by

$$F(b) = \sum_{i=1}^n \beta_i[K(b)] D_{c_i}[K(b)].$$

This is clearly a compact map. Note that $\beta_i[K(b)] \neq 0$ implies that $K(b) \in Q_{c_i}(\varepsilon_i)$, so that $D_{c_i}[K(b)] \in Q$; thus, each value $F(b)$ is a convex combination of points in Q, so F maps into Q. By Schauder's fixed point theorem, F therefore has a fixed point $b_0 \in Q$, and in particular, $b_0 \neq 0$.

Consider now the completely continuous linear operator $W : E \to E$ given by

$$W = \sum_{i=1}^n \beta_i[K(b_0)] D_{c_i} \circ K.$$

The set $L = \{y \in E \mid W(y) = y\}$ is a closed linear subspace; it is not $\{0\}$ because $b_0 \in L$; and it is not E because W is completely continuous and E is infinite-dimensional. Thus, L is a nontrivial closed linear subspace. Finally, it is an invariant subspace for T: if $y \in L$, then because T commutes with W we have $W(Ty) = T(Wy) = Ty$, so $Ty \in L$ and $T(L) \subset L$. □

As an immediate consequence, we obtain the following extension of a result of Aronszajn- Smith:

(6.2) COROLLARY. *Let E be an infinite-dimensional normed linear space, and $T : E \to E$ a continuous linear operator. If some iterate T^n is completely continuous and nonzero, then T has a nontrivial closed invariant subspace.*

PROOF. T commutes with T^n. □

Examining the proof of Lomonosov's theorem more carefully gives the following extension for complex normed vector spaces.

(6.3) COROLLARY. *Let E be an infinite-dimensional complex normed linear space and $T : E \to E$ a continuous linear operator such that $T \neq \lambda I$ for all $\lambda \in C$. If there is a nonzero completely continuous linear operator $K : E \to E$ such that $T \circ K = K \circ T$, then all members of* Comm(T) *have a common nontrivial closed invariant subspace.*

PROOF. We have seen in the proof of the theorem that if some $\mathscr{L}_y$ is not dense in E, then its closure $\overline{\mathscr{L}}_y$ is the desired subspace. We therefore need only consider the case where every $\mathscr{L}_y$ is dense.

In that case, we have formed an invariant subspace for T to be the eigenspace $\{y \in E \mid W(y) = y\}$ of a completely continuous operator $W : E \to E$. According to Riesz's theorem, every eigenspace of a completely continuous operator is necessarily finite-dimensional, so the invariant subspace L produced in the second part of the proof is in fact finite-dimensional. Since T is a linear transformation of L into itself and $T \neq \lambda I$ for all λ, some eigenspace $M = \{u \in E \mid T(u) = \xi u\}$ is nontrivial. If now $A \in \mathrm{Comm}(T)$, then for each $u \in M$ we have

$$\xi A(u) = A(\xi u) = AT(u) = T(Au),$$

so that $A(M) \subset M$ for each $A \in \mathrm{Comm}(T)$ and M is the required invariant subspace. □

7. Absolute Retracts and Generalized Schauder Theorem

In this section we establish a general fixed point theorem that is formulated in purely topological terms and contains the Schauder theorem as a special case. We begin with the relevant facts from the theory of retracts.

(7.1) DEFINITION. A space Y is called an *absolute retract* (or simply an AR) whenever (i) Y is metrizable and (ii) for any metrizable X and closed $A \subset X$ each $f : A \to Y$ is extendable over X. The class of absolute retracts is denoted by AR.

It is evident that if Y is an AR, then every space homeomorphic to Y is also an AR.

(7.2) PROPOSITION. *If Y is an AR and B is a retract of Y, then B is also an AR.*

PROOF. Let X be metrizable, $A \subset X$ closed, and $f : A \to B$ a continuous map. Let $r : Y \to B$ be a retraction. Since Y is an AR, the map f has an extension $F : X \to Y$ over X; then $r \circ F$ is the required extension for f. □

We shall now establish that every convex set in a normed linear space is an AR. This will be a consequence of a general extension theorem.

(7.3) LEMMA. *Let (X, d) be a metric space and $\{V_\lambda \mid \lambda \in \Lambda\}$ a nbd-finite open covering of X. Then there exists a partition of unity subordinate to the covering $\{V_\lambda \mid \lambda \in \Lambda\}$, i.e., a family $\{\varkappa_\lambda \mid \lambda \in \Lambda\}$ of continuous functions $\varkappa_\lambda : X \to I$ satisfying:*

(i) $\varkappa_\lambda(x) \geq 0$ *for all* $x \in X$.

(ii) $\varkappa_\lambda(x) \neq 0$ *if and only if* $x \in V_\lambda$,

(iii) *for each* $x \in X$ *there is a nbd* W *of* x *such that only a finite number of* $\varkappa_\lambda$ *are not identically zero in* W,

(iv) $\sum_{\lambda \in \Lambda} \varkappa_\lambda(x) = 1$ *for each* $x \in X$.

PROOF. For each $\lambda \in \Lambda$ we let

$$\varkappa_\lambda(x) = \frac{d(x, X - V_\lambda)}{\sum_{\lambda \in \Lambda} d(x, X - V_\lambda)}, \qquad x \in X,$$

and examine properties of the functions $\varkappa_\lambda$. We first observe that the sum in the denominator is always finite: for $d(x, X - V_\lambda) \neq 0$ if and only if $x \in V_\lambda$, and since the covering is nbd-finite, x lies in at most a finite number of V_λ. Further, since $\{V_\lambda \mid \lambda \in \Lambda\}$ is a covering, we have $\sum_{\lambda \in \Lambda} d(x, X - V_\lambda) \neq 0$ for each $x \in X$, and so $\varkappa_\lambda(x)$ is well defined for each $x \in X$. Now, each $\varkappa_\lambda$ is continuous; in fact, for any $x \in X$ there is a nbd meeting only a finite number of the sets of the covering; in this nbd $\varkappa_\lambda$ is explicitly given as a sum of a finite number of continuous functions, so $\varkappa_\lambda$ is continuous at each $x \in X$. It follows easily that the family $\{\varkappa_\lambda\}$ has properties (i)–(iv). □

(7.4) THEOREM (Dugundji extension theorem). *Let* X *be any metrizable space and* $A \subset X$ *a closed subset. Let* E *be any locally convex linear topological space. Then any* $f : A \to E$ *has an extension* $F : X \to E$ *with* $F(X) \subset \operatorname{conv}[f(A)]$.

PROOF. Let d be a metric for X. Cover $X - A$ by the balls

$$\{B(x, \tfrac{1}{2}d(x, A)) \mid x \in X - A\};$$

by Stone's theorem, this cover has a nbd-finite open refinement $\{V_\lambda \mid \lambda \in \Lambda\}$. For each V_λ choose a $B(v_\lambda, \frac{1}{2}d(v_\lambda, A)) \supset V_\lambda$; then choose an $a_\lambda \in A$ such that $d(v_\lambda, a_\lambda) \leq 2d(v_\lambda, A)$. We calculate the position of V_λ relative to A in terms of the points a_λ we have selected:

(i) $d(v_\lambda, A) \leq 2d(v, A)$ for each $v \in V_\lambda$; for

$$d(v_\lambda, A) \leq d(v_\lambda, v) + d(v, A) \leq \tfrac{1}{2}d(v_\lambda, A) + d(v, A),$$

so the result follows.

(ii) $d(a, a_\lambda) \leq 6d(a, v)$ for every $a \in A$ and $v \in V_\lambda$; for

$$\begin{aligned} d(a, a_\lambda) &\leq d(a, v) + d(v, v_\lambda) + d(v_\lambda, a_\lambda) \\ &\leq d(a, v) + \tfrac{1}{2}d(v_\lambda, A) + 2d(v_\lambda, A) \\ &\leq d(a, v) + d(v, A) + 4d(v, A). \end{aligned}$$

Using the points a_λ we have selected and the functions $\varkappa_\lambda$ of (7.3) we are ready to define the extension. We let

$$F(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \sum_{\lambda \in \Lambda} \varkappa_\lambda(x) f(a_\lambda) & \text{if } x \in X - A. \end{cases}$$

This function is evidently continuous at each point of the open set $X - A$, so only its continuity at the points of A needs to be proved.

Let $a \in A$, and let $W \ni f(a)$ be an open set. Since E is locally convex and f continuous on A, there is a convex C and a $\delta > 0$ such that $f[A \cap B(a, \delta)] \subset C \subset W$. We are going to show that

$$F[B(a, \delta/6)] \subset C \subset W,$$

which will prove the continuity of F at $a \in A$.

Let x be any point of $B(a, \delta/6) - A$; it belongs to only finitely many sets $V_{\lambda_1}, \dots, V_{\lambda_n}$. Then $d(x, a) < \delta/6$, so since $x \in V_{\lambda_i}$, we have $d(a, a_{\lambda_i}) < \delta$ by (ii); therefore, all the a_{λ_i} are in $A \cap B(a, \delta)$; consequently, all $f(a_{\lambda_i})$ are in C, and because $F(x) = \sum_{i=1}^{n} \varkappa_{\lambda_i}(x) f(a_{\lambda_i})$ is a convex combination of points of C, we conclude that $F(x) \in C$. Thus $F[B(a, \delta/6)] \subset W$, and F is continuous at a. As $F(X) \subset \operatorname{conv}[f(A)]$ is evident, the proof is complete. □

Since the values of the extension lie in the convex hull of $f(A)$ we immediately get

(7.5) THEOREM. *Let C be any convex subset of a locally convex linear topological space. Then for every metrizable space X and any closed $A \subset X$ each $f : A \to C$ has an extension $F : X \to C$. In particular, if C is metrizable, then C is an AR.* □

We now give two characterizations of the ARs in terms of a retraction property.

(7.6) THEOREM.

(i) *A metrizable space is an AR if and only if it is a retract of every metrizable space in which it is embedded as a closed set.*

(ii) *A metrizable space is an AR if and only if it is a retract of some normed linear space.*

PROOF. (i) Suppose Y is an AR, and that $Y \subset Z$ is a closed subset. Consider now the identity $1_Y : Y \to Y$; since $Y \subset Z$ is closed and Y is an AR, the map 1_Y extends over Z to an $r : Z \to Y$, and r is clearly a retraction. Conversely, suppose Y has the property in (i). Using the Arens Eells theorem, embed Y as a closed subset of a normed linear space E. Then E is an AR by (7.5), and Y is a retract of E by hypothesis; therefore, by (7.2), Y is an AR.

(ii) follows from (i) by combining the Arens–Eells embedding with (7.2) and (7.5). □

(7.7) THEOREM. *Let K be a closed ball in an infinite-dimensional normed linear space E, and let $\{U_i\}_{i \in I}$ be a family of mutually disjoint open balls that are contained in K. Then the set $K - \bigcup_{i \in I} U_i$ is an AR. In particular, the sphere ∂K is an AR.*

PROOF. We first assume that K is the unit ball in E and show that ∂K is a retract of K. Since by (4.3.12), $E \simeq E - \{0\}$, it follows from (7.5) that $E - \{0\}$ is an AR, and hence by (7.2) the unit sphere $S = \partial K$ in E is an AR. Consequently, there is a retraction $r : K \to \partial K$. A similar argument shows that there is a retraction $r : K \to K - \bigcup_{i \in I} U_i$, and by (7.5) the assertion follows. □

The generalized Schauder theorem will now be established. Using the factorization technique we first obtain the following preliminary result.

(7.8) THEOREM. *Let K be a compact metric space and $f : K \to K$ be a map. Assume that f can be factored as $K \xrightarrow{\alpha} X \xrightarrow{\beta} K$, where X is an AR. Then f has a fixed point.*

PROOF. Consider the diagram

$$\begin{tikzcd}
I^\infty \arrow[dd, dashed, "\varphi"'] & & \widehat{K} \arrow[ll, hook', "j"'] \arrow[dl, shift left, "s^{-1}"] \\
& K \arrow[ur, shift left, "s"] \arrow[dl, "\alpha"'] \arrow[d, "f"] & \\
X \arrow[r, "\beta"'] & K &
\end{tikzcd}$$

in which I^∞ is the Hilbert cube and $s : K \to \widehat{K}$ is a homeomorphism of K onto $\widehat{K} \subset I^\infty$ with inverse $s^{-1} : \widehat{K} \to K$. Because X is an AR, there is an extension $\varphi : I^\infty \to X$ of the map $\alpha s^{-1} : \widehat{K} \to X$ over I^∞, i.e., $\varphi j = \alpha s^{-1}$, where $j : \widehat{K} \hookrightarrow I^\infty$ is the inclusion. Consider now the composite

$$K \xrightarrow{js} I^\infty \xrightarrow{\beta\varphi} K.$$

We have

$$(\beta\varphi)(js) = \beta[\varphi j]s = \beta[\alpha s^{-1}]s = \beta\alpha = f.$$

Thus, f factors through the Hilbert cube I^∞ and consequently has a fixed point. □

As an immediate consequence, we have

(7.9) THEOREM (Generalized Schauder theorem). *Let X be an AR and $f : X \to X$ a compact map. Then f has a fixed point.*

PROOF. Because f factorizes as $X \xrightarrow{\alpha} K \xrightarrow{\beta} X$, where K is compact metric, and by (7.8), $\mathrm{Fix}(\alpha\beta) \neq \emptyset$, we infer that $f = \beta\alpha$ has a fixed point. □

Because every convex set in a normed linear space is an AR, the last result is a generalization of the Schauder fixed point theorem.

8. Fixed Points for Set-Valued Kakutani Maps

By a fixed point of a set-valued map $S : X \to 2^X$ is meant a point $x_0 \in X$ for which $x_0 \in Sx_0$. In this section we will establish some fixed point theorems for certain classes of convex-valued maps $S : C \to 2^C$, where C is a convex set in a normed space; maps of this type arise frequently in applied areas such as control theory and mathematical economics. The statement of these results will be seen to be formally similar to those for the single-valued case.

We begin with the formal definitions. If X and Y are topological spaces, a set-valued map $S : X \to 2^Y$ is called *compact* if the image

$$S(X) = \bigcup\{Sx \mid x \in X\}$$

is contained in a compact subset of Y; if Y is a linear topological space, we say that S is *finite-dimensional* whenever S is compact and $S(X)$ is contained in a finite-dimensional linear subspace of Y. Recall that a map $S : X \to 2^Y$ is called *upper semicontinuous* (written u.s.c.) if $\{x \mid Sx \subset W\}$ is open in X for each open $W \subset Y$.

(8.1) DEFINITION. Let X and Y be subsets of linear topological spaces and $S : X \to 2^Y$ be a set-valued map. If Y is convex, then S is called a *Kakutani map* (or simply a $\mathbb{K}$-*map*) provided S is u.s.c. with nonempty compact convex values.

The set of all $\mathbb{K}$-maps from X to 2^Y is denoted by $\mathbb{K}(X, Y)$. If $S^{-1} \in \mathbb{K}(Y, X)$, we write $S \in \mathbb{K}^*(X, Y)$.

a. Kakutani maps in normed linear spaces

We first establish a version of the Schauder theorem for compact Kakutani maps. Then we will also consider an extension of the transversality concept that is applicable in this more general context.

The proof of our first fixed point result is based on approximation considerations analogous to those required for the Schauder theorem: we first approximate the given compact S by a finite-dimensional set-valued function, and then this simpler set-valued function by a single-valued function. Both of these approximations have other uses, so are of interest in themselves.

(8.2) LEMMA. *Let X be a space, E a normed linear space, and $S : X \to 2^E$ a compact $\mathbb{K}$-map. Let $p : \overline{S(X)} \to L$ be a Schauder projection into a finite-dimensional linear subspace L such that $\|p(y) - y\| < \varepsilon$ for each $y \in \overline{S(X)}$, and for each $x \in X$, let $Tx = \operatorname{Conv} p(Sx)$. Then:*

(a) $\operatorname{Conv} T(X)$ *and each* Tx *are compact and convex,*
(b) $T : X \to 2^E$ *is a finite-dimensional* $\mathbb{K}$*-map,*
(c) $Tx \subset B(Sx, \varepsilon)$ *for each* $x \in X$,
(d) *if* $\overline{S(X)}$ *is contained in a convex* C, *then a* T *satisfying* (a)–(c) *can be chosen such that* $\operatorname{Conv} T(X) \subset C$.

PROOF. (a) Because each $p(Sx) \subset L$ is compact, its convex closure Tx is compact and convex. For the same reason, the compactness of $\overline{S(X)}$ implies that of $\operatorname{Conv} T(X)$.

(b) Clearly, only the fact that T is u.s.c. requires proof. For this purpose, choose any $x \in X$ and let $W \subset E$ be open with $Tx \subset W$; since Tx is compact, there is a $\delta > 0$ such that $B(Tx, \delta) \subset W$, and in particular, $p(Sx) \subset B(Tx, \delta/2)$. Being the composition of two point-compact u.s.c. set-functions, $x \mapsto p(Sx)$ is also point-compact and u.s.c., so there is a neighborhood $V = V(x)$ with $p(Sy) \subset B(Tx, \delta/2)$ for all $y \in V$. Since $\overline{B(Tx, \delta/2)}$ is convex, we find

$$T(y) = \operatorname{Conv} p(Sy) \subset B(Tx, \delta) \subset W$$

for all $y \in V$, so $T(V) \subset W$, and because x is arbitrary, T is u.s.c.

(c) We show that for each given $z \in Tx$ there is a $y \in Sx$ such that $z \in B(y, \varepsilon)$. We have $z = \sum_{i=1}^n \lambda_i w_i$ for suitable $w_i \in p(Sx)$ and real $0 \leq \lambda_i \leq 1$ with $\sum_{i=1}^n \lambda_i = 1$. For each i choose a $y_i \in Sx$ so that $p(y_i) = w_i$; then $y = \sum_{i=1}^n \lambda_i y_i \in Sx$, because Sx is convex, and

$$\Big\| z - \sum_{i=1}^n \lambda_i y_i \Big\| = \Big\| \sum_{i=1}^n \lambda_i (w_i - y_i) \Big\| = \Big\| \sum_{i=1}^n \lambda_i [p(y_i) - y_i] \Big\| < \Big(\sum \lambda_i \Big) \varepsilon = \varepsilon.$$

(d) We take the Schauder projection to be into the linear space spanned by the centers of finitely many ε-balls $\{B(c_i, \varepsilon) \mid c_i \in \overline{S(X)},\ i = 1, \ldots, n\}$ covering the compact $\overline{S(X)}$. □

The second approximation result is the general

(8.3) LEMMA. *Let* X *be a metric space,* E *a normed linear space, and* $S : X \to 2^E$ *a* $\mathbb{K}$*-map. Then for each* $\varepsilon > 0$ *there is a continuous map* $\varphi_\varepsilon : X \to \operatorname{conv} S(X)$ *with the property: for each* $x \in X$ *there is an* $\widehat{x} \in B(x, \varepsilon)$ *such that* $d(\varphi_\varepsilon(x), S\widehat{x}) < \varepsilon$.

PROOF. Fix $\varepsilon > 0$ and for each $x \in X$ let

$$U(x) = \{y \in X \mid Sy \subset B(Sx, \varepsilon)\},$$

which is open because S is u.s.c. Consider now the open cover $\{U(x) \cap B(x, \varepsilon) \mid x \in X\}$ of X; choose a barycentric neighborhood-finite refinement $\{V_\alpha\}$ (i.e., if $\bigcap V_{\alpha_i} \neq \emptyset$, then $\bigcup V_{\alpha_i} \subset$ some $U(x) \cap B(x, \varepsilon)$), and let $\{\varkappa_\alpha\}$

be a partition of unity subordinate to the cover $\{V_\alpha\}$. For each α, choose a $z_\alpha \in S(V_\alpha)$ and define $\varphi_\varepsilon : X \to \operatorname{conv} S(X)$ by

$$\varphi_\varepsilon(x) = \sum_\alpha \varkappa_\alpha(x) z_\alpha.$$

Clearly, φ_ε is continuous; we now verify that it has the required property. Let $x \in X$ be given, and let $V_{\alpha_1}, \dots, V_{\alpha_n}$ be all the sets V_α containing x; $\{V_\alpha\}$ being a barycentric refinement, there is some $U(\widehat{x}) \cap B(\widehat{x}, \varepsilon)$ such that

$$x \in \bigcup_{i=1}^{n} V_{\alpha_i} \subset U(\widehat{x}) \cap B(\widehat{x}, \varepsilon).$$

In particular, $x \in B(\widehat{x}, \varepsilon)$ and $\bigcup_{i=1}^{n} V_{\alpha_i} \subset U(\widehat{x}) = \{y \mid Sy \subset B(S\widehat{x}, \varepsilon)\}$; since each $z_{\alpha_i} \in S(V_{\alpha_i}) \subset B(S\widehat{x}, \varepsilon)$ and $B(S\widehat{x}, \varepsilon)$ is convex, we find that $\varphi_\varepsilon(x) \in B(S\widehat{x}, \varepsilon)$. Thus, $d(\varphi_\varepsilon(x), S\widehat{x}) < \varepsilon$ and $\widehat{x} \in B(x, \varepsilon)$, so the proof is complete. □

We can now establish a version of the Schauder theorem for compact Kakutani maps.

(8.4) THEOREM. *Let C be a convex (not necessarily closed) subset of a normed linear space, and let $S : C \to 2^C$ be a compact $\mathbb{K}$-map. Then S has a fixed point.*

PROOF. We first show that given any $\varepsilon > 0$, there are $\widehat{c} \in C$ and $\widehat{z} \in S\widehat{c}$ such that $\|\widehat{c} - \widehat{z}\| < 4\varepsilon$.

By (8.2) there is a u.s.c. map $T : C \to 2^C$ such that $\operatorname{Conv} T(X) \subset C$ and each Tx is compact, convex, and nonempty, with $Tx \subset B(Sx, \varepsilon)$ for each $x \in X$. For this T and the given $\varepsilon > 0$, choose a $\varphi : C \to \operatorname{Conv} T(X) \subset C$ satisfying the conditions of (8.3). Because $\operatorname{Conv} T(X)$ is compact, φ is a compact map, so by the Schauder theorem, there is a fixed point $\varphi(c) = c$. According to (8.3), there is a $\widehat{c} \in B(c, \varepsilon)$ with $d(c, T\widehat{c}) < \varepsilon$; choosing now a $\widehat{w} \in T\widehat{c}$ with $d(c, \widehat{w}) < 2\varepsilon$ and then a $\widehat{z} \in S\widehat{c}$ with $d(\widehat{w}, \widehat{z}) < \varepsilon$, we have $d(\widehat{c}, \widehat{z}) < 4\varepsilon$, which establishes our assertion.

Thus, for each $n = 1, 2, \dots$ there are $c_n \in C$ and $z_n \in Sc_n$ with $d(c_n, z_n) < 1/n$. Since the z_n belong to the compact $\overline{S(C)} \subset C$, there is a subsequence converging to some $c_0 \in C$; the corresponding c_n therefore also converge to c_0; because S is u.s.c., this implies $c_0 \in Sc_0$, and the proof is complete. □

Let C be a convex set in a normed linear space and (X, A) a closed pair in C. Denote by $\mathscr{K}(X, 2^C)$ the set of compact Kakutani maps $S : X \to 2^C$, and let $\mathscr{K}_A(X, 2^C) \subset \mathscr{K}(X, 2^C)$ be those that are fixed point free on A. By a homotopy between $S, T \in \mathscr{K}_A(X, 2^C)$ is meant an $H \in \mathscr{K}_{A\times I}(X \times I, 2^C)$ with $H(x, 0) = Sx$ and $H(x, 1) = Tx$ for each $x \in X$. The central lemma

(6.4.6) is valid with $\mathscr{K}_A(X, 2^C)$ replacing $\mathscr{K}_A(X, C)$ throughout. The proof, which is a repetition of that given, is left to the reader; it utilizes the facts that the composition of point-compact u.s.c. maps, as well as the sum $x \mapsto Sx + Tx$ and the intersection $x \mapsto Sx \cap Tx$ of two such maps are also point-compact and u.s.c. Thus, the transversality theorem (6.4.7) holds also for maps $\mathscr{K}_A(X, 2^C)$, and as before, with the aid of (8.4) leads to the Leray-Schauder principle for compact Kakutani maps and then to

(8.5) THEOREM (Nonlinear alternative). *Let C be a convex set in a normed linear space, and let $U \subset C$ be open with $0 \in U$ Then each $S \in \mathscr{K}_{\partial U}(\overline{U}, 2^C)$ has at least one of the following two properties:*
(a) *S has a fixed point,*
(b) *there exist $x \in \partial U$ and $\lambda \in (0, 1)$ such that $x \in \lambda Sx$.* □

As in the single-valued case, many fixed point results for compact Kakutani maps follow from (8.5) by imposing conditions that prevent occurrence of the second property.

b. Kakutani maps in locally convex spaces

We illustrate another technique for dealing with Kakutani maps by establishing a version of the Tychonoff theorem in arbitrary locally convex spaces:

(8.6) THEOREM. *Let C be a compact convex subset of a locally convex space E, and let $S : C \to 2^C$ be a $\mathbb{K}$-map. Then S has a fixed point.*

PROOF. Let $\{V_\alpha \mid \alpha \in \mathscr{A}\}$ be a base of closed convex symmetric neighborhoods of 0 for E, and for each $\alpha \in \mathscr{A}$ let $\operatorname{Fix}_\alpha S = \{x \in C \mid x \in V_\alpha + Sx\}$.

We first show that each $\operatorname{Fix}_\alpha S$ is closed in C: indeed, the set

$$\Delta_\alpha = \{(x, y) \mid y \in x + V_\alpha\}$$

is a closed neighborhood of the diagonal in $C \times C$, the graph G_S of S is closed in $C \times C$ because S is u.s.c., and $\operatorname{Fix}_\alpha S$, being the projection of the compact set $\Delta_\alpha \cap G_S$ into C, is therefore closed in C.

We next show that each $\operatorname{Fix}_\alpha S$ is nonempty. Let $\{x_1, \dots, x_k\}$ be a finite set such that $\bigcup_{i=1}^k x_i + V_\alpha$ covers C, let $K = \operatorname{Conv}\{x_1, \dots, x_k\}$, and define $S_K : K \to 2^K$ by $x \mapsto (Sx + V_\alpha) \cap K$. Each $S_K(x)$ is convex, because V_α is convex; and it is nonempty because V_α is a symmetric neighborhood of 0, so from $Sx \subset C \subset K + V_\alpha$ we find $Sx \cap (K + V_\alpha) \neq \emptyset$ and therefore $(Sx + V_\alpha) \cap K \neq \emptyset$. Thus, Theorem (8.4) applies to give an $x \in (Sx + V_\alpha) \cap K$, and therefore $\operatorname{Fix}_\alpha S \neq \emptyset$.

Since $\operatorname{Fix}_\alpha S \subset \operatorname{Fix}_\beta S$ whenever $V_\alpha \subset V_\beta$, the family $\{\operatorname{Fix}_\alpha S \mid \alpha \in \mathscr{A}\}$ of nonempty compact sets has the finite intersection property, and therefore there is a point $x_0 \in \bigcap_\alpha \operatorname{Fix}_\alpha S$. For this x_0 we must have $x_0 = Sx_0$:

otherwise, x_0 would not belong to $Sx_0 + V_\alpha$ for an appropriate V_α. This completes the proof. □

c. Coincidences in spaces with sufficiently many linear functionals

To illustrate still another technique for dealing with Kakutani maps we now derive a coincidence theorem for inward and outward maps in the sense of Fan in spaces with sufficiently many linear functionals.

We begin with the terminology. Let E be a linear topological space with sufficiently many linear functionals. We recall that an *open half-space* in E is a set of the form $H = \{x \in E \mid \varphi(x) < t\}$ for some nontrivial $\varphi \in E^*$ and some real number t. Two sets $M, N \subset E$ are said to be *strictly separated* by a closed hyperplane if there exist $\varphi \in E^*$ and $t \in \boldsymbol{R}$ such that $M \subset \{x \in E \mid \varphi(x) < t\}$ and $N \subset \{x \in E \mid \varphi(x) > t\}$.

Let $X \subset E$. We say that a map $A : X \to 2^E$ is *inward* (respectively *outward*) *in the sense of Fan* if for any $x \in X$ and any $\varphi \in E^*$ satisfying $\varphi(x) \le \varphi(y)$ for all $y \in X$ there is a point $u \in Ax$ such that $\varphi(u) \ge \varphi(x)$ (respectively $\varphi(u) \le \varphi(x)$).

(8.7) THEOREM. *Let X be a nonempty compact convex subset of E and $A, B : X \to 2^E$ be two $\mathbb{K}$-maps. If A is inward and B is outward in the sense of Fan, then for some $(x_0, y_0) \in X \times E$ we have $y_0 \in Ax_0 \cap Bx_0$.*

PROOF. Suppose that the assertion of the theorem is false, i.e., $Ax \cap Bx = \emptyset$ for each $x \in X$. Since E has sufficiently many linear functionals, the convex compact sets Ax and Bx can be strictly separated by a closed hyperplane. Precisely, for each $x \in X$, we can find $\varphi_x \in E^*$ and a number $t_x \in \boldsymbol{R}$ such that

$$Ax \subset \{u \in E \mid \varphi_x(u) < t_x\},$$
$$Bx \subset \{v \in E \mid \varphi_x(v) > t_x\}.$$

Since A, B are u.s.c. on X, there is a neighborhood N_x of x in X such that

$$(*) \qquad \begin{cases} Ay \subset \{u \in E \mid \varphi_x(u) < t_x\} & \text{for } y \in N_x, \\ By \subset \{v \in E \mid \varphi_x(v) > t_x\} & \text{for } y \in N_x. \end{cases}$$

For each $\varphi \in E^*$ we let

$$P(\varphi) = \{y \in X \mid \varphi(u) < \varphi(v) \text{ for all } u \in Ay \text{ and } v \in By\}$$

and denote by $Q(\varphi)$ the interior of $P(\varphi)$ relative to X. From $(*)$ it follows that $N_x \subset P(\varphi_x)$ and therefore $x \in Q(\varphi_x)$. This implies that the family $\{Q(\varphi_x) \mid x \in X\}$ of open sets covers X, and by the compactness of X we get a finite set $\{\varphi_{x_1}, \dots, \varphi_{x_n}\} \subset E^*$ such that

$$X = \bigcup_{i=1}^{n} Q(\varphi_{x_i}).$$

Let $\{\lambda_i\}_{i=1}^n$ be a partition of unity on X subordinate to this covering; letting $\varphi_i = \varphi_{x_i}$ for $i \in [n]$, we define $\psi : X \to E^*$ by

$$\psi(y) = \sum_{i=1}^{n} \lambda_i(y)\varphi_i \quad \text{for } y \in X.$$

If $\lambda_i(y) \neq 0$, then $y \in Q(\varphi_i) \subset P(\varphi_i)$, so we have

$$\varphi_i(u) < \varphi_i(v) \quad \text{for all } u \in Ay \text{ and } v \in By.$$

Consequently, for each $y \in X$ we have

$$(**) \qquad [\psi(y)]u < [\psi(y)]v \quad \text{for all } u \in Ay \text{ and } v \in By.$$

We now apply Theorem (1.5) for $\lambda = 0$ to the function $f : X \times X \to \boldsymbol{R}$ given by

$$f(x,y) = [\psi(y)](y-x) = \sum_{i=1}^{n} \lambda_i(y)\varphi_i(y-x), \qquad (x,y) \in X \times X;$$

clearly, f is continuous, and $x \mapsto f(x,y)$ is quasi-concave on X for each $y \in X$. Since for no $x_0 \in X$ could we have $f(x_0, x_0) > 0$, by (1.5) we get a point $y_0 \in X$ such that

$$f(x, y_0) = [\psi(y_0)](y_0 - x) \leq 0 \qquad \text{for all } x \in X.$$

Consequently, if we let $l = \psi(y_0)$, we have $l(y_0) = \min\{l(x) \mid x \in X\}$; i.e., $\{x \in E \mid l(y_0) = l(x)\}$ is a supporting hyperplane of X at y_0. Because A is inward and B is outward by hypothesis, there are $u_0 \in Ay_0$ and $v_0 \in By_0$ such that

$$(***) \qquad [\psi(y_0)]u_0 \geq [\psi(y_0)]y_0 \geq [\psi(y_0)]v_0,$$

which contradicts $(**)$. The proof is complete. □

Theorem (8.7) can be specialized by taking the identity map on X as A or B. This yields the following fixed point theorem for inward (or outward) $\mathbb{K}$-maps.

(8.8) THEOREM. *Let X be a nonempty compact convex subset of E, and let $A : X \to 2^E$ be a $\mathbb{K}$-map that is inward (or outward) in the sense of Fan. Then A has a fixed point.* □

9. Theorem of Ryll-Nardzewski

If $\mathscr{F}$ is any family of maps of a space X into itself, by a *fixed point for* $\mathscr{F}$ is meant a point $x_0 \in X$ such that $T(x_0) = x_0$ for each $T \in \mathscr{F}$. Our aim in this section is to prove the theorem of Ryll-Nardzewski on fixed points of families of affine self-maps of convex sets, which has numerous consequences

and applications. The proof will be based entirely on functional analysis and is independent of the previous material in this book; it is given here to indicate the use of some different techniques in fixed point theory. Although the Markoff Kakutani theorem (3.3.2) and the Kakutani theorem (3.4.3) are also concerned with fixed points for families of affine maps, they were treated separately because their direct proof does not require the more sophisticated background in functional analysis needed for the Ryll-Nardzewski theorem.

To formulate the theorem, we need the general

(9.1) DEFINITION. Let $\mathscr{F}$ be a family of self-maps of a set X in some linear topological space. The family $\mathscr{F}$ is called *noncontracting* on X if for any distinct points x, y of X, zero does not belong to the closure of the set $\{Tx - Ty \mid T \in \mathscr{F}\}$. The family $\mathscr{F}$ is called *distal* on X if for any distinct x, y in X, there is an open covering $\{V_\alpha \mid \alpha \in \mathscr{A}\}$ of X such that $T(y) \notin \bigcup\{V_\alpha \mid Tx \in V_\alpha\}$ for each $T \in \mathscr{F}$

We will need the following

(9.2) LEMMA. *Let $\mathscr{F}$ be a family of self-maps of a compact set X in a locally convex space E. The following conditions are equivalent:*

(1) *$\mathscr{F}$ is distal on X,*

(2) *for each net $\{T_\beta \in \mathscr{F} \mid \beta \in \mathscr{B}\}$ and any pair x, y of distinct points of X, if $T_\beta x \to u$ and $T_\beta y \to v$, then $u \neq v$,*

(3) *$\mathscr{F}$ is noncontracting on X.*

PROOF. (1)$\Rightarrow$(2). Suppose $u = v$. In any cover of X, the set containing u must contain almost all $T_\beta x$, $T_\beta y$, so that $\mathscr{F}$ cannot be distal on X.

(2)$\Rightarrow$(3). Assume that 0 belongs to the closure of $\{Tx - Ty \mid T \in \mathscr{F}\}$. We can then construct a net $\{T_\beta \mid \beta \in \mathscr{B}\}$ with $T_\beta x - T_\beta y \to 0$. By the compactness of X, we can assume that $T_\beta x \to u$ and $T_\beta y \to v$; then $u = v$, contradicting (2).

(3)$\Rightarrow$(1). If x, y are distinct points of X, then by (3) there is an open (in E) nbd W of 0 that contains no $Tx - Ty$, $T \in \mathscr{F}$. Choose a nbd U of 0 with $U - U \subset W$; then $\{X \cap (U + p) \mid p \in X\}$ is an open cover of X, and for no $T \in \mathscr{F}$ do Tx and Ty belong to a common $U + p$: if $\{Tx, Ty\} \subset U + p$, then $Tx - Ty = (Tx - p) - (Ty - p) \subset U - U \subset W$, which is impossible.□

If $\mathscr{F}$ is a family of self-maps of X, a subset $A \subset X$ is called *$\mathscr{F}$-invariant* if $T(A) \subset A$ for all $T \in \mathscr{F}$; a closed nonempty $A \subset X$ that is $\mathscr{F}$-invariant and has no proper closed $\mathscr{F}$-invariant subset is called a *minimal closed $\mathscr{F}$-invariant subset*. The proof of the following result relies on an extended version of the Krein–Milman theorem in locally convex spaces: if the convex closure $\operatorname{Conv} A$ of a set A is compact, then $\operatorname{Conv} A$ has extreme points, and if A itself is also compact, then those extreme points belong to A.

(9.3) THEOREM. *Let C be a nonempty compact convex set in a locally convex space E, and let $\mathscr{F}$ be a semigroup of continuous affine maps of C into itself. If $\mathscr{F}$ is distal on each minimal closed $\mathscr{F}$-invariant set, then $\mathscr{F}$ has a fixed point.*

PROOF. Let $\mathscr{K}$ be the collection of all nonempty compact convex subsets that are $\mathscr{F}$-invariant; since $C \subset \mathscr{K}$, this family is not empty. Partially order $\mathscr{K}$ by inclusion; since each descending chain $\{K_\alpha\}$ has the lower bound $\bigcap K_\alpha$, the Kuratowski–Zorn lemma gives a minimal $C_0 \subset C$ in $\mathscr{K}$.

Now let $\mathscr{K}_0$ be the family of all nonempty compact subsets of C_0 that are $\mathscr{F}$-invariant; again by the Kuratowski Zorn lemma there is a closed minimal $\mathscr{F}$-invariant $X \subset C_0$. We are going to prove that X consists of a single point.

Assume $x \neq y$ were two distinct points in X. Because $(x+y)/2 \in C_0$ and C_0 is $\mathscr{F}$-invariant, we have $A = \{T((x+y)/2) \mid T \in \mathscr{F}\} \subset C_0$, so $\bar{A} \subset C_0$ is compact. Moreover, $\bar{A}$ is $\mathscr{F}$-invariant, and because each T is affine, the convex closure $\operatorname{Conv} \bar{A} \subset C_0$ is also $\mathscr{F}$-invariant, so by the minimal property of C_0 we have $\operatorname{Conv} \bar{A} = C_0$. Now let z be an extreme point of C_0; because $\bar{A}$ is compact, the extended Krein Milman theorem shows that $z \in \bar{A}$, so there is a net $T_\alpha((x+y)/2) \to z$. Since $T_\alpha x$ and $T_\alpha y$ belong to the compact set X, we may assume that $T_\alpha x \to u \in X$ and $T_\alpha y \to v \in X$, so that

$$z = \lim \tfrac{1}{2}[T_\alpha x + T_\alpha y] = (u+v)/2,$$

and because z is an extreme point, $u = v = z$. This means that for each open covering $\{V_\alpha\}$ of X, any set V_{α_0} that contains u will contain almost all the $T_\alpha x$, $T_\alpha y$, and consequently, $\mathscr{F}$ is not distal on the closed minimal $\mathscr{F}$-invariant set X. The assumption that X has more than one point has led to a contradiction with our hypothesis. Thus, X must consist of a single point x_0, and since x_0 is $\mathscr{F}$-invariant, we have $T(x_0) = x_0$ for all $T \in \mathscr{F}$, so it is a fixed point for $\mathscr{F}$ This completes the proof. □

As a first consequence, we obtain

(9.4) THEOREM (F. Hahn). *Let C be a compact convex subset of a locally convex space E, and let $\mathscr{F}$ be a semigroup of continuous affine self-maps of C. If $\mathscr{F}$ is distal on C, then $\mathscr{F}$ has a fixed point.*

PROOF. It is clear that a family $\mathscr{F}$ distal on C is distal on any $\mathscr{F}$-invariant subset, so this follows immediately from (9.3). □

We use this result to extend Kakutani's theorem (3.4.3) to locally convex spaces. Recall that if C is a subset of a linear topological space, and $\mathscr{F}$ a family of self-maps of C, then $\mathscr{F}$ is *equicontinuous* on C if for each neighborhood W of 0, there is a neighborhood V of 0 such that whenever $x - y \in V$

then $Tx - Ty \in W$ for all $T \in \mathscr{F}$. The members of an equicontinuous family are, clearly, necessarily continuous.

(9.5) THEOREM (Kakutani). *Let C be a compact convex subset of a locally convex space, and let $\mathscr{F}$ be a group of affine transformations of C into itself. If $\mathscr{F}$ is equicontinuous on C, then $\mathscr{F}$ has a fixed point.*

PROOF. This will follow from (9.4) once we show that $\mathscr{F}$ is distal on C. In what follows U, V, W denote open neighborhoods of the origin.

Assume $\mathscr{F}$ were not distal on C; then there must be a pair x, y of distinct points in C with the property: for each neighborhood U there is a $T_U \in \mathscr{F}$ such that $T_U(x)$ and $T_U(y)$ belong to a common set of the open cover $\{U + c \mid c \in C\}$ of C. Choose now a nbd W so that $x \notin y + W$; then for each V there is a U with $U - U \subset V$, and we have $T_U x - T_U y \in V$, yet for the function $T_U^{-1} \in \mathscr{F}$ we have

$$T_U^{-1}(T_U x) - T_U^{-1}(T_U y) = x - y \notin W.$$

Thus, $\mathscr{F}$ cannot be equicontinuous on C. The equicontinuous family $\mathscr{F}$ is therefore distal on C, and by (9.4), the proof is complete. □

The Ryll-Nardzewski theorem is a generalization of (9.4) that involves an interplay between the given natural topology of a locally convex space (which we will call the strong topology, for emphasis) and its weak topology; it will be obtained as a consequence of Theorem (9.4). The main fact needed for the proof is the Mazur theorem in locally convex spaces: A convex set is weakly closed if and only if it is strongly closed; in particular, a strongly closed convex subset of a weakly compact space is weakly compact.

(9.6) THEOREM (Ryll-Nardzewski). *Let C be a nonempty weakly compact convex set in a locally convex space E, and let $\mathscr{F}$ be a semigroup of weakly continuous affine self-maps of C. If $\mathscr{F}$ is strongly noncontracting on C, then $\mathscr{F}$ has a fixed point.*

PROOF. We must show that $\bigcap\{\text{Fix}(T) \mid T \in \mathscr{F}\} \neq \emptyset$, and we begin by reducing the problem. Noting that each set Fix(T) is weakly closed, therefore weakly compact, it is enough to show that each finite intersection of the sets Fix(T) is nonempty.

Let then $T_1, \dots, T_n$ be finitely many members of $\mathscr{F}$, and let $\mathscr{S}$ be the subsemigroup generated by $T_1, \dots, T_n$; clearly, $\mathscr{S}$ is countable, and it suffices to show that $\mathscr{S}$ has a fixed point.

We reduce the problem further. Pick any $c_0 \in C$ and consider the convex closure $Q = \text{Conv}\{T(c_0) \mid T \in \mathscr{S}\}$; because $\mathscr{S}$ is countable, Q is strongly separable. Moreover, because each T is affine, Q is $\mathscr{S}$-invariant, and because Q is a closed convex subset of C, it is weakly closed, therefore weakly compact. Thus, replacing C by Q and $\mathscr{F}$ by $\mathscr{S}$, it is enough to prove the

theorem with the additional hypothesis that C is strongly separable. We now begin the proof.

We work in the locally convex space E with the weak topology, and the result will follow from (9.3) if we can show that $\mathscr{F}$ is weakly distal on each given weakly closed minimal $\mathscr{F}$-invariant set $X \subset C$.

Let then X be such a set, and let $x \neq y$ be two elements of X. Because $\mathscr{F}$ is strongly noncontracting, there exists a strongly open nbd V of 0 that contains no element of $\{Tx - Ty \mid T \in \mathscr{F}\}$. Choose a convex nbd W of 0 so that $\overline{W} - \overline{W} \subset V$. Then $\overline{W}$ is a strongly closed convex body; and since $C \supset X$ is strongly separable, a countable number of translates $\overline{W}_i \equiv \overline{W} + x_i$, $i \in N$, $x_i \in X$, cover X. Since each $\overline{W}_i$ is strongly closed and convex, it is also weakly closed. So $\{X \cap \overline{W}_i \mid i \in N\}$ is a countable weakly closed cover of the weakly compact set X. By Baire's theorem, at least one of these sets contains a weakly open set U, say $U \subset X \cap (\overline{W} + x_0)$.

We show that the family $\{T^{-1}(U) \mid T \in \mathscr{F}\}$ of weakly open sets satisfies the requirement of Definition (9.1) for the points $x \neq y$. First, these sets must cover X: otherwise, $X - \bigcup\{T^{-1}(U) \mid T \in \mathscr{F}\}$ would be a weakly compact $\mathscr{F}$-invariant proper subset of X, contradicting the minimality of X. Next, for no $S \in \mathscr{F}$ do Sx and Sy belong to a common set $T^{-1}(U)$: otherwise, TSx and TSy would belong to $U \subset X \cap (\overline{W} + x_0)$, so that $TSx - TSy = (TSx - x_0) - (TSy - x_0) \in \overline{W} - \overline{W} \subset V$, and since $TS \in \mathscr{F}$ and $\mathscr{F}$ is strongly noncontracting, this would contradict the choice of V.

Thus, the requirements of (9.1) are satisfied in the weak topology of X, and since x, y are arbitrary, this shows $\mathscr{F}$ is weakly distal on X. Thus, by (9.3) (applied to E with the weak topology), the family $\mathscr{F}$ has a fixed point on X, and as we have observed, this is enough to complete the proof. □

10. Miscellaneous Results and Examples

A. *Fixed points and coincidences for set-valued maps*

In subsections A, B, and C set-valued transformations (called simply *maps*) are denoted by capital letters; small letters stand for ordinary (i.e., single-valued) functions. Unless otherwise stated, we write "compact convex set" instead of "nonempty compact convex subset of a linear topological space"

Let $\mathscr{S}$ be a class of maps. We let $\mathscr{S}(X, Y) = \{T : X \to 2^Y \mid T \in \mathscr{S}\}$, $\mathscr{S}^* = \{T \mid T^{-1} \in \mathscr{S}\}$, $\mathscr{S}_c = \{T = T_k \circ \cdots \circ T_1 \mid T_i \in \mathscr{S}$ for all $i \in [k]$ and some $k \in N\}$ and $\mathscr{F}_{\mathscr{S}} = \{X \mid \mathrm{Fix}(T) \neq \emptyset$ for all $T \in \mathscr{S}(X, X)\}$. $\mathscr{S}^*$ is called the *dual class* to $\mathscr{S}$: clearly, $(\mathscr{S}^*)^* = \mathscr{S}$.

(A.1) A class $\mathscr{R}$ of maps between topological spaces is called *regular* if:

(i) maps in $\mathscr{R}$ have nonempty values,

(ii) if $s : X \to Y$ is continuous and $T \in \mathscr{R}(Y, Z)$, then $T \circ s \in \mathscr{R}(X, Z)$,

(iii) if S and T are in $\mathscr{R}$, then so is their Cartesian product $S \times T$.

Let $\mathscr{R}$ be a regular class of maps. Show: If $X \times Y \in \mathscr{F}_{\mathscr{R}}$ and $S \in \mathscr{R}(X,Y)$, $T \in \mathscr{R}^*(X,Y)$, then S and T have a coincidence.

(A.2) (*Maps of type* $\mathbb{B}$) If X and Y are spaces, we say that $S : X \to 2^Y$ is a *Browder map* or simply a $\mathbb{B}$-*map* provided (i) Y is convex and nonempty, (ii) for each compact $K \subset X$ there exists a finite set $\{y_1, \dots, y_n\} \subset Y$ and a continuous selection $s : K \to Y$ of S (i.e., $sx \in Sx$ for each $x \in K$) such that $s(K) \subset C \equiv \mathrm{Conv}\{y_1, \dots, y_n\}$. The class of $\mathbb{B}$-maps is denoted by $\mathbb{B}$. Show: Every $\mathbb{F}$-map is a $\mathbb{B}$-map, i.e., $\mathbb{F} \subset \mathbb{B}$ (Browder [1968]).

(A.3) (*Geometric results for maps of type* $\mathbb{B}$) Show:

(a) $\mathbb{B}$ is a regular class of maps and $\mathbb{B}_c = \mathbb{B}$.

(b) If Y is a compact convex set and $T \in \mathbb{B}(Y,Y)$ or $T \in \mathbb{B}^*(Y,Y)$, then $\mathrm{Fix}(T) \neq \emptyset$.

(c) If X, Y are compact convex sets and $S \in \mathbb{B}(X,Y)$, $T \in \mathbb{B}^*(X,Y)$, then their graphs intersect: $G_S \cap G_T \neq \emptyset$.

(A.4) (*Intersection theorem for* $\mathbb{B}$-*maps*) Let $\{X_i \mid i \in I\}$ be an arbitrary family of compact convex sets. We let $X^i = \prod_{j \neq i} X_j$. If $X = \prod_{i \in I} X_i$, then $X = X_i \times X^i$ for each $i \in I$, and every $x \in X$ can be written as $x = (x_i, x^i)$, where $x_i \in X_i$ and $x^i \in X^i$ Given $T_i : X^i \to 2^{X_i}$, its graph G_{T_i} is a subset of X. Show: If for each $i \in I$ we are given $T_i \in \mathbb{B}(X^i, X_i)$, then $\bigcap_{i \in I} G_{T_i} \neq \emptyset$.

[For each $i \in I$ choose, for an appropriate compact convex $C_i \subset X_i$, a continuous selection $t_i : X^i \to C_i$ of T_i; consider the map of $\prod_{i \in I} C_i$ into itself given by $x \mapsto \{t_i(x^i)\}$ and use the Tychonoff fixed point theorem.]

(A.5) (*Maps of type* Φ) Let X, Y be nonempty subsets of linear topological spaces and assume that Y is convex. We say that $S : X \to 2^Y$ is a Φ-*map* (written $S \in \Phi(X,Y)$) provided the values of S are convex and S admits a selection $\widehat{S} : X \to 2^Y$ with nonempty values and open fibers. Show:

(a) $\mathbb{F} \subset \Phi \subset \mathbb{B}$.

(b) If $S \in \Phi(X,Y)$ and X is paracompact, then S admits a continuous selection $s : X \to Y$

(A.6) Let X be a nonempty convex subset of a linear topological space and $S \in \mathbb{F}(X,X)$. Assume there exists a compact $K \subset X$ and a compact convex $C \subset X$ such that $Sx \cap C \neq \emptyset$ for each $x \in X - K$. Show: $\mathrm{Fix}(S) \neq \emptyset$.

(The results (A.3)–(A.6) are due to Ben-El-Mechaiekh et al. [1982], [1987].)

(A.7) Let X be a compact convex set, and let $f, g : X \times X \to \boldsymbol{R}$ satisfy:

(i) $g(x,y) \le f(x,y)$ for all $(x,y) \in X \times X$,

(ii) $x \mapsto f(x,y)$ is quasi-concave on X for each $y \in X$,

(iii) $y \mapsto g(x,y)$ is l.s.c. on X for each $x \in X$.

Show:

(A) Given any $\lambda \in \boldsymbol{R}$, either (a) there is a $y_0 \in X$ such that $g(x, y_0) \le \lambda$ for all $x \in X$, or (b) there is a $w \in X$ such that $f(w,w) > \lambda$.

(B) The following minimax inequality holds:

$$\inf_{y \in X} \sup_{x \in X} g(x,y) \le \sup_{x \in X} f(x,x).$$

[Use the fixed point theorem for Φ-maps.]

(A.8) Let X, Y be compact convex sets, and let $f, s, t, g : X \times Y \to \boldsymbol{R}$ satisfy:

(i) $f(x,y) \le s(x,y) \le t(x,y) \le g(x,y)$ for all $(x,y) \in X \times Y$,

(ii) $y \mapsto f(x,y)$ is l.s.c. on Y for each $x \in X$.

(iii) $x \mapsto s(x, y)$ is quasi-concave on X for each $y \in Y$,
(iv) $y \mapsto t(x, y)$ is quasi-convex on Y for each $x \in X$,
(v) $x \mapsto g(x, y)$ is u.s.c. on X for each $y \in Y$

Show:

(A) Given any $\lambda \in \boldsymbol{R}$, either (a) there is a $y_0 \in Y$ such that $f(x, y_0) \leq \lambda$ for all $x \in X$, or (b) there is an $x_0 \in X$ such that $g(x_0, y) > \lambda$ for all $y \in Y$

(B) The following minimax inequality holds:

$$\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).$$

[Use the coincidence theorem for Φ-maps and Φ^*-maps.]

(For (A.7) cf. Yen [1981]; for (A.8) see Ben-El-Mechaiekh et al. [1982].)

(A.9) (*Fan matching theorem*) Let X be a nonempty convex subset of a linear topological space and $\emptyset \neq Y \subset X$. Assume $T\ \ Y \to 2^X$ is a surjective map with open values such that:

(a) for some $\emptyset \neq Y_0 \subset Y$, the complement $X - \bigcup\{Ty \mid y \in Y_0\}$ is compact (or empty),
(b) Y_0 is contained in a compact convex subset of X.

Show: There exists a finite set $\{x_1, \ldots, x_k\} \subset X$ such that $\operatorname{conv}\{x_1, \ldots, x_k\} \cap \bigcup_{i=1}^k Tx_i \neq \emptyset$ (Fan [1984]).

(A.10) (*Schauder–Tychonoff theorem for* $\mathbb{F}$*-maps and* Φ*-maps*) Let X, Y be nonempty convex sets in linear topological spaces. A map $S: X \to 2^Y$ is *compact* if $S(X) \subset K$ for some compact $K \subset Y$ Show: If X is a convex subset of a locally convex linear space and $S: X \to 2^X$ is a compact $\mathbb{F}$-map (or more generally a Φ-map), then $\operatorname{Fix}(S) \neq \emptyset$.

[First prove that if K is a compact subset of a linear topological space, then its convex hull $\operatorname{conv} K$ is paracompact. Then use (A.5)(b) and the Schauder–Tychonoff theorem.]

(A.11) (*Nonlinear alternative for* $\mathbb{F}$*-maps and* Φ*-maps*) Let U be an open subset of a locally convex space E with $0 \in U$, and $S: \overline{U} \to 2^E$ be a compact $\mathbb{F}$-map (or more generally, a compact Φ-map). Show: either (i) $\operatorname{Fix}(S) \neq \emptyset$, or (ii) there are $x \in \partial U$ and $\lambda \in (0, 1)$ such that $x \in \lambda Sx$.

[Prove that the linear span $L(K)$ of a compact set K in a linear topological space is paracompact and use the nonlinear alternative for single-valued compact maps (6.9.E.8).]

B. Selecting families and the generalized Nash theorem

In this subsection we deal with families $\{T_i\}_{i \in I}$ of set-valued maps $T_i : X \to 2^{Y_i}$, where X is a subset of a linear topological space and $\{Y_i\}_{i \in I}$ is a family of convex sets. By a *selecting family* for $\{T_i\}_{i \in I}$ is meant a family $\{s_i\}_{i \in I}$ of continuous maps $s_i : X \to Y_i$ such that for each $x \in X$ there exists an $i \in I$ with $s_i(x) \in T_i(x)$.

(B.1) A family $\{T_i\}_{i \in I}$ is called a *Fan family* (or simply an $\mathbb{F}$*-family*) provided:

(i) all maps T_i have convex values and open fibers,
(ii) for each $x \in X$, $T_i(x) \neq \emptyset$ for some $i \in I$.

Prove: If X is paracompact, then any $\mathbb{F}$-family $\{T_i\ \ X \to 2^{Y_i}\}_{i \in I}$ admits a selecting family $\{s_i : X \to Y_i\}_{i \in I}$.

[Choose a family $\{A_i\}_{i \in I}$ of closed subsets of X such that $A_i \subset \{x \in X \mid T_i(x) \neq \emptyset\}$ for $i \in I$ and $\bigcup_{i \in I} A_i = X$ (cf. Engelking's book [1989], p. 301). Then define $S_i = X \to 2^{Y_i}$ by $S_i(x) = T_i(x)$ for $x \in A_i$ and $S_i(x) = Y_i$ otherwise; observing that every S_i has nonempty convex values and open fibers, apply the Michael theorem (see Appendix, (B.14)) to get a selection s_i of S_i. Prove that $\{s_i\}_{i \in I}$ is the desired selecting family.]

(B.2) Let $X = \prod_{i \in I} X_i$, where each X_i is compact convex, and assume $\{T_i : X \to 2^{X_i}\}$ is an $\mathbb{F}$-family. Show: There exists an index $i \in I$ and a point $x \in X$ such that $x_i \in T_i(x)$.

[Establish first the assertion for I finite. Then (using compactness of X) find a finite set $J \subset I$ such that for each $j \in J$ there exists a finite subset $\{x_j^1, x_j^2, \ldots, x_j^{n_j}\} \subset X_j$ with $X = \bigcup_{j \in J} \bigcup_{k=1}^{n_j} T_j^{-1}(x_j^k)$; conclude by reducing the case of arbitrary I to that in which I is finite.]

(B.3) Let X and $\{X_i\}_{i \in I}$ be as in (B.2) and assume $\{f_i : X \times X_i \to \boldsymbol{R}\}_{i \in I}$ is a family of real-valued functions satisfying:

(i) $y_i \mapsto f_i(x, y_i)$ is quasi-concave on X_i for each $x \in X$ and $i \in I$,

(ii) $x \mapsto f_i(x, y_i)$ is l.s.c. on X for each $y_i \in X_i$ and $i \in I$.

Show: There exists a point $\widehat{x} \in X$ such that for each $i \in I$,

$$\sup_{y_i \in X_i} f_i(\widehat{x}, y_i) \le \sup_{x \in X} f_i(x, x_i).$$

(B.4) (*Generalized Nash theorem*) Let $\{X_i\}_{i \in I}$ be a family of compact convex sets and $\{g_i\}_{i \in I}$ be a family of continuous real-valued functions on $X = \prod_{i \in I} X_i$ such that $y_j \mapsto g_j(y_j, x^j)$ is quasi-concave on X_j for each $x^j \in X^j$. Show: There exists a point $\widehat{x} \in X$ such that for each $j \in I$ we have $g_j(\widehat{x}) = \max_{y_j \in X_j} g_j(y_j, \widehat{x}^j)$.

[Define $\{f_i : X \times X_i \to \boldsymbol{R}\}_{i \in I}$ by $f_i(x, y_i) = g_i(y_i, x^i) - g_i(x)$ for $x \in X$, $y_i \in X_i$, $i \in I$, and by observing that (i), (ii) of (B.3) are satisfied, apply (B.3) to the family $\{f_i\}$.]

(The results (B.1)–(B.3) and the proof of (B.4) are due to Deguire–Lassonde [1995].)

C. Applications to economics and game theory

(C.1) (*Maximal elements of binary relations*) If X is a set and $S : X \to 2^X$, we let $\prec_S$ be the binary relation on X given by $x \prec_S y \Leftrightarrow y \in Sx$ (cf. (3.5.A.3)).

Let C be compact convex, and let $S, T : C \to 2^C$ satisfy:

(i) S has open fibers,

(ii) $Sx \subset Tx$ for each $x \in C$,

(iii) $y \notin \operatorname{conv}(Ty)$ for $y \in C$.

Show: There exists a maximal element in $(C, \prec_S)$.

[Use (1.1.3), (5.8.2), and prove that $Sx_0 = \emptyset$ for some $x_0 \in C$.]

(C.2) Let X, Y be nonempty convex sets in linear topological spaces, one of them being compact. Show: If $S \in \mathbb{F}(X, Y)$ and $T \in \mathbb{F}^*(X, Y)$, then there is an $x_0 \in X$ such that $Sx_0 \cap Tx_0 \neq \emptyset$.

(C.3) Let X, Y be nonempty convex sets in linear topological spaces and assume that either X or Y is compact. Let $f : X \times Y \to \boldsymbol{R}$ satisfy:

(i) $y \mapsto f(x, y)$ is quasi-convex and l.s.c. on Y for each $x \in X$,

(ii) $x \mapsto f(x, y)$ is quasi-concave and u.s.c. on X for each $y \in Y$.

Prove: $\sup_x \inf_y f(x, y) = \inf_y \sup_x f(x, y)$.

(C.4) Let $\{X_i\}_{i \in I}$ be a family of compact convex sets and $\{f_i\}_{i \in I}$ a family of real-valued functions on $X = \prod_{i \in I} X_i$ satisfying:

(i) $x^j \mapsto f_j(x_j, x^j)$ is l.s.c. on X^j for each $x_j \in X_j$,

(ii) $x_j \mapsto f_j(x_j, x^j)$ is quasi-concave on X_j for each $x^j \in X^j$,

(iii) for each $j \in I$ and $x^j \in X^j$, there is a $y_j \in X_j$ such that $f_j(y_j, x^j) > 0$.

Show:

(a) There exists $\widehat{x} \in X$ such that $f_j(\widehat{x}) > 0$ for each $j \in I$ (see Fan's survey [1973]).

(b) Using (a) deduce the generalized Nash theorem (B.4) (Ma [1969]).

[(a): For each $j \in I$ define $S_j : X^j \to 2^{X_j}$ by $S_j(x^j) = \{y_j \in X_j \mid f_j(y_j, x^j) > 0\}$. Prove that $S_j \in \mathbb{F}(X^j, X_j)$ and apply the intersection theorem (A.4) for $\mathbb{F}$-maps.]

(C.5)* (*Coincidences for* $\mathbb{K}$*-maps and* $\mathbb{F}^*$*-maps*) Let X, Y be nonempty convex sets in linear topological spaces with sufficiently many linear functionals and assume that Y is compact and $S \in \mathbb{K}(X, Y)$ Let $T : X \to 2^Y$ be a set-valued map. Prove:

(a) If $T \in \mathbb{F}^*(X, Y)$, then $G_S \cap G_T \neq \emptyset$.

(b) If X is compact and T has open values and nonempty convex fibers $T^{-1}(y)$ for $y \in T(X)$, then $G_S \cap G_T \neq Y$

(C.6) (*Coincidences for* $\mathbb{K}$*-maps and* $\mathbb{K}^*$*-maps*) Let X and Y be nonempty compact convex sets in linear topological spaces with sufficiently many linear functionals. Let $S \in \mathbb{K}(X, Y)$ and $T \in \mathbb{K}^*(X, Y)$. Show: $G_S \cap G_T \neq \emptyset$.

[Prove that $\mathbb{K}$ is a regular class of maps; then apply (A.3).]

(C.7) (*Generalized Walras excess demand theorem*) Let X, Y be nonempty convex sets in linear topological spaces with sufficiently many linear functionals and assume that Y is compact. Let $S \in \mathbb{K}(X, Y)$ and let $f : X \times Y \to \boldsymbol{R}$ satisfy:

(i) $y \mapsto f(x, y)$ is u.s.c. on Y for each $x \in X$,

(ii) $x \mapsto f(x, y)$ is quasi-convex on X for each $y \in Y$

(iii) $f|G_S \geq 0$.

Prove:

(A) There exists a $\widehat{y} \in Y$ such that $f(x, \widehat{y}) \geq 0$ for all $x \in X$.

(B) If X is compact, then there exists a $y_0 \in S(X)$ such that $f(x, y_0) \geq 0$ for all $x \in X$.

[For (A), define $T : X \to 2^Y$ by $Tx = \{y \in Y \mid f(x, y) < 0\}$, show that $T \in \mathbb{F}^*(X, Y)$, and apply (C.5)(a) to get a contradiction. For (B), use (C.5)(b).]

(C.8) Let X, Y be nonempty convex sets in linear topological spaces and assume that Y is compact. Let $T \in \mathbb{K}(Y, X)$, and let $f : X \times Y \to \boldsymbol{R}$ satisfy either (I) or (II) below:

$$\text{(I)} \qquad \begin{cases} y \mapsto f(x, y) \text{ is l.s.c. and quasi-convex on } Y \text{ for each } x \in X, \\ x \mapsto f(x, y) \text{ is u.s.c. and quasi-concave on } X \text{ for each } y \in Y, \end{cases}$$

$$\text{(II)} \qquad \begin{cases} y \mapsto f(x, y) \text{ is convex on } Y \text{ for each } x \in X, \\ x \mapsto f(x, y) \text{ is concave and u.s.c. on } X \text{ for each } y \in Y \end{cases}$$

Prove: If for each $y \in Y$,

$$(\alpha) \qquad \max_{x \in Ty} f(x, y) \geq 0,$$

then for some $(x_0, y_0) \in G_T$ we have

$$(\beta) \qquad \min_{y \in Y} f(x_0, y) \geq 0.$$

[Assume that property (I) holds; to show $(\alpha) \Rightarrow (\beta)$, define $\varphi : Y \times Y \to \boldsymbol{R}$ by $\varphi(y, z) = \max_{x \in Ty} f(x, z)$ and note that $\varphi(y, y) \geq 0$ for all $y \in Y$, $z \mapsto \varphi(y, z)$ is quasi-convex on Y for each $y \in Y$, and $y \mapsto \varphi(y, z)$ is u.s.c. on Y for each $z \in Y$ By the dual form of (1.5), get a point $y_0 \in Y$ with $\varphi(y_0, z) = \max_{x \in Ty_0} f(x, z) \geq 0$ for all $z \in Y$. Next apply

the von Neumann Sion theorem (1.4) to the function $\hat{f} = f|Ty_0 \times Y : Ty_0 \times Y \to \boldsymbol{R}$ to establish

$$\max_{x\in Ty_0} \min_{z\in Y} f(x,z) = \min_{z\in Y} \max_{x\in Ty_0} f(x,z) \geq 0.$$

Conclude by getting $x_0 \in Ty_0$ with $\min_{z\in Y} f(x_0, z) \geq 0$. The proof under assumption (II) is analogous; to justify the implication $(\alpha)\Rightarrow(\beta)$, use the dual form of the Fan Kneser theorem instead of (1.4).]

(C.9)* (*Generalized Gale Debreu theorem*) Let E be a normed linear space, E^* its dual space, and $\langle\cdot,\cdot\rangle$ the pairing between E^* and E. Given $A \subset E$, we let

$$A^+ = \{y \in E^* \mid \langle y, a\rangle \geq 0 \text{ for all } a \in A\}.$$

Assume that X is a nonempty compact convex subset of E and let $T \in \mathbb{K}(X, E^*)$ be such that $\max_{y\in Tx}\langle y, x\rangle \geq 0$ for each $x \in X$. Show: For some $\hat{x} \in X$, we have $T\hat{x} \cap X^+ \neq \emptyset$.

[Apply (C.7)(B) with $S(x) = \{y \in E^* \mid \langle y, x\rangle \geq 0\} \cap Tx$, $Y = \operatorname{Conv} S(X) \subset E^*$, and $f(x,y) = \langle y, x\rangle$ for $(x,y) \in X \times Y$.]

(In connection with (C.5) (C.9), see Granas Liu [1986].)

D. Topological transversality and ordinary differential equations

The results of this subsection illustrate the scope of the topological existence principles developed in Chapter II. In each instance the proof of existence is based on finding a priori estimates for solutions to a suitable family of related problems.

Classical problems with invertible left-hand side

(D.1) (*Cauchy problem*) Let $f : [0,T]\times \boldsymbol{R} \to \boldsymbol{R}$ be continuous and satisfy $|f(t,y)| \leq \psi(|y|)$ for all t and y, where $\psi : [0,\infty) \to (0,\infty)$ is continuous. Set

$$\int_0^\infty \frac{du}{\psi(u)} = T_\infty.$$

Show: If $T < T_\infty$, then the problem

$$(\mathscr{P}) \qquad y' = f(t,y), \quad y(0) = 0,$$

has a solution defined on $[0,T]$.

[To establish an a priori bound for solutions y to

$$(\mathscr{P}_\lambda) \qquad y' = \lambda f(t,y), \quad y(0) = 0,$$

where $0 \leq \lambda \leq 1$, defined on $[0,T]$ with $T < T_\infty$, note that

$$|y|' = \frac{y\cdot y'}{|y|} \leq |y'|$$

on the set where $y \neq 0$. Supposing that y is a solution of $(\mathscr{P}_\lambda)$ satisfying $|y(t)| \geq |r|$, determine $a \in [0,T]$ such that $|y| > |r|$ on $(a,t]$ and $|y(a)| = |r|$. Then, using $|y|' \leq |y'| \leq \psi(|y|)$ on $(a,t]$, get the inequality

$$\int_0^\infty \frac{du}{\psi(u)} = T_\infty > T \geq t - a \geq \int_a^t \frac{|y(s)|'}{\psi(|y(s)|)}\, ds = \int_0^{|y(t)|} \frac{du}{\psi(u)},$$

and derive the existence of a constant M such that $|y(t)| < M$.]

(D.2) (*Dirichlet problem*) Assume that a continuous $f : [0,1] \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ has a decomposition $f = g + h$ such that:

(a) $yg(t, y, p) \geq 0$ for all $(t, y, p) \in [0,1] \times \boldsymbol{R} \times \boldsymbol{R}$,

(b) $|g(t, y, p)| \leq C(t, y)|p|^2 + D(t, y)$ with C and D bounded on bounded sets,

(c) $|h(t, y, p)| \leq M(|y|^\alpha + |p|^\beta)$ for some $0 \leq \alpha, \beta < 1$ and $M > 0$.

Show: The Dirichlet problem

$$(\mathscr{D}) \qquad y'' = f(t, y, y'), \quad y(0) = y(1) = 0,$$

has a solution.

[We seek a priori bounds in $C^1[0,1]$ on solutions y to

$$(\mathscr{D}_\lambda) \qquad y'' = \lambda[g(t, y, y') + h(t, y, y')], \quad y(0) = y(1) = 0,$$

where $0 \leq \lambda \leq 1$. First multiplying both sides of $(\mathscr{D}_\lambda)$ by y and integrating from 0 to 1, use (a) and (b) to get the estimate

$$\|y'\|^2 \leq \int_0^1 |y| \cdot |h(t, y, y')| \, dt \leq \frac{1}{2} \int_0^1 |y|^2 \, dt + \frac{1}{2} \int_0^1 |h(t, y, y')|^2 \, dt.$$

Next, using (c) and the Hölder inequality establish the estimate

$$\|y'\|^2 - \tfrac{1}{2}\|y\|^2 \leq M^2(\|y\|^{2\alpha} + \|y'\|^{2\beta}).$$

Now using Wirtinger's inequality, $\pi^2\|y\|^2 \leq \|y'\|^2$, valid for all continuously differentiable functions y with $y(0) = y(1) = 0$, derive the inequality

$$\frac{1}{2}\|y'\|^2 + \left(\frac{\pi^2}{2} - \frac{1}{2}\right)\|y\|^2 \leq M^2(\|y\|^{2\alpha} + \|y'\|^{2\beta}),$$

and (because α and β are less than 1) deduce the existence of constants M_0 and M_1 such that $\|y\| < M_0$ and $\|y'\| < M_1$.]

(For the results (D.1)–(D.2) see Granas–Guenther–Lee [1991].)

Classical problems with noninvertible left-hand side

(D.3) (*Periodic problem, second order*) Let $f : [0,1] \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ be continuous, $f(0, y, p) = f(1, y, p)$, and assume that there are constants $M_0 > 0$ and M_1 such that:

(i) $yf(t, y, 0) > 0$ for $|y| > M_0$,

(ii) $\varrho = \inf\{|f(t, y, p)| \mid (t, y) \in [0,1] \times [-M_0, M_0] \text{ and } |p| > M_1\} > 0$.

We seek C^2 solutions to the problem

$$(\mathscr{P}) \qquad \begin{cases} y'' = f(t, y, y'), \\ y \in \mathscr{B} = \{u \in C^2[0,1] \mid u(0) = u(1),\ u'(0) = u'(1)\}. \end{cases}$$

(a) Construct a bounded continuous function $\varphi : [0,1] \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ such that

$$\varphi(t, y, p) = \operatorname{sgn} f(t, y, p)$$

for $(t, y) \in [0,1] \times [-M_0, M_0]$ and $|p| \geq M_1$.

(b) Let $\varepsilon = \varrho/(2(M_0 + 1))$ and consider the family of problems

$$(\mathscr{P}_\lambda) \qquad \begin{cases} y'' - \varepsilon y = \lambda[f(t, y, y') - \varepsilon y] + (1 - \lambda)\varphi(t, y, y')y'^2, \\ y \in \mathscr{B}. \end{cases}$$

where $0 \leq \lambda \leq 1$ and φ is the function constructed in (a). Establish a priori bounds for the possible solutions of $(\mathscr{P}_\lambda)$:

$$\|y\|_0 \leq M_0, \quad \|y'\|_0 \leq M_0' = \max\{1, M_1, \varrho\}.$$

(c) Show: The problem $(\mathscr{P})$ has a solution (Granas Guenther Lee).
[Use the fact that the linear operator $L : \mathscr{B} \to C[0,1]$ given by $y \mapsto y'' - \varepsilon y$ is bijective (Hartman [1964]).]

(D.4) (*Periodic problem, second order*) Consider the problem

$$(\mathscr{P}) \qquad \begin{cases} y'' = \alpha(t,y,y')y - \beta(t,y,y'), & t \in [0,1], \\ y \in \mathscr{B} = \{u \in C^1[0,1] \mid u(0) = u(1),\ u'(0) = u'(1)\}, \end{cases}$$

where $\alpha, \beta : [0,1] \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ are continuous, $\alpha(0,y,p) = \alpha(1,y,p)$, $\beta(0,y,p) = \beta(1,y,p)$, and the following conditions are satisfied:

1° $\alpha(t,y,p) > 0$ for all $(t,y,p) \in I \times \boldsymbol{R}^2$;

2° $|\beta(t,y,p)|/(y\alpha(t,y,0)) < 1$ for all $t \in [0,1]$ and $|y| > M_0$, where $M_0 > 0$ is a constant;

3° $|\beta(t,y,p)| \to \infty$ and $\alpha(t,y,p)/|\beta(t,y,p)| \to 0$ as $|p| \to \infty$ uniformly for $(t,y) \in [0,1] \times [-M_0, M_0]$.

Prove:

(a) The problem $(\mathscr{P})$ has a solution (Nirenberg [1960]).

(b) If $\beta(t,y,0) \geq 0$ for all $(t,y) \in [0,1] \times \boldsymbol{R}$, then every solution to $(\mathscr{P})$ is nonnegative.

(D.5) Let $a(t,y,p)$ and $b(t,y,p)$ be continuous functions of $(t,y,p) \in \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}$ with period 1 in t such that the following conditions hold on $\boldsymbol{R} \times \boldsymbol{R}^+ \times \boldsymbol{R}$:

(i) $a(t,y,p) > 0$, $b(t,y,0) \geq 0$,

(ii) $|b(t,y,p)| \to \infty$ and $a(t,y,p)/|b(t,y,p)| \to 0$ as $|p| \to \infty$ uniformly for (t,y) in a compact set,

(iii) there is a constant M_0 such that

$$\frac{b(t,y,0)}{ya(t,y,0)} \leq 1 \qquad \text{for } y > M_0 \text{ and } 0 \leq t \leq 1.$$

Show: The differential equation

$$y'' = a(t,y,y')y - b(t,y,y')$$

has at least one nonnegative periodic solution.

[Define $\alpha, \beta : \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ by $\alpha(t,y,p) = a(t,|y|,p)$, $\beta(t,y,p) = b(t,|y|,p)$; then consider the differential equation $y'' = \alpha(t,y,y')y - \beta(t,y,y')$ and apply (D.4)(a), (b).]

(D.6) Let $f(t) \geq 0$ be continuous with period 1. Show: $y'' = y^2 - y'^2 - f(t)$ has a nonnegative periodic solution.
[Write $y'' = (1+y)y - [y + y'^2 + f(t)]$ and apply (D.5).]

(D.7) Let $a_k(t)$ $(k = 0,1,\dots,n)$ and $b_l(t)$ $(l = 1,\dots,m)$ be continuous with period 1. Show: The differential equation

$$y'' = \sum_{k=0}^{n} a_k y^k - \sum_{l=1}^{m} b_l y'^l = (a_n y^{n-1} + \cdots + a_1 + 1)y - \Big(y - a_0 + \sum_{l=1}^{m} b_l y'^l\Big)$$

has a nonnegative periodic solution if $a_k \geq 0$, $k = 1,\dots,n$, $a_n > 0$, $a_0 \leq 0$, $b_m \neq 0$.
[Apply (D.5) with $a = a_n y^{n-1} + a_{n-1} y^{n-2} + \cdots + a_1 + 1$ and $\beta = y - a_0 + \sum_{l=1}^{m} b_l y'^l$.]

Carathéodory solutions

In what follows, by an *L^p-Carathéodory solution* of a differential equation $y' = f(t, y)$ on $[0, 1]$ is meant a function $y \in W^{1,p}[0, 1]$ such that $y'(t) = f(t, y(t))$ for almost all $t \in [0, 1]$.

Let $p \geq 1$. Recall that a function $g : [0, 1] \times \boldsymbol{R}^k \to \boldsymbol{R}$ is an *L^p-Carathéodory function* if:

(a) $z \mapsto g(t, z)$ is continuous for almost all $t \in [0, 1]$,
(b) $t \mapsto g(t, z)$ is measurable for all $z \in \boldsymbol{R}^k$,
(c) for each $r > 0$ there exists an $h_r \in L^p[0, 1]$ such that $|z| \leq r$ implies $|g(t, z)| \leq h_r(t)$ for almost all $t \in [0, 1]$.

(D.8) A function $f : [0, 1] \times \boldsymbol{R} \to \boldsymbol{R}$ is called *integrably bounded* if for some nonnegative $\alpha \in L^1[0, 1]$ we have, for all $y \in \boldsymbol{R}$,

$$|f(t, y)| \leq \alpha(t) \quad t\text{-almost everywhere.}$$

Prove: If $f : [0, 1] \times \boldsymbol{R} \to \boldsymbol{R}$ is an integrably bounded L^p-Carathéodory function then the initial value problem

$$(\mathscr{P}) \qquad y' = f(t, y), \quad y(0) = 0,$$

has an L^p-Carathéodory solution on $[0, 1]$ (Schauder [1927b]).

(D.9) (*Cauchy problem*) Let $f : [0, 1] \times \boldsymbol{R} \to \boldsymbol{R}$ be an L^p-Carathéodory function that satisfies

$$yf(t, y) \leq \alpha(t)(|y|^2 + 1) \quad \text{for all } y \in \boldsymbol{R} \text{ and almost every } t \in [0, 1],$$

where $\alpha \in L^1[0, 1]$ is nonnegative. Show: The initial value problem

$$(\mathscr{I}) \qquad y' = f(t, y), \quad y(0) = 0,$$

has an L^p-Carathéodory solution on $[0, 1]$.

[Establish a priori bounds in $C[0, 1]$ for solutions y to

$$(\mathscr{I}_\lambda) \qquad y' = \lambda f(t, y), \quad y(0) = 0,$$

where $\lambda \in [0, 1]$.]

Carathéodory problems with noninvertible left-hand side

(D.10) (*Periodic problem, first order*) Let $f : [0, 1] \times \boldsymbol{R} \to \boldsymbol{R}$ be L^p-Carathéodory, $\alpha \in C[0, 1]$ be nonnegative, and $r > 0$. Assume that:

(a) $yf(t, y) \leq \alpha(t)(|y|^2 + 1)$ for almost all $t \in [0, 1]$ and all $y \in \boldsymbol{R}$.
(b) $\int_0^1 y(s) f(s, y(s))\, ds \leq 0$ for all $y \in C[0, 1]$ such that $y(0) = y(1)$ and $\min\{|y(t)| \mid t \in [0, 1]\} > r$.

Show: The periodic problem

$$(\mathscr{P}) \qquad y' = f(t, y), \quad y(0) = y(1),$$

has an L^p-Carathéodory solution on $[0, 1]$.

[Find a priori bounds in $C[0, 1]$ for solutions y to

$$(\mathscr{P}_\lambda) \qquad y' + y = \lambda[f(t, y) + y], \quad y(0) = y(1),$$

where $\lambda \in [0, 1]$. To this end, letting y be a solution of $(\mathscr{P}_\lambda)$ for some λ, use (a) to establish the estimate

$$|y(\nu)|^2 + 1 \leq (|y(\mu)|^2 + 1) \exp(4\|\alpha\|_1).$$

where $|y(\mu)| = \min_{t\in[0,1]}|y(t)|$ and $|y(\nu)| = \max_{t\in[0,1]}|y(t)|$. Then use (b) to get $|y(\mu)| \leq r$, leading to the a priori bound $\|y\|_0 \leq (r^2+1)^{1/2}\exp(2\|\alpha\|_1)$.]

(D.11) (*Periodic problem, second order*) By an *L^p-Carathéodory solution* of a differential equation $y'' = f(t,y,y')$ on $[0,1]$ is meant a function $y \in W^{2,p}[0,1]$ such that $y''(t) = f(t,y(t),y'(t))$ for almost all $t \in [0,1]$.

Let $f: \mathbf{R} \to \mathbf{R}$ be continuous and $h: [0,1]\times\mathbf{R} \to \mathbf{R}$ be L^p-Carathéodory. Assume that $f(0) = 0$ and there is a constant $M > 0$ such that $yh(t,y) \geq 0$ for all $y \geq M$ and almost all $t \in [0,1]$. Show: The periodic problem

$$(\mathscr{Q}) \qquad y'' = f(y') + h(t,y), \quad y(0) = y(1), \quad y'(0) = y'(1),$$

has an L^p-Carathéodory solution on $[0,1]$.

[Find a priori bounds in $C^1[0,1]$ for solutions y to

$$(\mathscr{Q}_\lambda) \qquad y'' - y = \lambda[f(y') + h(t,y) - y], \quad y(0) = y(1), \quad y'(0) = y'(1),$$

where $\lambda \in [0,1]$. To this end, establish first the estimate $\|y\|_0 \leq M$. Next, multiplying $(\mathscr{Q}_\lambda)$ by y'' and using periodicity get the estimate

$$\|y''\|_2^2 \leq N\Big(\int_0^1 y''^2\,dt\Big)^{1/2} + M\Big(\int_0^1 y''^2\,dt\Big)^{1/2} = (N+M)\|y''\|_2,$$

where $N = \max\{|h(t,y)| \mid t \in [0,1] \text{ and } y \in [-M,M]\}$; from this derive $\|y'\|_0 \leq N + M$.]

(The results (D.9)-(D.11) are due to Granas Guenther–Lee [1991].)

Cauchy problem for systems in the complex domain

(D.12) Let $B_T = B(0,T)$ be an open ball in $\mathbf{C}$, let $f: \overline{B}_T \times \mathbf{C}^n \to \mathbf{C}^n$ be continuous, and analytic on $B_T \times \mathbf{C}^n$, and assume there is a continuous $\varphi: [0,\infty) \to (0,\infty)$ such that

$$T < \int_0^\infty \frac{ds}{\varphi(s)} \quad \text{and} \quad \|f(z,u)\| \leq \varphi(\|u\|) \quad \text{for all } (z,u).$$

(a) Show: For every $z \in \overline{B}_T - \{0\}$, and every continuous $u: \overline{B}_T \to \mathbf{C}^n$ that is analytic on B_T, the following estimate holds:

$$\frac{d}{dt}\|u(tz)\| \leq |z|\;\|u'(tz)\| \quad \text{for all } t \in [0,1] \text{ with } \|u(tz)\| > 0.$$

(b) Using (a), show that the solutions of

$$(\mathscr{P}_\lambda) \qquad u'(z) = \lambda f(z,u(z)), \quad z \in B_T, \quad u(0) = 0,$$

for $\lambda \in [0,1]$ are a priori bounded.

(c) Deduce that the problem $(\mathscr{P}_1)$ has a solution (Frigon [1990]).

[Find a priori bounds in the sup norm of the Banach space $E = \{u : \overline{B}_T \to \mathbf{C}^n \mid u$ is continuous, and analytic on $B_T\}$ for solutions u of $(\mathscr{P}_\lambda)$, $\lambda \in [0,1]$. If for some $z \in B_T$, $u(z) \neq 0$, determine $\tau \in [0,1)$ such that $\|u(tz)\| > 0$ on $(\tau,1]$, and $\|u(\tau z)\| = 0$. Then, using (a) and the growth condition, get the inequality

$$\int_0^{\|u(z)\|} \frac{ds}{\varphi(s)} \leq |z|(t-\tau) \leq T < \int_0^\infty \frac{ds}{\varphi(s)},$$

and deduce the existence of a constant M such that $\|u(z)\| < M$.]

E. Approximating sequences

Let X and Y be two metric spaces and $\{A_n\}$ a sequence of compact operators $A_n : X \to Y$. (i) $\{A_n\}$ is called *collectively compact* provided $\overline{\bigcup_{n\geq 1} A_n(X)}$ is compact. (ii) $\{A_n\}$ is *equicontinuous* provided for each $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x_1, x_2) < \delta \Rightarrow d[A_n(x_1), A_n(x_2)] < \varepsilon$ for all $x_1, x_2 \in X$ and all $n = 1, 2, \ldots$. Any sequence $\{A_n\}$ of collectively compact and equicontinuous operators $A_n : X \to Y$ converging pointwise to an operator $A : X \to Y$ is called an *approximating sequence* for A (Anselone [1976]).

(E.1) Let $\{A_n : X \to Y\}$ be an approximating sequence for $A : X \to Y$. Show:
 (a) A is continuous and compact.
 (b) $A_n \to A$ uniformly on compact sets.
 (c) $x_n \to x \Rightarrow A_n x_n \to A_n x$.

(E.2) Assume that $X \subset Y$ is closed, and let $\{A_n : X \to Y\}$ be an approximating sequence for $A : X \to Y$. Show:
 (a) If $A_n x_n = x_n$ for $n = 1, 2, \ldots$, then $\{x_n\}$ has a subsequence converging to a fixed point of A.
 (b) The set $\bigcup_{n\geq 1} \mathrm{Fix}(A_n) \cup \mathrm{Fix}(A)$ is compact.
 (c) If U is an arbitrary nbd of $\mathrm{Fix}(A)$, then $\mathrm{Fix}(A_n) \subset U$ for almost all n.

(E.3) Let C be convex in a normed linear space E and let $U \subset C$ be open in C; given $A : \overline{U} \to C$ we let $\partial A = A|\partial U$. Let $A \in \mathscr{K}_{\partial U}(\overline{U}, C)$ and $A_n \in \mathscr{K}_{\partial U}(\overline{U}, C)$ $(n = 1, 2, \ldots)$ be a sequence of operators such that $\{\partial A_n\}$ is an approximating sequence for ∂A. Show: A_n is homotopic to A in $\mathscr{K}_{\partial U}(\overline{U}, C)$ for almost all n.

(E.4) Let $A \in \mathscr{K}_{\partial U}(\overline{U}, C)$ be an essential operator and $\{A_n\}$ be an approximating sequence for A with each $A_n \in \mathscr{K}_{\partial U}(\overline{U}, C)$. Show:
 (i) $\mathrm{Fix}(A_n) \neq \emptyset$ for almost all n.
 (ii) $\sup_{x \in \mathrm{Fix}(A_n)} d[x, \mathrm{Fix}(A)] \to 0$ as $n \to \infty$.

(E.5) Let $U = \{x \in C \mid \|x - x_0\| < \varepsilon\}$ and $A \in \mathscr{K}_{\partial U}(\overline{U}, C)$ be an essential operator with a unique $x_0 = Ax_0$. Let $\{A_n\}$ be an approximating sequence for A. Then for all sufficiently large n, the equation $x = A_n(x)$ has at least one solution, and any sequence $\{x_n\}$ of such solutions converges in norm to x_0.

(E.6) Assume that an open $U \subset C$ contains the origin $0 \in E$, $A : \overline{U} \to C$ is a compact fixed point free operator, and $\{A_n : \overline{U} \to C\}$ an approximating sequence for A. Show: For all sufficiently large n, there are $x_n \in \partial U$ and $\lambda_n \in (0, 1)$ such that $x_n = \lambda_n A_n(x_n)$.

F. Fixed points for $\mathcal{ES}$(compact)-maps. Generalized Schauder-Tychonoff theorem

(F.1) Let Q be a class of normal spaces. A space Y is an *extensor space* for Q provided for any pair (X, A) in Q with A closed in X any $f_0 : A \to Y$ extends over X to $f : X \to Y$; the class of extensor spaces for Q is denoted by ES(Q). Show:
 (a) $Q' \subset Q$ implies ES(Q) $\subset$ ES(Q').
 (b) A retract of an ES(Q) is an ES(Q).
 (c) The Cartesian product of any collection of ES(Q) is an ES(Q).

(F.2) Prove: Every absolute retract is an ES(compact).
 [Using the Urysohn embedding, show that a normed linear space is an ES(compact); then apply (F.1)(b) and (7.6).]

(F.3) Prove: Each Tychonoff cube and each product of lines are ES(normal).

(F.4) Let K be a compact space and $f : K \to K$ a map of the form $K \xrightarrow{\alpha} X \xrightarrow{\beta} K$ with $X \in \mathrm{ES}(\text{compact})$. Show: f has a fixed point.

[First, using the Tychonoff embedding theorem show that f factorizes through some Tychonoff cube; then apply an argument similar to that in (7.8).]

(F.5) Let $X \in \mathrm{ES}(\text{compact})$ and $f : X \to X$ be a compact map. Show: f has a fixed point.

(F.6) A space Y is an *approximate extensor space for compact spaces* (written $Y \in \mathrm{AES}(\text{compact})$) provided given any compact pair (X, A), any $\alpha \in \mathrm{Cov}(Y)$ and $f_0 : A \to Y$, there exists an $f : X \to Y$ such that $f|A$ is α-close to f. Prove: If C is a convex set in a locally convex space, then $C \in \mathrm{AES}(\text{compact})$.

(F.7) Let K be a compact space and assume that a map $f : K \to K$ can be factorized as $K \xrightarrow{\alpha} X \xrightarrow{\beta} K$ with $X \in \mathrm{AES}(\text{compact})$. Show: f has a fixed point.

(F.8) (*Generalized Schauder Tychonoff theorem*) Let $X \in \mathrm{AES}(\text{compact})$ and $F : X \to X$ be a compact map. Show: F has a fixed point.

(F.9) (*$\mathcal{ES}$(compact)-maps*) We say that $f : X \to Y$ is an *$\mathcal{ES}$(compact)-map* if for any compact pair (Z, A) and any $g : A \to X$ there exists a $\varphi : Z \to Y$ such that $\varphi|A = fg$; the class of all $\mathcal{ES}$(compact)-maps is denoted by $\mathcal{ES}$(compact). Prove:

(i) $X \in \mathrm{ES}(\text{compact}) \Leftrightarrow 1_X \in \mathcal{ES}(\text{compact})$.

(ii) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ and either f or g is an $\mathcal{ES}$(compact)-map, then so also is $g \circ f$.

(iii) If either X or Y is an ES(compact) space, then every $f : X \to Y$ is an $\mathcal{ES}$(compact)-map.

(F.10) Let K be a compact space and $f : K \to K$ be an $\mathcal{ES}$(compact)-map. Show: (i) f has a fixed point; (ii) the generalized Schauder theorem (7.9) and (F.5) are special cases of (i).

(F.11) (*Approximate $\mathcal{ES}$(compact)-maps*) We say that $f : X \to Y$ is an *approximate $\mathcal{ES}$(compact)-map* if for any compact pair (Z, A), any $g : A \to X$, and each $\alpha \in \mathrm{Cov}(Y)$, there exists $\varphi_\alpha : Z \to Y$ such that $\varphi_\alpha|A$ is α-close to fg; the class of all approximate $\mathcal{ES}$(compact)-maps is denoted by $\mathcal{AES}$(compact). Prove:

(i) $X \in \mathrm{AES}(\text{compact}) \Leftrightarrow 1_X \in \mathcal{AES}(\text{compact})$.

(ii) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ and either f or g is an $\mathcal{AES}$(compact)-map, then so also is $g \circ f$.

(iii) If either X or Y is an AES(compact) space, then every $f : X \to Y$ is an $\mathcal{AES}$(compact)-map.

(The results (F.9)–(F.12) are due to Gauthier–Granas [2003].)

(F.12) Let K be a compact space and $f : K \to K$ be an $\mathcal{AES}$(compact)-map. Show: (i) f has a fixed point; (ii) (F.7), (F.8) and the Schauder-Tychonoff theorem (1.13) are special cases of (i).

G. Analytic proof of the Brouwer theorem

(G.1) Let $K = \{x \in E \mid \|x\| \leq 1\}$ be the closed ball in a Banach space E and $r : K \to \partial K$ be a Lipschitz retraction. Define a deformation $d_t : K \to E$ by $x \mapsto x + t(r(x) - x)$. Show:

(a) $d_t : (K, \partial K) \to (K, \partial K)$ and $d_t|\partial K = \mathrm{id}_{\partial K}$ for all $t \in [0, 1]$.

(b) For all sufficiently small t, d_t is a homeomorphism of K onto K.

[For (b), use the elementary domain invariance and the fact that a contractive field maps closed sets to closed sets.]

(G.2) Let K be the closed unit ball in a Hilbert space H with scalar product $(\ ,\ )$ and assume that $f : K \to K$ is a map without fixed points. Define $\beta : K \to H$ by

$$x \mapsto x - \frac{1-(x,x)}{1-(x,f(x))} f(x).$$

Show:

(i) β is well defined and continuous.
(ii) $\beta(x) \neq 0$ for all $x \in K$.
(iii) The map $r : K \to \partial K$ given by $x \mapsto \beta(x)/\|\beta(x)\|$ is a retraction.
(iv) If $\dim H < \infty$ and f is C^1, then so is the map r.

(G.3) Let K^n be the closed unit ball in R^n, $\partial K^n = S^{n-1}$, and let $r : K^n \to S^{n-1}$ be a C^1 retraction. Define a deformation $d_t : K^n \to R^n$ by $x \mapsto x + tg(x)$, where $g(x) = r(x) - x$. Show:

(a) $d_t | S^{n-1} = \mathrm{id}_{S^{n-1}}$.
(b) For all sufficiently small t, d_t is a diffeomorphism of K^n onto K^n

[For (b), express the determinant $\det\{\delta^i_j + t\partial g_i/\partial x_j\}$ of the partial derivatives of d_t in the form of a polynomial with respect to t:

$$P_t(x) = 1 + \sum_{k=1}^{n} \gamma_k(x) t^k \quad \text{with } \gamma_k \in C(K^n);$$

use (G.1) and the inverse function theorem.]

(G.4) Using the notation of (G.3), we assign to $g(x) = r(x) - x$ a polynomial function $I_g : [0,1] \to R$ by setting

$$I_g(t) = \int_{K^n} P_t(x)\,dx = a_0 + \sum_{k=1}^{n} a_k t^k,$$

where $a_0 = \mathrm{vol}(K^n)$ and $a_k = \int_{K^n} \gamma_k(x)\,dx$. Show: The polynomial function $I_g(t)$ has the constant value $a_0 = \mathrm{vol}(K^n)$ for all $t \in [0,1]$.

[Prove that $I_g(t) = a_0$ for all sufficiently small t: use (G.3)(b) and the change of variables formula $\int_{K^n} P_t(x)\,dx = \mathrm{vol}(d_t(K^n)) = \mathrm{vol}(K^n) = a_0$.]

(G.5) Show: There is no C^1 retraction of K^n onto S^{n-1}

[Supposing $r : K^n \to S^{n-1}$ is such a retraction, deduce that $P_1(x) = \det\{\partial r_i(x)/\partial x_j\} \equiv 0$, and therefore $I_g(1) = 0$, contradicting (G.4).]

(G.6) Prove: Every continuous $f : K^n \to K^n$ has a fixed point.

[Assuming first that f is C^1, apply (G.5) and (G.2): for arbitrary f, apply the Weierstrass approximation theorem and the special case.]

(The above proof is essentially due to Milnor [1978]: for further improvements and simplifications see Rogers [1980], Groger [1981], Traynor [1996].)

H. Localization and multiplicity of fixed points in cones

In this subsection the following notation will be used. Let $(E, \|\ \|)$ be a normed linear space and $C \subset E$ a closed nonempty convex cone: the property $C \cap (-C) = 0$ is not required, so the case $C = E$ is included. For $\varrho > 0$, let $B_\varrho = \{x \in C \mid \|x\| < \varrho\}$, $S_\varrho = \{x \in C \mid \|x\| = \varrho\}$, $K_\varrho = B_\varrho \cup S_\varrho$. Given $r, R \in R$ with $0 < r < R$, let $A_{r,R} = \overline{B_R - K_r}$.

(H.1) Let U be an open bounded subset of C with $0 \in U$, and let $F \in \mathscr{K}_{\partial U}(\overline{U}, C)$.

(a) Assume that for all $x \in \partial U$ and all $\lambda \in [0,1]$ one of the following is satisfied:

1° $x \neq \lambda Fx$.

2° $\|Fx\| \leq \max\{\|x\|, \|x - Fx\|\}$.

3° $\langle Fx, x\rangle \leq \langle x, x\rangle$, where $\langle\ ,\ \rangle$ is a scalar product in E.

Show: F is essential.

(b) Assume that there is a $c \in C - \{0\}$ such that $x \neq Fx + \lambda c$ for all $x \in \partial U$ and all $\lambda \geq 0$. Show: F is inessential.

[For (b), assuming F is essential, find for each $n \in N$ a point $x_n \in U$ such that $x_n = Fx_n + nc$; then, using boundedness of U, get a contradiction.]

(H.2) (*Localization result*) Let $F \in \mathscr{K}_{S_R \cup S_r}(K_R, C)$.

(a) Suppose that F is essential (with respect to K_R) and $F|S_r$ is inessential (with respect to K_r). Show: F has a fixed point in $A_{r,R}$.

(b) Assume that (i) for all $x \in S_R$ and all $\lambda \geq 0$ one of the conditions 1°–3° of (H1)(a) holds, and (ii) there is a $c \in C - \{0\}$ such that $x \neq Fx + \lambda c$ for all $x \in S_r$ and all $\lambda \geq 0$. Prove: F has a fixed point in $A_{r,R}$.

[For (a), show that $F|A_{r,R} \in \mathscr{K}_{S_R \cup S_r}(A_{r,R}, C)$ is essential.]

(c) Let $C = E$, so that $F \in \mathscr{K}_{S_R \cup S_r}(K_R, E)$. Assume that (i) $Fx = -F(-x)$ for all $x \in S_R$, and (ii) there is a $c \in E - \{0\}$ such that $x \neq Fx + \lambda c$ for all $x \in S_r$ and all $\lambda \geq 0$. Prove: F has a fixed point in $A_{r,R}$.

(H.3) (*Multiplicity result*) (a) Let $F : A_{r,R} \to C$ and $G : K_R \to C$ be two fixed point free compact maps such that $F|S_R = G|S_R$. Show: $F|S_r$ is inessential.

[Letting $\varrho \in (0, r)$, consider the functions $\alpha, \beta, \gamma : A_{\varrho,r} \to R - \{0\}$ given by

$$\alpha(x) = (r - \varrho)\|x\|, \quad \beta(x) = (R - \varrho)r - (R - r)\|x\|, \quad \gamma(x) = \alpha(x)/\beta(x),$$

and let $\widehat{F} : K_R \to C$ be given by

$$\widehat{F}(x) = \begin{cases} (\varrho/R)G((R/\varrho)x) & \text{for } x \in K_\varrho, \\ \gamma(x)F((1/\gamma(x))x) & \text{for } x \in A_{\varrho,r}, \\ F(x) & \text{for } x \in A_{r,R}. \end{cases}$$

Prove that $\widehat{F}$ is a fixed point free compact extension of $F|A_{r,R}$ over K_R.]

(b) Let $F \in \mathscr{K}_{S_R \cup S_r}(K_R, C)$ be an inessential map such that $F|S_r$ is essential (with respect to K_r). Show: (i) F has at least two fixed points, $x_1 \in B_r$ and $x_2 \in A_{r,R}$, and (ii) $F|A_{r,R} \in \mathscr{K}_{S_R \cup S_r}(A_{r,R}, C)$ is essential.

[To prove that $\mathrm{Fix}(F|A_{r,R}) \neq \emptyset$, suppose $F : A_{r,R} \to C$ is fixed point free; use (a) to get a contradiction.]

(c) Let $F : K_R \to C$ be a compact map such that (i) $x \neq \lambda Fx$ for all $x \in S_r$ and $\lambda \in [0, 1]$, (ii) there is a $c \in C - \{0\}$ such that $x \neq Fx + \lambda c$ for all $x \in S_R$ and all $\lambda \geq 0$. Show: F has at least two fixed points, $x_1 \in B_r$ and $x_2 \in A_{r,R}$.

[Use (H.1) and (b).]

(d) Assume $C = E$ and let $F \in \mathscr{K}_{S_R \cup S_r}(K_R, E)$ be such that (i) $Fx = -F(-x)$ for all $x \in S_r$, (ii) there is a $c \in E - \{0\}$ such that $x \neq Fx + \lambda c$ for all $x \in S_R$ and all $\lambda \geq 0$. Prove: F has at least two fixed points, $x_1 \in B_r$ and $x_2 \in A_{r,R}$.

(The above results are due to Simon and Volkmann [1988].)

I. Selected problems

(I.1) Let E and F be Banach spaces and $L \in \mathcal{L}(E, F)$ be surjective. Let $T : E \to 2^F$ be a completely continuous Kakutani map.

(a) Let $K_\varrho = \{x \in E \mid \|x\| \le \varrho\}$. Show: Either (i) $Lx \in Tx$ for some $x \in K_\varrho$, or (ii) there are $y \in \partial K_\varrho$ and $\lambda \in (0,1)$ such that $Ly \in \lambda Ty$.

(b) Let $T : E \to 2^F$ be a completely continuous Kakutani map and set $\mathcal{E}_T = \{x \in E \mid Lx \in \lambda Tx$ for some $\lambda \in (0,1)\}$. Show: If the set $\mathcal{E}_T$ is bounded, then L and T have a point of coincidence (Granas [1993]).

(I.2) (*Generalized Leray-Schauder principle*) Let E be a Banach space, $C \subset E$ closed convex, and V open in $C \times [0,1]$; we let $V_\lambda = \{x \in C \mid (x,\lambda) \in V\}$ be the section of V at λ. Given $H : \overline{V} \to C$ we define $\widehat{H} : \overline{V} \to C \times [0,1]$ by $\widehat{H}(x,\lambda) = (H(x,\lambda), 0)$; clearly, if H is compact, then so is $\widehat{H}$.

Let $H : \overline{V} \to C$ be a compact map such that $H(x,\lambda) \ne x$ for all $(x,\lambda) \in \partial V$, and assume that one of the following conditions holds: (i) $\widehat{H} : \overline{V} \to C \times [0,1]$ is essential; (ii) $(1-t)x_0 + tH(x,0) \ne x$ for all $(x,0,t) \in \partial V \times (0,1)$ for some $x_0 \in V_0$. Show: For each $\lambda \in [0,1]$ the map $x \mapsto H(x,\lambda)$ has a fixed point in V_λ (Precup [1995]).

(I.3) Let X be normal, Y arbitrary, and $S : X \to 2^Y$ an open map (i.e., with open graph). Let $f : X \to Y$ be an S-selection on some closed $A \subset X$ and assume that f is homotopic to some S-selection $g : X \to Y$ by a homotopy Φ that is S-small on A (i.e., $\Phi(a \times I) \subset Sa$ for each $a \in A$). Show: $f|A$ extends to an S-selection $f^* : X \to Y$ such that

(i) $f^*(X) \subset \Phi(X \times I)$,

(ii) $f^* \simeq f$ rel A,

(iii) $f^* \simeq g$ by an S-small homotopy extending $\widehat{\Phi}|A \times I$.

[Consider $\Delta : X \times I \to X \times I$ given by $(x,t) \mapsto (x, \Phi(x,t))$; from $A \times I \subset \Delta^{-1}(\Gamma_S)$, deduce that $W \times I \subset \Delta^{-1}(\Gamma_S)$ for some nbd $W \supset A$, and hence $\Phi(w \times I) \subset S(w)$ for each $w \in W$. Take an Urysohn function $\lambda : X \to I$ with $\lambda|X - W = 1$, $\lambda|A = 0$, and show that $f^*(x) = \Phi(x, \lambda(x))$ is the required extension of f.]

(I.4) Let X be normal, Y arbitrary, and $T : X \to 2^Y$ a closed map (i.e., with graph closed in $X \times Y$). Let $A \subset X$ be closed and $f : X \to Y$ continuous with $f(a) \notin Ta$ for each $a \in A$. Assume that f is homotopic to a map $g : X \to Y$ with $g(x) \notin Tx$ for each $x \in X$ via a homotopy Φ such that $\Phi_t(a) \notin Ta$ for all $a \in A$ and $t \in I$. Show: $f|A$ has an extension $f^* : X \to Y$ with $f^*(x) \notin Tx$ for each x and such that:

(i) $f^*(X) \subset \Phi(X \times I)$,

(ii) $f^* \simeq f$ rel A,

(iii) $f^* \simeq g$ by a homotopy $\widehat{\Phi}$ with $\widehat{\Phi}_t(x) \notin Tx$ for all x and t.

(I.5) Let X be a nonempty compact convex subset of a locally convex space E and $K \subset E$ a nonempty closed convex set. Let $g : X \times X \to E$ be continuous and such that:

(i) for each $y \in X$ and any convex set $C \subset E$ the set $\{x \in X \mid g(x,y) \in C\}$ is convex,

(ii) for each $y \in X$ the set $\{x \in X \mid g(x,y) \in K\}$ is nonempty.

Show: There exists $\widehat{y} \in X$ such that $g(\widehat{y}, \widehat{y}) \in K$ (Lassonde [1983]).

[Let Δ be the diagonal in $X \times X$; supposing the assertion is not true, find a closed convex symmetric nbd U of 0 in E such that $g(\Delta) \cap (K+U) = \emptyset$. Consider $G : X \to 2^X$ given by $Gx = \{y \in X \mid g(x,y) \in K + U\}$; using the Fan–Browder theorem, get a contradiction by showing that $g(x_0, x_0) \in K + U$ for some $x_0 \in X$.]

(I.6) (*Fan–Iokhvidov theorem*) Let X be a nonempty compact convex subset of a locally convex space E, C be a convex closed subset of E and $f : X \to E$ a continuous map such that $f(X) \subset X + C$. Show: There exists $\widehat{y} \in X$ such that $f(\widehat{y}) \in \widehat{y} + C$ (Iokhvidov [1964], Fan [1966]).

[Let $g : X \times X \to E$ be defined by $g(x,y) = f(x) - y$ and apply (I.5).]

(I.7) (*Himmelberg theorem*) Let C be a nonempty convex set in a locally convex space E, and let $T \in K(C, C)$ be a compact Kakutani map. Show:

(a) If V is a closed convex nbd of 0 in E, then there exists a point $x_V \in C$ such that $Tx_V \cap (x_V + V) \neq \emptyset$.

[Observe that the map $x \mapsto (Tx - V) \cap C$ has closed values and open fibers; using the compactness of T, find an $A \in \langle C \rangle$ with $(Tx - V) \cap A \neq \emptyset$ for each $x \in C$. Apply (5.9.H.3).]

(b) The map $T \in K(C, C)$ has a fixed point (Himmelberg [1972]; the above proof is due to Lassonde [1990]).

(I.8) (*Fixed points for upper demicontinuous maps*) Let E be a locally convex space, $X \subset E$ a nonempty subset and $A : X \to 2^E$ a set-valued map with nonempty values. We say that A is *upper demicontinuous* if for any $x \in X$ and any open half-space $H \subset E$ containing Ax there is a nbd N_x of x in X such that $A(N_x) \subset H$.

(a) Let X be compact convex and $A, B : X \to 2^E$ be upper demicontinuous set-valued maps with nonempty closed convex values such that for each $x \in X$ one of the sets Ax, Bx is compact. Assume furthermore that one of the following conditions is satisfied:

(i) For each $(\varphi, x) \in E^* \times X$ with $\varphi(x) = \min\{\varphi(z) \mid z \in X\}$ there are $u \in Ax$, $v \in Bx$ such that $\varphi(u) \leq \varphi(v)$.

(ii) A is inward and B is outward in the sense of Fan.

Show: A and B have a coincidence.

(b) Let X be compact convex and $A : X \to 2^E$ be upper demicontinuous with nonempty closed convex values. Show: If A is either inward or outward in the sense of Fan, then A has a fixed point.

(The above results are due to Fan [1972].)

(I.9) (*Krein–Rutman theorem*) Let E be a Banach space, $C \subset E$ a closed cone in E, and let $\preceq$ be the partial order in E induced by C ($x \preceq y \Leftrightarrow y - x \in C$). Let $F : E \to E$ be a completely continuous operator such that:

(a) $x \preceq y \Rightarrow F(x) \preceq F(y)$,

(b) $F(\alpha x) = \alpha F(x)$ for $\alpha \geq 0$, $x \in E$,

(c) there exists $v \in C$ with $\|v\| = 1$ such that $F(v) \succeq \beta v$ for some $\beta > 0$.

Show: There exist $x_0 \in C - \{0\}$ and $\lambda \geq \beta$ such that $F(x_0) = \lambda x_0$ (Krein–Rutman [1948]).

[Given $\varepsilon \in (0, 1/2)$, let $C_\varepsilon = \{x \in E \mid x \succeq \varepsilon\|x\|v\}$ and fix $l \in E^*$ with $l(v) > 0$, $\|l\| = 1$. Let $X_\varepsilon = \{x \in C_\varepsilon \mid \|x\| \leq 1,\ l(x) \geq \varepsilon\|F(v)\|\}$ and

$$F_\varepsilon(x) = \frac{F(x) + 2\varepsilon\|F(x)\|v}{\|F(x) + 2\varepsilon\|F(x)\|v\|} \qquad \text{for } x \in X_\varepsilon.$$

Then prove successively: (i) X_ε is closed, convex and nonempty, (ii) $F_\varepsilon : X_\varepsilon \to X_\varepsilon$ is well defined and compact, (iii) there exists $x_\varepsilon \in X_\varepsilon$ such that $x_\varepsilon = F_\varepsilon(x_\varepsilon) = \lambda_\varepsilon x_\varepsilon - \mu_\varepsilon v$, where $\mu_\varepsilon > 0$, $\lambda_\varepsilon \geq k > 0$. Use compactness of F_ε to conclude the proof.]

11. Notes and Comments

Applications of the topological KKM-principle

The topological KKM-principle has numerous significant applications in diverse fields. We list some typical applications (which cannot be obtained from the geometric KKM-principle) that are discussed in Section 1:

1° geometric results concerning set-valued maps of Ky Fan;
2° minimax inequalities (of Fan and F. C. Liu);
3° fixed point results of Schauder–Tychonoff, Fan, and Iokhvidov;
4° theory of games and mathematical economy.

The geometrical results (1.2), (1.3) and (1.17) were (in a somewhat different setting) established in Fan [1962], [1964], and [1966], respectively; Browder [1968] formulated Theorem (1.2) in terms of set-valued maps, and using the selection technique, deduced it directly from the Brouwer theorem. The minimax inequality (1.7) is taken from Fan's survey [1972], and Theorem (1.8) from Liu [1978]. Theorem (1.4), established by Sion [1958] with the aid of the Knaster-Kuratowski–Mazurkiewicz theorem, evolved from earlier results of von Neumann [1928], [1937]. The direct simple proof of (1.4) given in the text is a modification of a proof by Fan [1964].

Among the diverse applications of the Fan minimax inequality (1.7), we mention the following existence theorem in potential theory:

Let X be a compact space and $G : X \times X \to \boldsymbol{R}$ be a continuous nonnegative function such that $G(x,x) > 0$ for $x \in X$. Then there exists a positive Radon measure μ on X such that:
(a) $\int G(x,y)\,d\mu(y) \geq 1$ *for all* $x \in X$,
(b) $\int G(x,y)\,d\mu(y) = 1$ *for each* x *in the support of* μ.

For a proof and other applications of (1.7), see Fan's survey [1972].

The Schauder–Tychonoff theorem (1.13) is one of the best known classical results in fixed point theory. Its proof, which uses (1.12), is a modification of that given by Lassonde [1983]. For locally convex spaces, Theorem (1.14) was established by Fan [1969]; the proof in the text (valid for spaces with sufficiently many linear functionals) is due to Kaczynski [1983]. The Nash equilibrium theorem was established in $\boldsymbol{R}^n$ by Nash [1951]; the proof of its extended version presented in the text follows that given in Fan's survey [1972]. Among other noteworthy results related to the Nash theorem we mention Gale–Mas-Colell [1975] and Deguire–Lassonde [1995] (cf. (10.B.4)).

For more general results and other applications of KKM-maps the reader is referred to the survey by Fan [1972], Lassonde [1983], Gwinner [1981], and "Miscellaneous Results and Examples"

Topological KKM-theory was extended to spaces with a generalized convex structure (see Horvath [1987], [1993], Khamsi [1996], van de Vel [1993], and also section I(c) in "Additional References").

Applications of the antipodal theorem

Theorem (2.1) is due to Kuratowski–Steinhaus [1953]; the proof of (2.1) follows Borsuk [1953]. Theorem (2.2) was conjectured by Steinhaus; its proof

(based on the antipodal theorem) was first given by Banach (see Steinhaus [1945]). Theorem (2.3) is a special case of a theorem of P.A. Smith stating that no nontrivial finite group can act freely on $\boldsymbol{R}^n$ (see Smith's survey [1960] for more details and further references). Theorem (2.4) is from Krein–Krasnosel'skiĭ-Milman [1948]; it has important applications in perturbation theory of linear operators and also in approximation theory.

The remarkable feature of the Krein–Krasnosel'skiĭ–Milman theorem is that it implies in fact the Borsuk–Ulam theorem; this was discovered by A.L. Brown [1979].

We now list some further applications of the antipodal theorem:

(i) (*Algebra*) Dai and Lam [1984] applied the Borsuk–Ulam theorem to show that the quotient $\boldsymbol{R}[x_1,\dots,x_n]/(1+x_1^2+\cdots+x_n^2)$ has level n, i.e., -1 cannot be written as a sum of fewer than n squares.

(ii) (*Graph theory*) Using a version of the antipodal theorem due to Fan [1952], Lovász [1978] established the Kneser conjecture: *Given any decomposition of the collection $\mathscr{S}$ of all n-element subsets of a $(2n+k)$-element set S into classes $\mathscr{S}_1,\dots,\mathscr{S}_{k+1}$, at least one of these classes contains two disjoint n-element sets* (in connection with this result see also Bárány [1978]). For some other combinatorial applications of the antipodal theorem see Alon [1987].

(iii) (*Dimension theory*) Using the antipodal theorem one can show that the n-skeleton of the $(2n+2)$-simplex cannot be embedded in $\boldsymbol{R}^{2n}$ (see Flores [1935]).

(iv) (*Approximation theory*) Some applications of the antipodal theorem in approximation theory can be found in Tikhomirov [1960].

(v) (*Game theory*) Using a version of the Borsuk–Ulam theorem, Simon–Spież–Toruńczyk [1995] established the existence of equilibria for certain games arising in mathematical economy.

For more details about applications of the antipodal theorem the reader is referred to Steinlein's survey [1985] and also to "Additional References", where further information may be found.

Applications of topological transversality

The Leray–Schauder continuation method is at present one of the most basic tools of nonlinear analysis. In the proof of Theorem (4.2) (Granas–Guenther–Lee [1977]) we indicate in a simple setting how this method may be used in the study of nonlinear differential equations; Theorem (4.3) was proved by Bernstein [1912], to whom the basic idea of "a priori" bounds is due.

In subsection D of "Miscellaneous Results and Examples", further applications of topological transversality are given. To focus on the essential

features of the topological methods, the discussion is restricted to scalar equations of the first or second order; however, the general techniques also apply to systems of equations and more general boundary conditions. A significant feature of these exercises is that by recasting a differential equation as an integral equation, both the classical and Carathéodory cases can be treated in a classical setting, without recourse to Sobolev space formulations.

For other applications to ordinary differential equations see, for example, Cesari [1960], the surveys by Mawhin [1976] and Granas–Guenther-Lee [1980], and also Lee–O'Regan [1995], where further references can be found. For an application to nonlinear integral equations the reader is referred to Guenther–Lee [1993].

Invariant subspace problem and other applications

(i) (*Operator theory*) The application of the Schauder theorem to the invariant subspace problem presented in Section 6 was given by V. Lomonosov; Theorem (6.1), due to Lomonosov [1973], extends an earlier result of Aronszajn-Smith [1954]. We remark that the invariant subspace problem for arbitrary bounded linear operators was solved in the negative by Enflo and independently by Read; for details we refer to Beauzamy [1988]. For other applications to operator theory, see Krein [1964] and also the survey of Fan [1972], where further references can be found.

(ii) (*Descriptive set theory*) Engelking–Holsztyński–Sikorski [1966] observed that the proof of the existence of Borel sets of arbitrarily high classes (in a metric space) can be considerably simplified by using the fact that the Hilbert cube I^∞ has the fixed point property.

(iii) (*Linear algebra*) The Brouwer theorem yields a simple proof of the Perron–Frobenius theorem: *Every square matrix $\|a_{ij}\|$ with $a_{ij} \in \boldsymbol{R}^+$ has at least one nonnegative real eigenvalue* (Alexandroff–Hopf [1935]). For topological proofs of some generalizations of this result see Fan [1958].

(iv) (*Geometry*) The Brouwer theorem permits proving the following result: *Let (X, d) be a metric space, let $\varphi : [0,1] \to X$ be a continuous arc with $\varphi(0) \neq \varphi(1)$, and let $\beta_0, \beta_1, \ldots, \beta_s$ be $s+1$ strictly positive numbers. Then there exists a partition $0 = t_0 < t_1 < \cdots < t_{s+1} = 1$ of the interval $[0,1]$ and a constant $K > 0$ such that*

$$\beta_j = Kd(\varphi(t_j), \varphi(t_{j+1})) \quad \textit{for } j = 0, 1, \ldots, s$$

(see Urbanik [1954]).

(v) (*Optimal control theory*) The Brouwer theorem implies the following result: *Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be differentiable at 0 and such that $f(0) = 0$. If the derivative $f'(0)$ is invertible, then for some nbd V of 0 we have $0 \in \operatorname{Int} f(V)$.* For further details see Halkin [1975].

Dugundji extension theorem

Theorem (7.4) and its consequence (7.5) established by Dugundji [1951] represent strict generalizations of the Tietze extension theorem for continuous real-valued functions. These results are at the roots of the theory of ANRs to be developed in §11.

One can get various related general extension theorems by trading less structure on the domain for more structure on the range. We give two examples of such results:

1° *If X is paracompact, $A \subset X$ closed, and E a Banach space, then every $f : A \to E$ is extendable to an $F : X \to E$* (Arens theorem).

2° *If X is normal, $A \subset X$ closed and E a separable Banach space, then every $f : A \to E$ extends to an $F : X \to E$.*

Dugundji [1951] showed that the explicit form of the extension in (7.4) permits proving the following theorem ([1]): *Let X be metric, $A \subset X$ closed, and let* $\mathrm{CB}(X)$ *denote the Banach space of all bounded continuous real-valued functions with the sup norm. Then there exists a linear operator* $L : \mathrm{CB}(A) \to \mathrm{CB}(X)$ *that makes correspond to each* $f \in \mathrm{CB}(A)$ *an extension* $Lf \in \mathrm{CB}(X)$. This theorem naturally extends to Banach-space-valued functions.

More details and references concerning Dugundji type results can be found in the books of Bessaga–Pełczyński [1975] and Hu [1965].

Absolute retracts and generalized Schauder theorem

The theory of metric ANRs provides a framework within which fixed point results can be expressed in purely topological terms. The first result of this type was proved for compact ARs by Borsuk [1931a]. The generalized Schauder theorem (7.8) as well as Theorem (F.8) are due to Granas; they are special cases of more general results to be presented in Chapters IV and V.

A link between topological fixed point theory and the functional-analytic fixed point results is provided by Theorems (7.4) and (7.5) together with various embedding results into linear topological spaces. For example, the fact that a convex set in a normed space is an AR (Dugundji [1951]) implies that the Schauder theorem (6.3.2) is a special case of (7.8). The fact that the unit sphere in an infinite-dimensional normed space is an AR (first proved, using a different method, by Dugundji [1951]) permits a simple proof of the Birkhoff–Kellogg theorem (Granas [1962]) and also implies the result of Göhde [1959] that the nonconvex set in (7.7), being an AR, has the fixed point property for compact maps.

([1]) For A separable, this theorem was established by Borsuk [1933b].

In connection with Theorems (7.4), (7.5), we make the following remarks:

(a) There exists a compact convex subset C of a locally convex space E that is not a retract of E (Klee [1953] and E. Michael [1953]).

(b) There exists a linear metric space that is not an absolute retract (Cauty [1994]).

In connection with Theorem (7.8), we mention that there exists a noncompact AR having the fixed point property (Klee [1960]).

Fixed points for Kakutani maps

The first result for such maps was established by Kakutani [1941], who extended the Brouwer theorem and applied the result to prove a version of the von Neumann minimax principle in $\boldsymbol{R}^n$. Kakutani's theorem was extended to Banach spaces by Bohnenblust and Karlin [1950] and to locally convex spaces by Glicksberg [1952] and Fan [1952]. An analogue of the Schauder–Tychonoff theorem for compact Kakutani maps was established by Himmelberg [1972].

Photo by I. Nauioka

J. Dugundji and S. Kakutani, Montreal, 1973

The proof of (8.4) given in the text combines the (graph) approximation technique developed for compact Kakutani maps in Cellina [1969] and the suitably modified approximation technique used in the single-valued case (Lemma (8.2)) (for another technique based on pointwise approximation of Kakutani maps see Olech [1968]).

We remark that Borsuk's antipodal theorem remains valid for compact Kakutani maps (Granas [1959b], Cellina–Lasota [1969]), as do the topological transversality theorem (Granas [1976]) and the Dugundji extension

theorem (7.4) (Ma [1972]). Other results can be found in Fan's survey [1972]; see also Ma [1972].

Fixed point results for Kakutani maps prove useful in many branches of mathematics; for instance, they find applications in game theory (Bohnenblust Karlin [1950], Glicksberg [1952], Fan [1952]), optimal control theory (Lasota Opial [1965], [1968]), differential inclusions (T. Pruszko [1984], Frigon [1990a], Deimling [1991]), partial differential equations (Chang [1980], the lecture notes by Lasry Robert [1976]), and classical functional analysis (Day [1962]).

For numerous applications of the Kakutani maps in mathematical economy the reader is referred to the books by Aubin [1991] and Border [1985] (see also Florenzano [1981]). More general fixed point results for set-valued maps will be treated in Chapter VI.

Ryll-Nardzewski theorem

This remarkable result (considerably extending Kakutani's theorem) was announced in Ryll-Nardzewski [1962]; a detailed proof based on probabilistic arguments was presented in Ryll-Nardzewski [1966]. The first geometric proof was given by Namioka–Asplund [1967]. A simple proof of (9.4) (which is a special case of (9.6)) was found by F. Hahn [1967] who also showed that Theorem (9.5) of Kakutani [1938] follows directly from (9.6). Theorem (9.3) is a special case of a result of Namioka. The proof of the Ryll-Nardzewski theorem presented in the text is a corrected version of that given in the book of Dugundji–Granas [1982]; the latter proof is valid only under the assumption of metrizability of the space E (see also Hansel–Troallic [1976]). For a noteworthy extension of the Ryll-Nardzewski theorem the reader is referred to Namioka [1983].

Let G be a locally compact group. Following Eberlein [1949], call a function $f \in \mathrm{CB}(G)$ *weakly almost periodic* ($f \in W(G)$) if its orbit under left translations $\mathcal{O}(f) = \{L_x f \mid x \in G\}$ is relatively compact with respect to the weak topology in $\mathrm{CB}(G)$. Solving a problem raised by Eberlein [1949], Ryll-Nardzewski [1962] applied his fixed point theorem to prove that *for every locally compact group G there exists a left-invariant mean on $W(G)$.* For the class of almost periodic functions $\mathrm{AP}(G) \subset W(G)$ such a result (originally due to von Neumann] [1934]) can be obtained with the aid of the Kakutani theorem. For details see Burckel's book [1972].

The Ryll-Nardzewski theorem has found applications in the theory of finite von Neumann algebras (Yeadon [1971], Strătilă- Zsidó [1979]) and in linear functional analysis (existence of invariant linear functionals under the action of the group of isometries in a Banach space; Fan [1976]).

III.

Homology and Fixed Points

In this chapter we develop the algebraic and geometric notions needed to formulate and prove the main result, the Lefschetz–Hopf theorem for polyhedra. We further illustrate the use of homology by studying the special case of maps $S^n \to S^n$, showing that the Brouwer degree of a map not only completely characterizes its homotopy behavior, but also gives considerable information about special topological features that such a map may have. We come full circle with the beginning of the last chapter by deriving Borsuk's antipodal theorem within this homological framework.

§8. Simplicial Homology

This paragraph is devoted to an exposition of only those notions of simplicial homology theory that provide the framework for the main results of the chapter.

1. Simplicial Complexes and Polyhedra

(1.1) DEFINITION. A *simplicial complex* is a finite family $\mathscr{K} = \{\sigma\}$ of simplices in some $\boldsymbol{R}^n$ such that:

(a) if σ is in $\mathscr{K}$, then so also is every face of σ,

(b) if $\sigma, \tau \in \mathscr{K}$, then $\sigma \cap \tau$ is either empty or a face common to both σ and τ.

A *polyhedron* is a pair $(K, \mathscr{K})$, where $\mathscr{K}$ is a simplicial complex and $K = \bigcup\{\sigma \mid \sigma \in \mathscr{K}\}$.

If $(K, \mathscr{K})$ is a polyhedron, we frequently say that $\mathscr{K} = \operatorname{sd} K$ is the *simplicial decomposition* of the polyhedron, and call $K = |\mathscr{K}|$ the *support* of $\mathscr{K}$. Clearly, the same support can be taken with different simplicial decompositions (consider, for example, the space of a 1-simplex); they are then different polyhedra.

The *dimension* of $(K, \mathscr{K})$ is $\max\{\dim \sigma \mid \sigma \in \mathscr{K}\}$; the *mesh* of $\mathscr{K}$ is $\max\{\delta(\sigma) \mid \sigma \in \mathscr{K}\}$. By a *subpolyhedron* of $(K, \mathscr{K})$ is meant a pair $(L, \mathscr{L})$, where $\mathscr{L}$ is any subfamily of $\mathscr{K}$ satisfying (a), and L is the support of $\mathscr{L}$; since $\mathscr{L}$ clearly also satisfies (b), the pair $(L, \mathscr{L})$ is indeed a polyhedron. Particularly important is the subset $\mathscr{K}^r \subset \mathscr{K}$ consisting of all $\sigma \in \mathscr{K}$ having dimension $\leq$ some fixed $r \geq 0$; the subpolyhedron determined by $\mathscr{K}^r$ is called the *r-skeleton* of $(K, \mathscr{K})$, and is denoted by $(K, \mathscr{K})^r$. The subpolyhedron $(K, \mathscr{K})^0$ is called the set of *vertices* of the polyhedron.

To illustrate, let σ be a fixed simplex. Taking the simplicial structure to be the set $\mathscr{F}(\sigma)$ of all its faces gives a polyhedron $(\sigma, \mathscr{F}(\sigma))$; the subset $\mathscr{F}' = \mathscr{F}(\sigma) - \sigma$, being supported by $\dot{\sigma}$, gives the subpolyhedron $(\dot{\sigma}, \mathscr{F}')$ of $(\sigma, \mathscr{F})$, and if $\dim \sigma = s$, it is the $(s-1)$-skeleton of $(\sigma, \mathscr{F})$. Observe that for any $(K, \mathscr{K})$ and any $\sigma \in \mathscr{K}$, the polyhedron $(\sigma, \mathscr{F}(\sigma))$ is a subpolyhedron of $(K, \mathscr{K})$.

Let $(K, \mathscr{K})$ be a polyhedron. For any $x \in K$, the *carrier* $\sigma(x)$ of x is the simplex $\sigma \in \mathscr{K}$ of lowest dimension that contains x. It is unique, being $\bigcap\{\sigma \in \mathscr{K} \mid x \in \sigma\}$. Clearly, (i) σ is the carrier of x if and only if $x \in \sigma - \dot{\sigma}$ and (ii) a point x belongs to some simplex σ if and only if its carrier is a face of σ.

Let K be the support of a finite polyhedron. Although K is in general the support of many different simplicial decompositions, the topology of K can be recovered from any given simplicial decomposition: it is the weak topology generated by the family $\mathscr{K}$ Consequently, for any topological space Y a map $f : K \to Y$ is continuous if and only if $f|\sigma : \sigma \to Y$ is continuous for each $\sigma \in \mathscr{K}$

We now describe a family of open sets in K that is obtained directly from a given $\mathscr{K}$ Note that if p is a vertex of $(K, \mathscr{K})$, then $\{\sigma \in \mathscr{K} \mid p \notin \sigma\}$ determines a subpolyhedron of $(K, \mathscr{K})$.

(1.2) DEFINITION. Let $(K, \mathscr{K})$ be a polyhedron. For any vertex p of $\mathscr{K}$, the set $K - \bigcup\{\sigma \in \mathscr{K} \mid p \notin \sigma\}$ is called the *star* of p, and is denoted by $\operatorname{St} p$.

$\operatorname{St} p$ is therefore an open set; clearly, it contains p but no other vertex of $(K, \mathscr{K})$. Moreover, $\{\operatorname{St} p \mid p \in (K, \mathscr{K})^0\}$ is an open cover of K: given any $x \in K$, let p_0 be a vertex of its carrier $\sigma(x)$; because x belongs only to simplices having $\sigma(x)$ as face, we find $x \notin \bigcup\{\sigma \in \mathscr{K} \mid p_0 \notin \sigma\}$, so $x \in \operatorname{St} p_0$. This open cover has important properties.

(1.3) THEOREM. *Let $p_0, \dots, p_n$ be vertices of $(K, \mathscr{K})$. Then:*

(a) *$\bigcap_{i=0}^n \operatorname{St} p_i \neq \emptyset$ if and only if $(p_0, \dots, p_n)$ is a simplex of $\mathscr{K}$,*

(b) *if $\sigma = (p_0, \dots, p_n)$ is a simplex of $\mathscr{K}$, then $\bigcap_{i=0}^n \operatorname{St} p_i$ is the set of all $x \in K$ carried by simplices having σ as face.*

PROOF. (a) Assume that $p_0, \ldots, p_n$ do not span a simplex $\sigma \in \mathscr{K}$. Then for each $\sigma \in \mathscr{K}$ at least one of $p_0, \ldots, p_n$ is not a vertex of σ, so

$$\bigcup\{\sigma \mid p_0 \notin \sigma\} \cup \cdots \cup \bigcup\{\sigma \mid p_n \notin \sigma\} = K$$

and $\bigcap_{i=0}^n \operatorname{St} p_i = \emptyset$. The argument is reversible.

(b) If $x \in \bigcap_{i=0}^n \operatorname{St} p_i$, then

$$x \notin \bigcup\{\sigma \mid p_0 \notin \sigma\} \cup \cdots \cup \bigcup\{\sigma \mid p_n \notin \sigma\};$$

the carrier of x must therefore contain all the vertices $p_0, \ldots, p_n$. This argument is also reversible. □

We can also use $\mathscr{K}$ to introduce coordinates in the support K analogous to barycentric coordinates in a simplex. Observe that if $\{p_0, \ldots, p_T\} = (K, \mathscr{K})^0$, then each $x \in K$ can be written uniquely as $x = \sum_{i=0}^T \mu_i(x) p_i$, where $0 \le \mu_i(x) \le 1$ for each $i = 0, \ldots, T$ and $\sum_{i=0}^T \mu_i(x) = 1$ for each $x \in K$; indeed, we have only to define, for each i,

$$\mu_i(x) = \begin{cases} \text{barycentric coordinate of } x \text{ relative to } p_i \\ \qquad \text{if } p_i \text{ is a vertex of the carrier of } x, \\ 0 \quad \text{otherwise.} \end{cases}$$

Thus, for each $x \in K$, the nonzero $\mu_i(x)$ are the barycentric coordinates of x in its carrier. Since for each i and any $\sigma \in \mathscr{K}$, $\mu_i|\sigma$ is either identically zero or the barycentric coordinate function of σ relative to p_i, we conclude that each $\mu_i : K \to \boldsymbol{I}$ is in fact continuous. It is easy to see, using (1.3), that $\operatorname{St} p_i = \{x \mid \mu_i(x) > 0\}$.

It follows from this representation of the points of K that for any topological space Y, each map $f : Y \to K$ is uniquely expressible as $f(y) = \sum_{i=0}^T \mu_i \circ f(y) p_i$, and if f is continuous, then each function $\mu_i \circ f : Y \to \boldsymbol{I}$ is continuous. The construction of continuous maps of spaces into polyhedra is frequently based on

(1.4) THEOREM. *Let $(K, \mathscr{K})$ be a polyhedron in $\boldsymbol{R}^n$ with vertices $\{p_0, \ldots$ $\ldots, p_T\}$, and let Y be an arbitrary topological space. Let $\varphi_i : Y \to \boldsymbol{I}$ be a family of functions, one for each vertex p_i, with $\sum_{i=0}^T \varphi_i(y) p_i \in K$ for each $y \in Y$. If each φ_i is continuous, then $y \mapsto \sum_{i=0}^T \varphi_i(y) p_i$ is a continuous map $f : Y \to K$.*

PROOF. Regarded as a map of Y into $\boldsymbol{R}^n$, f is evidently continuous; since $f(Y) \subset K \subset \boldsymbol{R}^n$, the continuity of f as a map of Y into K follows. □

The homotopy of two maps of a space Y into a polyhedron can sometimes be detected by their behavior relative to the given $\mathscr{K}$. One frequently used condition is

(1.5) THEOREM. *Let $(K, \mathscr{K})$ be a polyhedron, Y an arbitrary space, and $f, g : Y \to K$ continuous. Assume that for each $y \in Y$, there is some simplex $\sigma \in \mathscr{K}$ containing both $f(y)$ and $g(y)$ (such maps are called contiguous). Then $f \simeq g$.*

PROOF. For each $y \in Y$, the line segment joining $f(y)$ to $g(y)$ lies in some $\sigma \in \mathscr{K}$, therefore in K. These segments are given by

$$H(y,t) = \sum_{i=0}^{T}[t\mu_i(f(y)) + (1-t)\mu_i(g(y))]p_i;$$

since the functions $t\mu_i \circ f + (1-t)\mu_i \circ g : Y \times I \to I$ are continuous, the map $H : Y \times I \to K$ is continuous and provides the required homotopy. □

2. Subdivisions

Given any simplicial decomposition $\mathscr{K}$ of K, we shall construct another simplicial decomposition $\mathscr{K}'$ of K, called the barycentric subdivision of $\mathscr{K}$; its importance lies in that all the simplices of $\mathscr{K}'$ are "smaller" than those of $\mathscr{K}$, so by repetition of the construction, we can get simplicial decompositions of a given support that have arbitrarily small mesh.

The *barycenter* of an n-simplex $\sigma = (p_0, \ldots, p_n)$ is the point $[\sigma] = \frac{1}{n+1}\sum_{i=0}^{n} p_i$ (occasionally we use the notation b_σ); a zero-simplex is its own barycenter. We first describe the *barycentric subdivision* $\mathrm{Sd}^1 \sigma$ of the polyhedron $(\sigma, \mathscr{F}(\sigma))$, where σ is a single simplex; the vertices of $\mathrm{Sd}^1 \sigma$ will be the barycenters of all the faces of σ. The definition is by induction on the dimension of σ:

1. If $\dim \sigma = 0$, then $\mathrm{Sd}^1 \sigma = \sigma$.

2. Assume $\mathrm{Sd}^1 \sigma$ known for all σ of dimension $\leq n-1$. Let σ^n be an n-simplex, and let $\mathscr{L}$ be the collection of all $(n-1)$-simplices of $\mathrm{Sd}^1 \sigma^{n-1}$, as σ^{n-1} runs over all the $(n-1)$-faces of σ^n Then $\mathrm{Sd}^1 \sigma^n$ consists of the n-simplices

$$\{(y_0, \ldots, y_{n-1}, [\sigma^n]) \mid (y_0, \ldots, y_{n-1}) \in \mathscr{L}\}$$

and all faces of these simplices.

Observe that the barycentric subdivision of σ^n introduces no new vertices on $\dot{\sigma}^n$, so if each of two simplices σ, τ is barycentrically subdivided, the process will yield identical subdivisions on $\sigma \cap \tau$; the barycentric subdivision $\mathscr{K}' = \mathrm{Sd}^1 \mathscr{K}$ of $(K, \mathscr{K})$ is obtained by subdividing each $\sigma \in \mathscr{K}$ barycentrically, and consists of

$$\{\sigma' \mid \sigma' \in \mathrm{Sd}^1 \sigma \text{ and } \sigma \in \mathscr{K}\}.$$

For theoretical purposes, it is more convenient to describe $\mathrm{Sd}^1 \mathscr{K}$ in another manner.

(2.1) DEFINITION. Let $(K, \mathscr{K})$ be a polyhedron. $\mathrm{Sd}^1 \mathscr{K}$ consists of all simplices $([\sigma_0], \ldots, [\sigma_s])$, where $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_s$ is a strictly increasing sequence of simplices of $\mathscr{K}$

It is easy to verify that each set $\{[\sigma_0], \ldots, [\sigma_s]\}$ in the definition is in fact affinely independent. A proof that this construction is equivalent to that previously given can be based on the observation that each simplex $([\sigma_0], \ldots, [\sigma_s])$ is contained in σ_s, since σ_s is a convex set containing all the points $[\sigma_i]$, $i = 0, \ldots, s$.

(2.2) THEOREM. *Let $(K, \mathscr{K})$ be an n-polyhedron of mesh μ. Then the mesh of $\mathrm{Sd}^1 \mathscr{K}$ is $\leq \frac{n}{n+1}\mu$.*

PROOF. We need only estimate the length of any edge of a simplex of $\mathscr{K}'$ Let $p = [\sigma_0]$ and $q = [\sigma_1]$ be the vertices of such an edge; according to the definition, we have, say, $\sigma_0 \subset \sigma_1$, so that the vertices of σ_0 are among those of $\sigma_1 = (p_0, \ldots, p_s)$, say $\sigma_0 = (p_0, \ldots, p_r)$, where $r < s \leq n$. Then $p - q = \frac{1}{r+1} \sum_{i=0}^{r} (p_i - q)$, and for each $i = 0, \ldots, r$, also $p_i - q = \frac{1}{s+1} \sum_{j=0}^{s} (p_i - p_j)$, so that

$$\|p - q\| \leq \frac{1}{(r+1)(s+1)} \sum_{i=0}^{r} \sum_{j=0}^{s} \|p_i - p_j\| \leq \frac{(s+1)(r+1) - (r+1)}{(s+1)(r+1)} \mu,$$

because there are $(s+1)(r+1)$ terms in the double sum, but $r+1$ of them are zero, and all of them are $\leq \mu$. Thus,

$$\|p - q\| \leq \frac{s}{s+1} \mu \leq \frac{n}{n+1} \mu,$$

completing the proof. □

The process can be repeated. Defining the mth barycentric subdivision of $(K, \mathscr{K})$ inductively by

$$\mathrm{Sd}^0 \mathscr{K} = \mathscr{K}, \quad \mathrm{Sd}^m \mathscr{K} = \mathrm{Sd}^1(\mathrm{Sd}^{m-1} \mathscr{K}),$$

we have

(2.3) COROLLARY. *Let $\dim K \leq n$. Then*

$$\operatorname{mesh} \mathrm{Sd}^m \mathscr{K} \leq \left(\frac{n}{n+1}\right)^m \operatorname{mesh} \mathscr{K},$$

so that the diameters of the simplices of $\mathrm{Sd}^m \mathscr{K}$ tend to zero under repeated barycentric subdivision. □

3. Simplicial Maps and Simplicial Approximations

If $(K, \mathscr{K})$ and $(L, \mathscr{L})$ are two polyhedra, it is natural to consider maps of K into L that respect the given simplicial decompositions.

(3.1) DEFINITION. A map $\varphi : (K, \mathscr{K}) \to (L, \mathscr{L})$ is called *simplicial* if:
(a) for each simplex $(p_0, \ldots, p_s) \in \mathscr{K}$, the points $\varphi(p_0), \ldots, \varphi(p_s)$ are vertices (not necessarily distinct) of some simplex of $\mathscr{L}$,
(b) the map φ is affine on each $\sigma \in \mathscr{K}$:

$$\varphi\Big(\sum_{i=0}^{s} \lambda_i p_i\Big) = \sum_{i=0}^{s} \lambda_i \varphi(p_i), \quad \sum_{i=0}^{s} \lambda_i = 1, \ \lambda_i \geq 0.$$

It is to be emphasized that φ is a piecewise linear map: it is uniquely determined by its values on the vertices of $\mathscr{K}$ Therefore, we often write $\varphi : \mathscr{K} \to \mathscr{L}$. Clearly, a constant map, sending all of K to a vertex of L, is simplicial. Moreover, because the barycentric coordinate functions λ_i are continuous, any simplicial map $\varphi : (K, \mathscr{K}) \to (L, \mathscr{L})$ is continuous on each simplex $\sigma \in \mathscr{K}$, and therefore it is continuous.

We relate the simplicial maps to arbitrary continuous maps of K into L by

(3.2) DEFINITION. Let $f : K \to L$ be any continuous map. A simplicial map $\varphi : (K, \mathrm{Sd}^r \mathscr{K}) \to (L, \mathscr{L})$ for some $r \geq 0$ is called a *simplicial approximation* to f if $f(\mathrm{St}\, p) \subset \mathrm{St}\, \varphi(p)$ for each vertex $p \in \mathrm{Sd}^r \mathscr{K}$.

We establish

(3.3) THEOREM. *Let $f : K \to L$ be any continuous map. Then there exist simplicial approximations $\varphi : (K, \mathrm{Sd}^r \mathscr{K}) \to (L, \mathscr{L})$ to f for all sufficiently large $r \geq 0$. Moreover, every simplicial approximation to f is homotopic to f.*

PROOF. Let $\delta > 0$ be a Lebesgue number for the open cover

$$\{f^{-1}(\mathrm{St}\, q) \mid q \in (L, \mathscr{L})^0\}$$

of K. By (2.3), the mesh of $\mathrm{Sd}^r \mathscr{K}$ is $< \delta/2$ for all sufficiently large r; since the diameter of each $\mathrm{St}\, p$ is $< \delta$, each $\mathrm{St}\, p$ is contained in $f^{-1}(\mathrm{St}\, q)$ for at least one q. For each vertex $p \in \mathrm{Sd}^r \mathscr{K}$, choose $\varphi(p)$ to be any vertex of L such that $f(\mathrm{St}\, p) \subset \mathrm{St}\, \varphi(p)$. The map φ sends the vertices of each simplex $(p_0, \ldots, p_n) \in \mathrm{Sd}^r \mathscr{K}$ to the vertices of a simplex of $\mathscr{L}$: for

$$\emptyset \neq f\Big(\bigcap_{i=0}^{n} \mathrm{St}\, p_i\Big) \subset \bigcap_{i=0}^{n} f(\mathrm{St}\, p_i) \subset \bigcap_{i=0}^{n} \mathrm{St}\, \varphi(p_i),$$

and (1.3) applies.

Extending φ affinely over each simplex gives the desired simplicial approximation to f. To show that f and φ are homotopic, let $x \in K$, and let $(p_0, \ldots, p_n) = \sigma(x)$ be its carrier in $\mathrm{Sd}^r \mathscr{K}$ Then $\varphi(x) \in (\varphi(p_0), \ldots, \varphi(p_n)) \in \mathscr{L}$; on the other hand, $f(x) \in f(\bigcap_{i=0}^{n} \mathrm{St}\, p_i) \subset \bigcap_{i=0}^{n} \mathrm{St}\, \varphi(p_i)$, so by (1.3),

$f(x)$ is carried by some simplex having $(\varphi(p_0), \dots, \varphi(p_n))$ as face. Thus, $f(x)$ and $\varphi(x)$ are always in a common simplex of $\mathscr{L}$, so by (1.5) the proof is complete. □

Note that the simplicial approximation to f is in general not unique: it depends on the level r of subdivision that is used, and on the choice of the vertex $\varphi(p)$ for each vertex $p \in \mathrm{Sd}^r \mathscr{K}$. However, for each fixed r, any two simplicial approximations $\varphi, \psi : \mathrm{Sd}^r \mathscr{K} \to L$ to f are contiguous: for if $(p_0, \dots, p_n)$ is the carrier of any given $x \in \mathrm{Sd}^r \mathscr{K}$, then $f(x)$ belongs to $\bigcap_{i=0}^n \mathrm{St}\, \varphi(p_i)$ and also to $\bigcap_{i=0}^n \mathrm{St}\, \psi(p_i)$, so by (1.3), all the vertices $\varphi(p_i)$, $\psi(p_i)$ lie in some single simplex of $\mathscr{L}$, and both $\varphi(x), \psi(x)$ belong to that simplex. Note also that for each fixed r and simplicial approximation $\varphi : \mathrm{Sd}^r \mathscr{K} \to \mathscr{L}$ to f, always $d(f(x), \varphi(x)) \leq \mathrm{mesh}\, \mathscr{L}$, and that the homotopy path from $f(x)$ to $\varphi(x)$ lies in a single simplex of $\mathscr{L}$.

4. Vertex Schemes, Realizations, and Nerves of Coverings

Definition (1.1) suggests that a polyhedron may be described simply by specifying its set of vertices and those sets of vertices that span the simplices of $\mathscr{K}$; it was the observation that such a description can in fact be made in purely abstract terms, and suffices to determine the carrier uniquely up to homeomorphism, that suggested that algebraic methods may be useful in the study of polyhedra.

(4.1) DEFINITION. An *abstract complex* $\widehat{\mathscr{K}}$ is a finite set $\mathscr{A}$ of elements together with a family $\mathrm{sd}\, \mathscr{A}$ of subsets of $\mathscr{A}$ that satisfies the condition: if $s \in \mathrm{sd}\, \mathscr{A}$, then every subset of s also belongs to $\mathrm{sd}\, \mathscr{A}$

The elements of $\mathscr{A}$ are called the "vertices" of $\widehat{\mathscr{K}}$, and each set $\{p_0, \dots, p_n\}$ of $n+1$ elements that belongs to $\mathrm{sd}\, \mathscr{A}$ is called an "n-simplex" of $\widehat{\mathscr{K}}$.

Each polyhedron $(K, \mathscr{K})$ determines an abstract complex $\widehat{\mathscr{K}}$ called its *vertex scheme*: $\mathscr{A}$ is the set of vertices of $(K, \mathscr{K})$, and $\mathrm{sd}\, \mathscr{A}$ consists of those subsets of $(K, \mathscr{K})^0$ that span the simplices of $\mathscr{K}$.

Any polyhedron with vertex scheme isomorphic to $\widehat{\mathscr{K}}$ in obvious sense is called a *realization* of $\widehat{\mathscr{K}}$. We will now show that every abstract complex $\widehat{\mathscr{K}}$ has at least one realization K in some $\boldsymbol{R}^n$, and in fact, the topological nature of these realizations is determined entirely by the abstract complex $\widehat{\mathscr{K}}$

(4.2) THEOREM. *Every abstract complex $\widehat{\mathscr{K}}$ has a realization in some $\boldsymbol{R}^n$; and any two realizations of $\widehat{\mathscr{K}}$ are simplicially homeomorphic.*

PROOF. We describe a realization of $\widehat{\mathscr{K}}$, called the *standard realization*. Let $\{p_0, \dots, p_T\}$ be the vertices of $\widehat{\mathscr{K}}$, and let $e_0, \dots, e_T$ be the vertices of the

standard simplex $\Delta^T \subset \boldsymbol{R}^{T+1}$. Select a definite one-to-one correspondence $p_i \leftrightarrow e_i$ and define a subpolyhedron $(K, \mathscr{K}) \subset (\Delta^T, \mathscr{F}(\Delta^T))$ by taking

$$\mathscr{K} = \{(e_{i_0}, \ldots, e_{i_s}) \mid (p_{i_0}, \ldots, p_{i_s}) \in \operatorname{sd} \mathscr{A}\}.$$

This is clearly a polyhedron with vertex scheme isomorphic to $\widehat{\mathscr{K}}$, so it is a realization of $\widehat{\mathscr{K}}$. To prove the second part, we show that any realization of $\widehat{\mathscr{K}}$ is homeomorphic to the K we have constructed. Let b_i be the vertices of any realization P of $\widehat{\mathscr{K}}$, where $b_i \leftrightarrow p_i$; then $(b_0, \ldots, b_n) \in \operatorname{sd} P$ if and only if $(e_0, \ldots, e_n) \in \mathscr{K}$; defining $f(b_i) = e_i$, $i = 0, \ldots, T$, and extending affinely over each simplex yields the required homeomorphism $f : P \to K$. $\square$

The notion of an abstract complex provides a convenient means for defining polyhedra. One important such use is to define the nerve of a finite family of subsets of a space.

(4.3) DEFINITION. Let X be a topological space, and $H = \{H_\alpha \mid \alpha \in \mathscr{A}\}$ an indexed finite family of subsets of X. Let $\mathscr{K}$ be the abstract complex with the set $\mathscr{A}$ as vertices and with $\{\alpha_0, \ldots, \alpha_n\} \in \operatorname{sd} \mathscr{A}$ if and only if $H_{\alpha_0} \cap \cdots \cap H_{\alpha_n} \neq \emptyset$. The standard realization of $\mathscr{K}$ is called the *nerve* of the family $\{H_\alpha \mid \alpha \in \mathscr{A}\}$ and is denoted by $N(H)$.

As an example, let $(P, \operatorname{sd} P)$ be any polyhedron, and $\{\operatorname{St} p \mid p \in (P, \operatorname{sd} P)^0\}$ the covering by its vertex stars. By (1.3), the nerve of this family has precisely the same vertex scheme as $(P, \operatorname{sd} P)$, so that this nerve is homeomorphic to P.

Information about the structure of a space can be obtained by studying the nature of polyhedra "approximating" the space in some sense. The intuitive idea is that nerves of open coverings can be regarded as being such "approximating" polyhedra, with a finer covering giving a "better" approximation. For normal spaces, the relation of the space to the nerves of its finite open covers can be given in a very precise manner:

(4.4) THEOREM (Alexandroff). *Let X be normal, and $U = \{U_\alpha \mid \alpha \in \mathscr{A}\}$ a finite open cover of X. Then there exists a continuous map $\kappa : X \to N(U)$ such that $\kappa^{-1}(\operatorname{St} \alpha) \subset U_\alpha$ for each vertex α of $N(U)$.*

PROOF. Shrink the open cover U to an open cover $\{V_\alpha \mid \alpha \in \mathscr{A}\}$ having $\overline{V}_\alpha \subset U_\alpha$ for each $\alpha \in \mathscr{A}$; and let $\mu_\alpha : X \to \boldsymbol{I}$ be an Urysohn function with $\mu_\alpha | \overline{V}_\alpha = 1$ and $\mu_\alpha | (X - U_\alpha) = 0$ for each $\alpha \in \mathscr{A}$ Because each $x \in X$ belongs to some V_α, we have $\sum_\alpha \mu_\alpha(x) \neq 0$ for each x, so that the functions

$$\lambda_\alpha(x) = \frac{\mu_\alpha(x)}{\sum_\alpha \mu_\alpha(x)}$$

are continuous on X for each $\alpha \in \mathscr{A}$. Define $\kappa : X \to N(U)$ by

$$\kappa(x) = \sum_{\alpha} \lambda_\alpha(x) \cdot \alpha.$$

To see that $\kappa(x) \in N(U)$ for each $x \in X$, note that if $\lambda_{\alpha_1}, \dots, \lambda_{\alpha_n}$ are all the λ_α that do not vanish at x, then $x \in U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$, so that the $(\alpha_1, \dots, \alpha_n)$ span a simplex of $N(U)$ and $\kappa(x)$ belongs to that simplex. It follows from (1.4) that $\kappa : X \to N(U)$ is continuous. Finally, $\kappa(x) \in \operatorname{St}\alpha$ if and only if $\lambda_\alpha(x) > 0$, so that $\kappa^{-1}(\operatorname{St}\alpha) \subset U_\alpha$. □

These simple maps are called *standard maps* of X into $N(U)$; the κ stands for Kuratowski, who was the first to give the explicit formula.

5. Simplicial Homology

Let $\mathscr{K}$ be a finite simplicial complex, and linearly order its vertices. Then each simplex $(q_0, \dots, q_n)$ can be written uniquely as $|p_0, \dots, p_n|$, where $p_0 \prec p_1 \prec \cdots \prec p_n$, and is called an *oriented* n-simplex.

(5.1) DEFINITION. For each $n \geq 0$, the free abelian group $C_n(\mathscr{K})$ generated by the oriented n-simplices of $\mathscr{K}$ is called the *group of n-dimensional chains* of $\mathscr{K}$; clearly, $C_n(\mathscr{K}) = 0$ for $n > \dim \mathscr{K}$ For each $n \geq 1$, the *boundary operator* $\partial_n : C_n(\mathscr{K}) \to C_{n-1}(\mathscr{K})$ is the homomorphism defined on each generator by

$$\partial_n |p_0, \dots, p_n| = \sum_{i=0}^{n} (-1)^i |p_0, \dots, \widehat{p_i}, \dots, p_n|,$$

the caret over the p_i indicating that it is to be omitted; for $n = 0$, we define $\partial_0 : C_0(\mathscr{K}) \to 0$ to be the zero homomorphism.

A simplex with its vertices written in some order is called an *ordered* n-simplex. We relate an arbitrary ordered n-simplex $(p_{i_0}, \dots, p_{i_n})$, made up of vertices $p_0, \dots, p_n$ written in some order, to the generators of $C_n(\mathscr{K})$ by the convention

$$(p_{i_0}, \dots, p_{i_n}) = \begin{cases} +|p_0, \dots, p_n| & \text{if } \{p_{i_0}, \dots, p_{i_n}\} \text{ is an} \\ & \text{even permutation of } \{p_0, \dots, p_n\}, \\ -|p_0, \dots, p_n| & \text{if } \{p_{i_0}, \dots, p_{i_n}\} \text{ is an} \\ & \text{odd permutation of } \{p_0, \dots, p_n\}. \end{cases}$$

With this convention, each simplex, ordered as it is written, represents $+$ or $-$ a generator of $C_n(\mathscr{K})$; moreover, the boundary operator applied formally to that ordered simplex yields exactly the boundary of the generator it represents. Thus, the boundary operator can be used on any ordered simplex without first converting it to an oriented simplex.

It is straightforward to verify that any composition $\partial_n \circ \partial_{n+1}$ is zero, which is equivalent to the statement that $\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n$. The kernel of $\partial_n : C_n(\mathscr{K}) \to C_{n-1}(\mathscr{K})$ is denoted by $Z_n(\mathscr{K})$ and called the *group of n-cycles* of $\mathscr{K}$; note that $Z_0(\mathscr{K}) = C_0(\mathscr{K})$. The image of $\partial_{n+1} : C_{n+1}(\mathscr{K}) \to C_n(\mathscr{K})$ is called the *group $B_n(\mathscr{K})$ of n-boundaries*, and the quotient group $H_n(\mathscr{K}) = Z_n(\mathscr{K})/B_n(\mathscr{K})$ is the nth *homology group* of $\mathscr{K}$. The elements of $H_n(\mathscr{K})$ are called *homology classes*, the coset $z + B_n(\mathscr{K})$ being the homology class of the n-cycle z. Two n-cycles z, z' belonging to the same homology class are called *homologous*; this will occur if and only if $z - z' = \partial c_{n+1}$ for some $(n+1)$-chain c_{n+1}. Note that $H_n(\mathscr{K}) = 0$ for $n > \dim \mathscr{K}$, and $H_n(\mathscr{K}) = Z_n(\mathscr{K})$ for $n = \dim \mathscr{K}$

The entire construction can be put in abstract form. Let a *chain complex* be any indexed family $C_* = \{(C_n, \partial_n) \mid n = 0, 1, \ldots\}$ of abelian groups and homomorphisms $\partial_n : C_n \to C_{n-1}$, where $C_{-1} = 0$, ∂_0 is the zero homomorphism and $\partial_n \circ \partial_{n+1} = 0$ for all $n \geq 0$. A chain complex therefore appears as a sequence

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

of abelian groups and homomorphisms, in which the composition of any two of the homomorphisms is zero; and the group $H_n(C_*) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$ provides a measure of how much this sequence deviates from exactness at the group C_n.

Within this framework, we can define, for a simplicial complex $\mathscr{K}$, homology groups over an arbitrary abelian group, or field, G. For any such G, the finite formal sums $c_n = \sum g_i |\sigma_i^n|$ with $g_i \in G$ form an abelian group $C_n(\mathscr{K}; G)$, which is a vector space whenever G is a field. The boundary homomorphism $\partial_n : C_n(\mathscr{K}; G) \to C_{n-1}(\mathscr{K}; G)$ is defined as $\partial_n c_n = \sum g_i \partial |\sigma_i^n|$, and is a linear map when G is a field. Again $\partial_n \circ \partial_{n+1} = 0$, so that $\{(C_n(\mathscr{K}; G), \partial_n) \mid n = 0, 1, \ldots\}$ is a chain complex; the corresponding homology groups $H_n(\mathscr{K}; G)$ are called the homology groups of $\mathscr{K}$ *over* G. If G is a field, then $H_n(\mathscr{K}; G)$, being a quotient of a vector space by a linear subspace, is itself a vector space ([1]).

As the homology groups have been defined, they appear to depend on the triangulation used; we will see later that in fact, they are topological invariants of the support of $\mathscr{K}$ We now calculate some homology groups.

EXAMPLE 1. If $\mathscr{K}$ is a connected simplicial complex (i.e., $|\mathscr{K}|$ is connected), then $H_0(\mathscr{K}; G) = G$. We show this for $G = Z$, the proof for arbitrary G being the same. The connectedness of $\mathscr{K}$ is equivalent to the statement that for any two vertices p_0 and p_n there is a sequence $\sigma_1 = (p_0, p_1)$,

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([1]) Note that in this terminology, $H_n(\mathscr{K})$ is $H_n(\mathscr{K}; Z)$, and, in fact, the chain complex $\{(C_n(\mathscr{K}; G), \partial)\}$ is precisely $\{(C_n(\mathscr{K}) \otimes G, \partial \otimes 1)\}$, where $\otimes$ stands for the tensor product.

$\sigma_2 = (p_1, p_2), \ldots, \sigma_n = (p_{n-1}, p_n)$ of 1-simplices joining p_0 and p_n. Now fix a vertex p_0; for any vertex p_n we have a 1-chain $c_{0n} = \sum_{i=1}^{n} \sigma_i$ such that $\partial c_{0n} = p_n - p_0$. In particular, for any 0-chain $c = \sum n_i p_i \in C_0(\mathscr{K}) = Z_0(\mathscr{K})$, we find

$$\sum n_i p_i - \left(\sum n_i\right) p_0 = \sum n_i (p_i - p_0) = \sum n_i \partial c_{0i} = \partial\left(\sum n_i c_{0i}\right),$$

so that every 0-cycle $\sum n_i p_i$ is homologous to some cycle np_0. Now, np_0 and mp_0 cannot be homologous when $m \neq n$, since the boundary of any 1-chain, $\partial \sum n_i \sigma_i^1$, will have coefficients alternating in sign, therefore a coefficient sum equaling 0. Thus, every 0-cycle is homologous to a unique cycle np_0, and the assertion is proved. □

EXAMPLE 2. If $\mathscr{K} = \{p\}$, a single vertex, then $H_0(p; G) = G$ and $H_i(p; G) = 0$ for all $i > 0$.

EXAMPLE 3. Let $\mathscr{K}$ be any simplicial complex, and v a vertex not in $\mathscr{K}$. The *cone* $\mathscr{K}_v$ over $\mathscr{K}$ with vertex v is the simplicial complex having as vertices the vertex v and the vertices of $\mathscr{K}$, and simplices $\{(v, p_0, \ldots, p_n) \mid (p_0, \ldots, p_n) \in \mathscr{K}\}$ together with all faces of these simplices. Then $H_0(\mathscr{K}_v; G) = G$ and $H_i(\mathscr{K}_v; G) = 0$ for all $i > 0$. Since $\mathscr{K}_v$ is connected, the first statement is clear. To prove the second, we introduce some notation. Extend the given linear ordering of $\mathscr{K}$ to one of $\mathscr{K}_v$ by $v \prec$ the first vertex of $\mathscr{K}$, and write $\sigma = v \cdot \tau$ if $\sigma = |v, p_0, \ldots, p_n|$, $\tau = |p_0, \ldots, p_n|$, extending this notation in the obvious way for chains. Since $\partial(v \cdot \tau) = \tau - v \cdot \partial\tau$, we have $\partial(v \cdot c) = c - v \cdot \partial c$ for every chain. To establish our result, let $n > 0$, and let c_n be an n-chain in $\mathscr{K}_v$; then $c_n = v \cdot c_{n-1} + d_n$, where we have collected all simplices in the chain involving the vertex v, so $\partial c_n = c_{n-1} - v \cdot \partial c_{n-1} + \partial d_n$. If c_n is a cycle, then because no simplex with vertex v can offset any of the remaining terms, we must have $c_{n-1} + \partial d_n = 0$. Consider now the chain $c_{n+1} = v \cdot d_n$; then

$$\partial c_{n+1} = d_n - v \cdot \partial d_n = d_n + v \cdot c_{n-1} = c_n,$$

so every cycle bounds.

EXAMPLE 4. For the (closed) n-simplex σ^n, we have $H_0(\sigma^n; G) = G$ and $H_i(\sigma^n; G) = 0$ for all $i > 0$, for σ can be regarded as the cone over any one of its $(n-1)$-faces.

EXAMPLE 5. Let K^s be the s-skeleton of a simplex K, and note that only the groups $C_i(K)$ for $0 \leq i \leq s$ are used in forming the homology groups $H_i(K)$ for $0 \leq i \leq s-1$; since $C_i(K) = C_i(K^s)$ for $i \leq s$, it follows that $H_i(K) = H_i(K^s)$ for $0 \leq i \leq s-1$. Using this observation, we calculate the homology of $\dot{\sigma}^n$, the boundary of an n-simplex (which is homeomorphic to the $(n-1)$-sphere S^{n-1}). We shall show that whenever $n > 1$,

$$H_i(\dot{\sigma}^n; G) = \begin{cases} 0 & \text{for } i \neq 0, n-1, \\ G & \text{for } i = 0, n-1. \end{cases}$$

Since $\dot{\sigma}^n$ is the $(n-1)$-skeleton of σ^n, we find $H_i(\dot{\sigma}^n; G) = H_i(\sigma^n; G)$ for $0 \leq i \leq n-2$, so from the previous example, all the groups except for $i = n-1$ are as indicated. Since $\dim \dot{\sigma}^n = n-1$, we have $H_{n-1}(\dot{\sigma}^n; G) = Z_{n-1}(\dot{\sigma}^n; G)$, so that we need only determine the group of $(n-1)$-cycles. Let $\sigma^n = (p_0, \ldots, p_n)$ and consider the chain $c_n = 1\sigma^n$ in $C_n(\dot{\sigma}^n; Z)$; then $b_{n-1} = \partial c_n = \sum_{i=0}^{n}(-1)^i(p_0, \ldots, \widehat{p}_i, \ldots, p_n)$ is a cycle on $\dot{\sigma}^n$ because $\partial b_{n-1} = \partial\partial c_n = 0$; we will show that the cycles on $\dot{\sigma}^n$ are precisely all the chains gb_{n-1} for $g \in G$. In fact, any $(n-1)$-chain on $\dot{\sigma}^n$ can be written as

$$c_{n-1} = \sum_{i=0}^{n}(-1)^i g_i(p_0, \ldots, \widehat{p}_i, \ldots, p_n),$$

so

$$\begin{aligned}\partial c_{n-1} &= \sum_{i=0}^{n}(-1)^i g_i\Big\{\sum_{k<i}(-1)^k(p_0, \ldots, \widehat{p}_k, \ldots, \widehat{p}_i, \ldots, p_n) \\ &\qquad + \sum_{k>i}(-1)^{k-1}(p_0, \ldots, \widehat{p}_i, \ldots, \widehat{p}_k, \ldots, p_n)\Big\} \\ &= \sum_{i<k}\{(-1)^i g_i(-1)^{k-1} + (-1)^k g_k(-1)^i\}(p_0, \ldots, \widehat{p}_i, \ldots, \widehat{p}_k, \ldots, p_n) \\ &= \sum_{i<k}\{(-1)^{i+k}[g_k - g_i]\}(p_0, \ldots, \widehat{p}_i, \ldots, \widehat{p}_k, \ldots, p_n).\end{aligned}$$

Thus, c_{n-1} will be a cycle if and only if all $g_k - g_i = 0$, and therefore if and only if $c_{n-1} = gb_{n-1}$ for some $g \in G$. Thus, $H_{n-1}(\dot{\sigma}^n; G) = Z_n(\dot{\sigma}^n; G) \cong G$. The cycle b_{n-1} is called the *basic $(n-1)$-cycle* of $\dot{\sigma}^n$ in $Z_{n-1}(\dot{\sigma}^n; Z)$.

6. Chain Transformations and Chain Homotopies

Let $\mathscr{K}$, $\mathscr{L}$ be two simplicial complexes.

(6.1) DEFINITION. A collection $\tau = \{\tau_n\}$ of homomorphisms $\tau_n : C_n(\mathscr{K}) \to C_n(\mathscr{L})$, one for each dimension $n \geq 0$, and such that $\partial_{n+1} \circ \tau_{n+1} = \tau_n \circ \partial_{n+1}$ is called a *chain transformation* or a *chain map*; this can be visualized as a diagram

$$\begin{array}{ccccccccccc} \longrightarrow & C_n(\mathscr{K}) & \xrightarrow{\partial_n} & C_{n-1}(\mathscr{K}) & \xrightarrow{\partial_{n-1}} & C_{n-2}(\mathscr{K}) & \longrightarrow \cdots & \xrightarrow{\partial_1} & C_0(\mathscr{K}) & \longrightarrow & 0 \\ & \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_{n-1}} & & \downarrow{\scriptstyle \tau_{n-2}} & & & \downarrow{\scriptstyle \tau_0} & & \\ \longrightarrow & C_n(\mathscr{L}) & \xrightarrow[\partial_n]{} & C_{n-1}(\mathscr{L}) & \xrightarrow[\partial_{n-1}]{} & C_{n-2}(\mathscr{L}) & \longrightarrow \cdots & \xrightarrow[\partial_1]{} & C_0(\mathscr{L}) & \longrightarrow & 0 \end{array}$$

in which each square is commutative.

Chain transformations are important because they induce homomorphisms $\tau_{*n} : H_n(\mathscr{K}) \to H_n(\mathscr{L})$ of the homology for each n, by the rule

$$\tau_{*n}[z + B_n(\mathscr{K})] = \tau_n(z) + B_n(\mathscr{L}).$$

For τ_n maps $Z_n(\mathscr{K})$ into $Z_n(\mathscr{L})$, because $\partial\tau(z) = \tau\partial z = \tau 0 = 0$ if $\partial z = 0$. and $\tau\partial c = \partial\tau c$ shows that $\tau(B_n(\mathscr{K})) \subset B_n(\mathscr{L})$, so we need only pass to the quotient.

The composition $\lambda \circ \tau$ of chain maps is easily seen to be a chain map, and a direct calculation shows that $(\lambda \circ \tau)_* = \lambda_* \circ \tau_*$.

EXAMPLE 1. Let $f : \mathscr{K} \to \mathscr{L}$ be a simplicial map. For each $n \geq 0$, define $f_{n\#} : C_n(\mathscr{K}) \to C_n(\mathscr{L})$ by setting

$$f_{n\#}|p_0, \ldots, p_n| = \begin{cases} (f(p_0), \ldots, f(p_n)) & \text{if all the } f(p_i) \text{ are distinct,} \\ 0 & \text{otherwise,} \end{cases}$$

on the generators. It is straightforward to verify that $f_\# = \{f_{n\#}\}$ is a chain map. The homomorphism $f_* : H_n(\mathscr{K}) \to H_n(\mathscr{L})$ defined by $f_\#$ is called the *homomorphism induced by* f. Thus, every simplicial map induces a homomorphism in homology. If both $\mathscr{K}$ and $\mathscr{L}$ are connected, then $f_* : H_0(\mathscr{K}) \cong H_0(\mathscr{L})$; and under composition, $(g \circ f)_* = g_* \circ f_*$.

EXAMPLE 2. Let $\mathscr{K}'$ be the first barycentric subdivision of $\mathscr{K}$. We shall define a chain map

$$\mathrm{Sd} : C_*(\mathscr{K}) \to C_*(\mathscr{K}').$$

We observe that if b_σ is the barycenter of σ, and if B' is the complex representing the barycentric subdivision of $\partial\sigma$, then the simplices of $\mathscr{K}'$ carried by σ are exactly the cone $b_\sigma \cdot B'$. We define Sd by induction: set $\mathrm{Sd}\, p_0 = p_0$ for $p_0 \in C_0(\mathscr{K})$, and if Sd is defined for all $k < n$, set

$$\mathrm{Sd}\,\sigma^n = b_\sigma \cdot \mathrm{Sd}\,\partial\sigma^n$$

on each generator $|\sigma^n|$ of $C_n(\mathscr{K})$.

We now show that $\partial\,\mathrm{Sd} = \mathrm{Sd}\,\partial$ by induction: this is certainly true for $n = 0$, and if true for dimension $n - 1$, then using $b = b_\sigma$,

$$\begin{aligned} \partial\,\mathrm{Sd}\,\sigma^n &= \partial(b \cdot \mathrm{Sd}\,\partial\sigma^n) = \mathrm{Sd}\,\partial\sigma^n - b \cdot \partial\,\mathrm{Sd}\,\partial\sigma^n \\ &= \mathrm{Sd}\,\partial\sigma^n - b \cdot \mathrm{Sd}\,\partial\partial\sigma^n \quad \text{(by the induction hypothesis)} \\ &= \mathrm{Sd}\,\partial\sigma^n. \end{aligned}$$

More generally, if $\mathscr{K}^{(m)}$ is the mth barycentric subdivision of $\mathscr{K}$, we obtain a chain map $\mathrm{Sd}^m : C_*(\mathscr{K}) \to C_*(\mathscr{K}^{(m)})$ by iteration of the chain maps

$$C_*(\mathscr{K}) \xrightarrow{\mathrm{Sd}} C_*(\mathscr{K}') \xrightarrow{\mathrm{Sd}} C_*(\mathscr{K}'') \to \cdots \to C_*(\mathscr{K}^{(m)});$$

observe that for each $\sigma \in \mathscr{K}$, the chain $\mathrm{Sd}^m\,\sigma$ is carried by σ.

(6.2) DEFINITION. Let $\tau, \mu : C_*(\mathscr{K}) \to C_*(\mathscr{L})$ be two chain maps. A *chain homotopy* joining τ and μ is a collection $D = \{D_n\}$ of homomorphisms $D_n : C_n(\mathscr{K}) \to C_{n+1}(\mathscr{L})$, one for each $n \geq 0$, such that

$$\tau_n - \mu_n = \partial_{n+1} D_n + D_{n-1}\partial_n \qquad \text{for all } n \geq 1.$$

We say that τ and μ are *chain homotopic* if such a D exists.

It is simple to verify that chain homotopy is an equivalence relation in the set of all chain maps $C_*(\mathscr{K}) \to C_*(\mathscr{L})$; moreover, if $\tau, \mu : C_*(\mathscr{K}) \to C_*(\mathscr{L})$ are chain homotopic, and $\alpha, \beta : C_*(\mathscr{L}) \to C_*(\mathscr{P})$ are chain homotopic, then $\alpha \circ \tau$ and $\beta \circ \mu$ are chain homotopic maps of $C_*(\mathscr{K})$ into $C_*(\mathscr{P})$. The importance of the concept stems from the following: chain homotopic maps always induce the same homomorphisms $\tau_*, \mu_* : H_*(\mathscr{K}) \to H_*(\mathscr{L})$: for, if z is any cycle in $\mathscr{K}$, then $\tau(z) - \mu(z) = \partial D z$, so $\tau(z)$ is homologous to $\mu(z)$ for each z.

EXAMPLE 3. Let $f, g : \mathscr{K} \to \mathscr{L}$ be two contiguous simplicial maps. Then $f_* = g_* : H_*(\mathscr{K}) \to H_*(\mathscr{L})$. For let

$$D|p_0, \ldots, p_n| = \sum_{i=0}^{n} (-1)^i (f(p_0), \ldots, f(p_i), g(p_i), \ldots, g(p_n))$$

on each generator of $C_n(\mathscr{K})$. Each term on the right is made of vertices (possibly repeated) of some simplex in $\mathscr{L}$ precisely because the maps are contiguous; we set any term with repeated vertices equal to zero. A routine calculation shows that $g_\# - f_\# = D\partial + \partial D$ for the induced chain maps, so the result follows.

Let $\mathscr{K}'$ be the first barycentric subdivision of $\mathscr{K}$, and let $\pi : \mathscr{K}' \to \mathscr{K}$ be a *pseudo-identity map*, defined on the vertices of $\mathscr{K}'$ by

$$\pi(p') = \text{some vertex of the carrier of } p'.$$

This map is clearly simplicial; indeed, $\operatorname{St} p' \subset \operatorname{St} \pi p'$, so that any such map π is a simplicial approximation to the identity map of $|\mathscr{K}|$ Consider now the chain maps

$$C_*(\mathscr{K}) \overset{\mathrm{Sd}}{\to} C_*(\mathscr{K}') \overset{\pi}{\to} C_*(\mathscr{K}),$$

where we write π instead of $\pi_\#$ for simplicity.

(6.3) THEOREM.
(a) $\pi \circ \mathrm{Sd} : C_*(\mathscr{K}) \to C_*(\mathscr{K})$ *is the identity map of* $C_*(\mathscr{K})$.
(b) $\mathrm{Sd} \circ \pi : C_*(\mathscr{K}') \to C_*(\mathscr{K}')$ *is chain homotopic to the identity map of* $C_*(\mathscr{K}')$.

PROOF. We prove (a) by induction, the assertion being trivial for $n = 0$. Assume that it is true for dimensions $< n - 1$, and let σ^n be any generator

of $C_n(\mathscr{K})$. It is clear from the definitions that $\pi \circ \mathrm{Sd}\,\sigma^n = \alpha\sigma^n$, where α is some integer, possibly zero. Now

$$\alpha\partial\sigma^n = \partial[\pi \circ \mathrm{Sd}\,\sigma^n] = \pi\partial\,\mathrm{Sd}\,\sigma^n = \pi \circ \mathrm{Sd}\,\partial\sigma^n,$$

so by the induction hypothesis, $\alpha = 1$ and $\pi\,\mathrm{Sd}\,\sigma^n = \sigma^n$ for every generator of $C_n(\mathscr{K})$, completing the inductive step.

We begin the proof of (b) by observing that each simplex $\sigma' \in \mathscr{K}'$ is of the form $([\sigma_0], \dots, [\sigma_t])$, where $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_t$, so by defining $s(\sigma') = \sigma_t$ we see that $s(\sigma')$ is a simplex of $\mathscr{K}$ containing σ', and $s(\partial\sigma') \subset s(\sigma')$ for each σ' We will construct, by induction, a chain homotopy $D : C_n(\mathscr{K}') \to C_{n+1}(\mathscr{K}')$ such that the obviously defined support $|D(\sigma')|$ of $D(\sigma')$ is contained in $s(\sigma')$ for each σ'; the proof relies strongly on the fact that the barycentric subdivision on each simplex of $\mathscr{K}$ is the cone over its subdivided boundary with the barycenter of the simplex as apex.

We define $D : C_0(\mathscr{K}') \to C_1(\mathscr{K}')$: for each generator p_0' of $C_0(\mathscr{K}')$ we have $p_0' - \mathrm{Sd}\,\pi(p_0') = p_0' - \pi p_0'$, where $\pi p_0'$ is some vertex of $s(p_0')$; since the simplex $s(p_0')$ appears in $\mathscr{K}'$ as the cone over its barycenter, we find from Example 3 of Section 5 that $p_0' - \pi p_0' = \partial\check{c}_1$ for some $\check{c}_1$ with $|\check{c}_1| \subset s(p_0')$, so setting $D(p_0') = \check{c}_1$ gives $p_0' - \mathrm{Sd}\,\pi p_0' = \partial D(p_0')$, as required.

Assume $D : C_i(\mathscr{K}') \to C_{i+1}(\mathscr{K}')$ has been constructed for all $i < n$ with $|D(\sigma')| \subset s(\sigma')$ for each σ'. Choose any generator $\check{\sigma}^n$ of $C_n(\mathscr{K}')$ and consider the chain $\check{c}_n = \check{\sigma}^n - \mathrm{Sd}\,\pi\check{\sigma}^n - D\partial\check{\sigma}^n$, which lies in $s(\check{\sigma}^n)$. We have

$$\partial\check{c}_n = \partial\check{\sigma}^n - \mathrm{Sd}\,\pi\partial\check{\sigma}^n - \partial D\partial\check{\sigma}^n = D\partial\partial\check{\sigma}^n = 0,$$

the next to last equality by the induction hypothesis, so $\check{c}_n$ is a cycle in $s(\check{\sigma}^n)$, therefore bounds in $s(\check{\sigma}^n)$: $\check{c}_n = \partial\check{c}_{n+1}$. Letting $D\check{\sigma}^n = \check{c}_{n+1}$, so that $\partial D\check{\sigma}^n = \check{c}_n$, and repeating for each generator $\check{\sigma}^n$ completes the induction.□

It is clear that this can be iterated. Let $\mathscr{K}^{(m)}$ be the mth barycentric subdivision of $\mathscr{K}$, and $\mathrm{Sd}^m : C_*(\mathscr{K}) \to C_*(\mathscr{K}^{(m)})$ the iteration of the chain mapping Sd. Define $\pi : \mathscr{K}^{(m)} \to \mathscr{K}$ to be an iteration of pseudo-identity maps

$$\mathscr{K}^{(m)} \to \mathscr{K}^{(m-1)} \to \cdots \to \mathscr{K}$$

Then $\pi \circ \mathrm{Sd}^m : C_*(\mathscr{K}) \to C_*(\mathscr{K})$ is the identity and $\mathrm{Sd}^m \circ \pi : C_*(\mathscr{K}^{(m)}) \to C_*(\mathscr{K}^{(m)})$ is chain homotopic to the identity.

As an application, let $\mathscr{K}$ be a finite simplicial complex; evidently, $C_n(\mathscr{K})$ and $C_n(\mathscr{K}^{(m)})$ are in general not isomorphic. However, we have

(6.4) THEOREM (Invariance of homology under barycentric subdivision). *For any finite simplicial complex $\mathscr{K}$, we have $\mathrm{Sd}_*^m : H_*(\mathscr{K}) \cong H_*(\mathscr{K}^{(m)})$, and $\pi_* : H_*(\mathscr{K}^{(m)}) \cong H_*(\mathscr{K})$ is the inverse.*

PROOF. Theorem (6.3) gives $\pi_* \circ \mathrm{Sd}_*^m = \mathrm{id}$ and $\mathrm{Sd}_*^m \circ \pi_* = \mathrm{id}$. □

7. Induced Homomorphism

Let $\mathscr{K}$ and $\mathscr{L}$ be two simplicial complexes. A simplicial map $\varphi : \mathscr{K} \to \mathscr{L}$ induces a homomorphism $\varphi_* : H_*(\mathscr{K}) \to H_*(\mathscr{L})$. We are now going to construct a unique homomorphism $f_* : H_*(\mathscr{K}) \to H_*(\mathscr{L})$ for each continuous (not necessarily simplicial!) map $f : K \to L$, in such a way that if f is simplicial, then f_* is the homomorphism induced by the simplicial map f. This will be a consequence of

(7.1) LEMMA. *Let $f : K \to L$ be continuous. For any two simplicial approximations $\varphi : \mathscr{K}^{(m)} \to \mathscr{L}$ and $\psi : \mathscr{K}^{(m+s)} \to \mathscr{L}$ of f, the homomorphisms $\varphi_* \operatorname{Sd}_*^m, \psi_* \operatorname{Sd}_*^{m+s} : H_*(\mathscr{K}) \to H_*(\mathscr{L})$ are identical.*

PROOF. Consider the diagram

$$\begin{tikzcd}
 & H_*(\mathscr{K}^{(m)}) \arrow[dd, "\operatorname{Sd}_*^s"] \arrow[dr, "\varphi_*"] & \\
H_*(\mathscr{K}) \arrow[ur, "\operatorname{Sd}_*^m"] \arrow[dr, "\operatorname{Sd}_*^{m+s}"'] & & H_*(\mathscr{L}) \\
 & H_*(\mathscr{K}^{(m+s)}) \arrow[ur, "\psi_*"'] &
\end{tikzcd}$$

The left triangle is commutative. For the right triangle, let $\pi : \mathscr{K}^{(m+s)} \to \mathscr{K}^{(m)}$ be a pseudo-identity map. Then $\varphi \circ \pi$ and ψ are both simplicial approximations to f, since

$$f(\operatorname{St} p') \subset f(\operatorname{St} \pi p') \subset \operatorname{St} \varphi\pi(p'),$$

so that $\varphi\pi$ and ψ are contiguous, and therefore $\psi_* = (\varphi\pi)_* = \varphi_* \circ \pi_*$; consequently, by (6.4), $\psi_* \operatorname{Sd}_*^s = \varphi_*$, and the proof is complete. □

On the basis of this result, we make

(7.2) DEFINITION. Let $f : K \to L$ be an arbitrary continuous map of simplicial complexes (i.e., of their underlying spaces), and let $\varphi : \mathscr{K}^{(m)} \to \mathscr{L}$ be any simplicial approximation to f The homomorphism $f_* : H_*(\mathscr{K}) \to H_*(\mathscr{L})$ is $\varphi_* \operatorname{Sd}_*^m : H_*(\mathscr{K}) \to H_*(\mathscr{L})$.

Observe that if $f : K \to L$ is itself a simplicial map, then the homomorphism $f_* : H_*(\mathscr{K}) \to H_*(\mathscr{L})$ given by this definition is precisely that induced by f itself.

This definition has the correct behavior under composition:

(7.3) THEOREM. *If $K \xrightarrow{f} L \xrightarrow{g} P$, then $g_* \circ f_* = (g \circ f)_*$; furthermore, $(\mathrm{id}_K)_* = \mathrm{id}_{H_*(\mathscr{K})}$.*

PROOF. Let $\psi : \mathscr{L}^{(s)} \to \mathscr{P}$ be a simplicial approximation to g, and $\varphi : \mathscr{K}^{(m)} \to \mathscr{L}^{(s)}$ a simplicial approximation to $f : K \to L$. With a pseudo-identity map $\pi : \mathscr{L}^{(s)} \to \mathscr{L}$, the map $\pi\varphi : \mathscr{K}^{(m)} \to \mathscr{L}$ is a simplicial approximation to $f : K \to L$. The homomorphism $g_* \circ f_*$ is therefore

$$\begin{aligned}\psi_* \mathrm{Sd}_*^s \circ \pi_* \circ \varphi_* \mathrm{Sd}_*^m &= \psi_* \varphi_* \mathrm{Sd}_*^m \qquad \text{(by Theorem (6.3))}\\ &= (\psi\varphi)_* \mathrm{Sd}_*^m,\end{aligned}$$

and $\psi\varphi : \mathscr{K}^{(m)} \to \mathscr{P}$ is easily seen to be a simplicial approximation to $g \circ f$. The second assertion is evident. □

We are now going to show that any two homotopic maps $f, g : P \to K$ induce the same homomorphism. For this purpose, we observe that if $\boldsymbol{I}$ is the unit interval and P is a polyhedron, then $P \times \boldsymbol{I}$ has a natural simplicial decomposition by taking for each $(p_0, \dots, p_n) \in \mathscr{P}$ the family of simplices

$$\{(p_0 \times 0, \dots, p_i \times 0, p_i \times 1, \dots, p_n \times 1) \mid i = 0, 1, \dots, n\}$$

together with all faces of such simplices. The desired result will be a consequence of

(7.4) LEMMA. *Let P be a polyhedron. If there are two continuous maps $\alpha \neq \beta : \boldsymbol{I} \to \boldsymbol{I}$ such that*

$$(\mathrm{id} \times \alpha)_* = (\mathrm{id} \times \beta)_* : H_*(\mathscr{P} \times \boldsymbol{I}) \to H_*(\mathscr{P} \times \boldsymbol{I}),$$

then for any polyhedron K, homotopic maps $f, g : P \to K$ induce the same homomorphisms $f_* = g_* : H_*(\mathscr{P}) \to H_*(\mathscr{K})$.

PROOF. For any $\tau \in \boldsymbol{I}$, let $k_\tau : \boldsymbol{I} \to \boldsymbol{I}$ be the constant map $k_\tau(\boldsymbol{I}) = \tau$. We first show that

$$(\mathrm{id} \times k_0)_* = (\mathrm{id} \times k_1)_* : H_*(\mathscr{P} \times \boldsymbol{I}) \to H_*(\mathscr{P} \times \boldsymbol{I});$$

for since $\alpha \neq \beta$, there is a $\tau \in \boldsymbol{I}$ with $\xi = \alpha(\tau) \neq \beta(\tau) = \eta$. Let $\mu : \boldsymbol{I} \to \boldsymbol{I}$ be an Urysohn function with $\mu(\xi) = 0$ and $\mu(\eta) = 1$. Then

$$\begin{aligned}\mathrm{id} \times k_0 &= (\mathrm{id} \times \mu) \circ (\mathrm{id} \times \alpha) \circ (\mathrm{id} \times k_\tau),\\ \mathrm{id} \times k_1 &= (\mathrm{id} \times \mu) \circ (\mathrm{id} \times \beta) \circ (\mathrm{id} \times k_\tau),\end{aligned}$$

so by passing to the induced homomorphisms in homology, the hypothesis of the lemma gives our preliminary result.

To prove the lemma, let $\varphi : f \simeq g : P \times \boldsymbol{I} \to P$, and let $j : P \to P \times \boldsymbol{I}$ be the embedding $j(p) = (p, 0)$. We have

$$\begin{aligned}f &= \varphi \circ (\mathrm{id} \times k_0) \circ j,\\ g &= \varphi \circ (\mathrm{id} \times k_1) \circ j.\end{aligned}$$

Passing to homology and using our preliminary result finishes the proof. □

(7.5) THEOREM. *Let P, K be finite polyhedra and $f, g : P \to K$ continuous. If $f \simeq g$, then $f_* = g_* : H_*(\mathscr{P}) \to H_*(\mathscr{K})$.*

PROOF. Being a finite polyhedron, $P \times \boldsymbol{I}$ has finitely generated homology, so there are at most countably many distinct endomorphisms $H_*(\mathscr{P} \times \boldsymbol{I}) \to H_*(\mathscr{P} \times \boldsymbol{I})$. However, there are uncountably many distinct continuous maps $\boldsymbol{I} \to \boldsymbol{I}$, so that $(\mathrm{id} \times \alpha)_* = (\mathrm{id} \times \beta)_*$ for some distinct $\alpha, \beta : \boldsymbol{I} \to \boldsymbol{I}$, and applying the lemma completes the proof. □

We are now in a position to show that the homology groups of a simplicial complex are invariant under homeomorphisms of the supports: although the homology groups are calculated using a specific simplicial subdivision, they are in fact functions only of the underlying space.

(7.6) THEOREM (Topological invariance of simplicial homology). *Let $\mathscr{P}$, $\mathscr{K}$ be simplicial complexes, and $|\mathscr{P}|$, $|\mathscr{K}|$ their underlying spaces. Let $h : |\mathscr{P}| \to |\mathscr{K}|$ be a homeomorphism. Then $h_* : H_*(\mathscr{P}) \cong H_*(\mathscr{K})$.*

PROOF. Let $g : |\mathscr{K}| \to |\mathscr{P}|$ be the inverse of h. Since $gh = \mathrm{id}_P$ and $hg = \mathrm{id}_K$, we find from (7.3) that $(gh)_* = \mathrm{id}_*$ and $(hg)_* = \mathrm{id}_*$ and then that $g_*h_* = \mathrm{id}$ and also $h_*g_* = \mathrm{id}$; since these last two conditions imply that h_* is an isomorphism and g_* its inverse, the proof is complete. □

8. Triangulated Spaces and Polytopes

To give the homology a broader scope, we define a generalization of the notion of a polyhedron that relaxes the restriction that the simplices be rectilinear: specifically, we consider homeomorphic images of polyhedra.

(8.1) DEFINITION. A homeomorphism h of a space X onto a polyhedron $(P, \mathscr{P})$ is called a *triangulation* of X. A space X together with a triangulation $h : X \to P$ is called a *polytope* (or a *triangulated space*) and is denoted by $(X; h, P)$.

Obviously, every polyhedron is a polytope. To give a simple example of a nonrectilinear polytope, let h be the central projection of the n-sphere $S^n \subset R^{n+1}$ onto the boundary of an $(n+1)$-simplex σ^{n+1} having the origin as barycenter; then $h : S^n \to \dot{\sigma}^{n+1}$ is a triangulation of S^n, and $(S^n; h, \dot{\sigma}^{n+1})$ is a polytope.

Let $(X; h, P)$ be a triangulation of the space X. Clearly, $h^{-1}(\sigma) \approx \sigma$ for each simplex $\sigma \in \mathscr{P}$; the family $\{h^{-1}(\sigma) \mid \sigma \in \mathscr{P}\}$ is called the *simplicial subdivision* of the polytope, the set $h^{-1}(\sigma)$ being called an *n-simplex* whenever $\dim \sigma = n$. The notions of vertex, subpolytope, star, and barycentric subdivision in $(X; h, P)$ are obtained as inverse images under h of the corresponding concepts in P: broadly speaking, we treat the sets $h^{-1}(\sigma) \subset X$ as

if they were rectilinear simplices. If $(X; h, P)$ and $(Y; g, K)$ are polytopes, a map $\varphi : X \to Y$ is called *simplicial* if $g\varphi h^{-1} : P \to K$ is simplicial; because h^{-1} is uniformly continuous, X has subdivisions of arbitrarily small mesh, and the simplicial approximation theorem carries over to polytopes.

If $(X; h, P)$ is a polytope, we define $H_*(X; h, P) = H_*(\mathscr{P})$; whenever $(X; g, K)$ is any other triangulation of X, we have $gh^{-1} : P \approx K$, so by (7.6), the group $H_*(X; h, P)$ is independent of the triangulation used, and will be denoted simply by $H_*(X)$. If $\varphi : X \to Y$ is a continuous map of the polytope $(X; h, P)$ into $(Y; g, K)$, we define the induced homomorphism $\varphi_* : H_*(X) \to H_*(Y)$ to be that induced by $g\varphi h^{-1} : P \to K$; it is trivial to verify that $(\psi\varphi)_* = \psi_*\varphi_*$ whenever $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$, so as in (7.6), homeomorphic polytopes have isomorphic homology. We remark that the converse is not true: a point and the unit interval have isomorphic homology, but they are not homeomorphic.

In the future, we will denote a polytope simply by X, whenever the triangulation used is clear from the context; we also write K for both a simplicial complex and its underlying space.

9. Relative Homology

The notion of homology in a simplicial complex K can be made relative to a subcomplex $L \subset K$: Using the factor groups $C_n(K)/C_n(L)$, a boundary operator $\widehat{\partial} : C_n(K)/C_n(L) \to C_{n-1}(K)/C_{n-1}(L)$ can be defined by $\widehat{\partial}[c_n + C_n(L)] = [\partial c_n + C_{n-1}(L)]$; the homology of the chain complex $\{C_n(K)/C_n(L); \widehat{\partial}\}$ is denoted by $H_*(K, L)$, and called the *homology of K mod L*, or the *homology of K relative to L*; and for any abelian group G, one gets the relative homology groups $H_n(K, L; G)$ over G by considering $C_n(K) \otimes G$ and its subgroup $C_n(L) \otimes G$.

The groups $H_*(K, L)$ can be expressed directly in terms of the chains in K: Let

$$Z_n(K, L) = \begin{cases} \{c \in C_n(K) \mid \partial c \in C_{n-1}(L)\}, & n > 0, \\ C_0(K) & n = 0; \end{cases}$$

this is called the group of *n-cycles of K mod L*. Let

$$\begin{aligned} B_n(K, L) &= B_n(K) + C_n(L) \\ &= \text{the subgroup of } C_n(K) \text{ generated by } B_n(K) \text{ and } C_n(L); \end{aligned}$$

this is called the group of *bounding n-cycles of K mod L*; the quotient group

$$H_n(K, L) = Z_n(K, L)/B_n(K, L)$$

is the nth *homology group of K mod L*. Note that if $L = \emptyset$, then $H_n(K, \emptyset) \equiv H_n(K)$.

EXAMPLE 1. Let $K = \{p,q\}$ be a complex consisting of two vertices, and let $L = \{q\}$. Then $H_0(p \cup q, q) = Z$ and $H_i(p \cup q, q) = 0$ for $i > 0$. The second assertion is obvious, since $C_i(p \cup q) = 0$ for $i > 0$. As for the first, we have $Z_0(p \cup q, q) = C_0(p) \oplus C_0(q)$ and $B_0(p \cup q, q) = C_0(q)$, and because $C_0(p) \cong Z$, the assertion follows. Similarly, $H_0(p \cup q, q; G) = G$ for any abelian group G.

Let K and P be two simplicial complexes, with subcomplexes $L \subset K$ and $Q \subset P$. A chain map $\tau : (C_n(K), C_n(L)) \to (C_n(P), C_n(Q))$ induces, by passing to the quotient, a chain map $\hat{\tau} : C_n(K)/C_n(L) \to C_n(P)/C_n(Q)$, and therefore a homomorphism $\tau_* : H_*(K,L) \to H_*(P,Q)$; expressed directly in terms of the n-cycles of K mod L, we have $\tau_*[c+B_n(K)+C_n(L)] = [\tau c+B_n(P)+C_n(Q)]$. In particular, every simplicial map $f : (K,L) \to (P,Q)$ induces a homomorphism $f_* : H_*(K,L) \to H_*(P,Q)$.

Let L be a subcomplex of K. One of the characteristic features of homology theory is a long exact sequence that relates the homologies of K, L and K mod L. To define this interrelation, let $i_* : H_n(L) \to H_n(K)$ be induced by the inclusion $i : L \hookrightarrow K$, and let $j_* : H_n(K) \to H_n(K,L)$ be induced by the inclusion $j : (K, \emptyset) \to (K,L)$; specifically,

$$j_*[c + B_n(K)] = [c + B_n(K) + C_n(L)];$$

and let $\partial_* : H_n(K,L) \to H_{n-1}(L)$ be the homomorphism

$$\partial_*[c + B_n(K) + C_n(L)] = [\partial c + B_{n-1}(L)].$$

It is straightforward to verify

(9.1) THEOREM (Homology sequence of a pair). *Let L be a subcomplex of K. Then the sequence*

$$\cdots \to H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{j_*} H_n(K,L) \xrightarrow{\partial_*} H_{n-1}(L) \xrightarrow{i_*} \cdots$$
$$\cdots \xrightarrow{i_*} H_0(K) \xrightarrow{j_*} H_0(K,L) \to 0$$

is exact, which means that the image of each homomorphism is precisely the kernel of the next homomorphism. Moreover, a simplicial map $f : (K,L) \to (P,Q)$ induces a homomorphism of the (K,L) exact sequence into that of (P,Q), which means that in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(L) \arrow[r, "i_*"] \arrow[d, "(f|L)_*"] & H_n(K) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(K,L) \arrow[r, "\partial_*"] \arrow[d, "f_*"] & H_{n-1}(L) \arrow[r] \arrow[d, "(f|L)_*"] & \cdots \\
\cdots \arrow[r] & H_n(Q) \arrow[r, "i_*"'] & H_n(P) \arrow[r, "j_*"'] & H_n(P,Q) \arrow[r, "\partial_*"'] & H_{n-1}(Q) \arrow[r] & \cdots
\end{tikzcd}$$

each square is commutative. □

By regarding the exact sequence of (9.1) as arising from a chain complex and a subcomplex, we are led to a generalization: Starting with subcomplexes $M \subset L \subset K$, we have the chain complex $\{C_n(K)/C_n(M)\}$ and subcomplex $\{C_n(L)/C_n(M)\}$; the quotient complex is $\{C_n(K)/C_n(L)\}$, so we obtain, more generally,

(9.2) COROLLARY (Homology sequence of a triple). *Let $M \subset L \subset K$ be simplicial complexes. Then there is an exact sequence*

$$\cdots \to H_n(L,M) \to H_n(K,M) \to H_n(K,L)$$
$$\xrightarrow{\hat{\partial}} H_{n-1}(L,M) \to \cdots \to H_0(K,L) \to 0,$$

where all unlabeled homomorphisms are induced by inclusion maps, and $\hat{\partial}$ is the composition $H_n(K,L) \xrightarrow{\partial_*} H_{n-1}(L) \xrightarrow{j_*} H_{n-1}(L,M)$. Moreover, a simplicial map $f : (K,L,M) \to (P,Q,R)$ induces a homomorphism of the exact sequence of the triple (K,L,M) into that of (P,Q,R). □

It is clear that with $M = \emptyset$, the exact sequence of (9.2) is precisely that of (9.1).

EXAMPLE 2. Let K be a simplicial complex, and p a vertex of K. Then

$$H_i(K,p) = H_i(K) \qquad \text{for all } i > 0,$$
$$H_0(K,p) \cong \operatorname{Ker}[H_0(K) \xrightarrow{\pi_*} H_0(p)],$$

where $\pi : K \to p$ is the constant map. For consider the exact sequence of the couple (K,p); if $i \geq 2$, we have $H_i(p) \to H_i(K) \to H_i(K,p) \xrightarrow{\partial_*} H_{i-1}(p)$ exact with the two end terms zero, so $H_i(K) \cong H_i(K,p)$ for $i \geq 2$. The terminal part of the exact sequence reads

$$0 \to H_1(K) \to H_1(K,p) \xrightarrow{\partial_*} H_0(p) \xrightarrow{i_*} H_0(K) \to H_0(K,p) \to 0.$$

Now observe that because the composition $p \xrightarrow{i} K \xrightarrow{\pi} p$ is the identity map, we have $\pi_* i_* = \mathrm{id}$, so $i_* : H_0(p) \to H_0(K)$ is monic and $H_0(K) = \operatorname{Im} i_* \oplus \operatorname{Ker} \pi_*$. Using this information in the exact sequence, we find that ∂_* is the zero homomorphism, so that $H_1(K) = H_1(K,p)$, and $H_0(K,p) = H_0(K)/\operatorname{Im} i_* \cong \operatorname{Ker} \pi_*$, as asserted.

In particular, if K is connected, then $H_0(K,p) = 0$. The exact sequence of the triple (K,L,p), p a vertex of L, is called the *reduced homology sequence* of (K,L); we remark that in calculations with exact sequences using connected subcomplexes with vanishing homology, it is more convenient to use the triple (K,L,p) because the cases $i = 0,1$ do not require special consideration.

Let A, B be any two subcomplexes of a simplicial complex K. The inclusion map

$$e : (A, A \cap B) \to (A \cup B, B)$$

is called an *excision*: the pair $(A, A \cap B)$ is formed from $(A \cup B, B)$ by removing from $A \cup B$ all of B that is not in A. The second basic feature of simplicial homology is the following result that the reader can easily verify:

(9.3) THEOREM. *Let A, B be any two subcomplexes of a simplicial complex K, and $e : (A, A \cap B) \to (A \cup B, B)$ the excision. Then*

$$e_* : H_n(A, A \cap B) \cong H_n(A \cup B, B)$$

for all $n \geq 0$.

PROOF (sketch). Consider the homomorphism

$$\mu : H_n(A \cup B, B) \to H_n(A, A \cap B)$$

induced by the chain map generated by $\mu(\sigma) = \sigma$ if $\sigma \in A$, $\mu(\sigma) = 0$ otherwise, and investigate $e_* \circ \mu$ and $\mu \circ e_*$. □

EXAMPLE 3. Let K be a simplicial complex. The *suspension* ΣK of K is the union of two distinct cones K_+, K_- over K with $K_+ \cap K_- = K$. For example, the suspension of the two-point complex $S^0 = \{p, q\}$ is a polyhedron homeomorphic to S^1; the suspension of S^1 is a polytope homeomorphic to S^2, and inductively, $\Sigma S^n \approx S^{n+1}$ To determine the homology of ΣK in terms of K, we consider the exact sequences of the triples $(\Sigma K, K_-, p)$ and $(K_+, K_+ \cap K_-, p)$, where p is a vertex of K. For any $n \geq 0$, we have

$$\begin{tikzcd}
H_n(K_-,p) \arrow[d] & \\
H_n(K_+ \cup K_-, p) \arrow[d, "j_*"'] & H_n(K_+, p) \arrow[d] \\
H_n(K_+ \cup K_-, K_-) \arrow[d] & H_n(K_+, K_+ \cap K_-) \arrow[l, "e_*"'] \arrow[d, "\partial_*"] \\
H_{n-1}(K_-,p) & H_{n-1}(K_+ \cap K_-, p) \arrow[d] \\
 & H_{n-1}(K_+, p)
\end{tikzcd}$$

where e_* is induced by excision. Since $H_i(K_+, p) = 0 = H_i(K_-, p)$ for all $i \geq 0$ by Example 2, the ∂_* and j_* are isomorphisms, and using (9.3), we

conclude that

$$\partial_* e_*^{-1} j_* : H_n(\Sigma K, p) \equiv H_n(K_+ \cup K_-, p) \cong H_{n-1}(K_+ \cap K_-, p)$$
$$\equiv H_{n-1}(K, p)$$

for all $n \geq 1$; clearly, $H_0(\Sigma K, p) = 0$ because ΣK is connected.

As an application, we find

$$H_n(S^k, p) \equiv H_n(\Sigma S^{k-1}, p) \cong H_{n-1}(S^{k-1}, p)$$

for $n \leq k$, so by iteration,

$$H_n(S^k, p) \cong H_0(S^{k-n}, p).$$

If $n < k$, then $H_0(S^{k-n}, p) = 0$ because S^{k-n} is connected; if $n = k$, then $H_0(S^0, p) = H_0(p \cup q, p) = Z$. Thus, for any $n \geq 1$, the homology of S^n is

$$H_i(S^n) = \begin{cases} Z & \text{for } i = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

With this result, it is easy to show that the boundary of an n-simplex, $n \geq 1$, is never a retract of the simplex, a proposition we have seen to be equivalent to Brouwer's fixed point theorem. For assume that $\partial\sigma^n$ is a retract of σ^n, and let $r : \sigma^n \to \partial\sigma^n$ be such a retraction. With $i : \partial\sigma^n \to \sigma^n$ we have $r \circ i = \text{id} : \partial\sigma^n \to \partial\sigma^n$, so in homology, $r_* i_* = \text{id}$. But this is impossible because $i_* : H_{n-1}(\partial\sigma^n) \to H_{n-1}(\sigma^n)$ is the zero homomorphism whereas $\text{id} : H_{n-1}(\partial\sigma^n) \to H_{n-1}(\partial\sigma^n)$ is the isomorphism $Z \to Z$. This contradiction completes the proof.

10. Miscellaneous Results and Examples

(10.1) Let K be a simplicial complex with $H_{n+1}(K) = 0$. Let K_1, K_2 be subcomplexes such that $K = K_1 \cup K_2$. Show: If a cycle z_n in $K_1 \cap K_2$ bounds in K_1 and also in K_2, then it bounds in $K_1 \cap K_2$.

(10.2) Let K be a simplicial complex. Two n-chains c_1^n, c_2^n in K are called *homologous*, written $c_1^n \sim c_2^n$, if $c_1^n - c_2^n = \partial c^{n+1}$ for some $(n+1)$-chain c^{n+1}. Let L be a subcomplex of K. Then: (i) a chain $c^n = \sum \alpha_i \sigma_i^n$ is *in* L (written $c^n \subset L$) if $\alpha_i = 0$ for each $\sigma_i^n \in K - L$; (ii) c^n is a *cycle mod* L if $\partial c^n \subset L$; (iii) c^n *bounds mod* L if $c^n \sim \tilde{c}^n$, where $\tilde{c}^n \subset L$.

(a) Prove the following statements:

1° The boundary of a cycle of K mod L bounds in L if and only if the cycle is homologous mod L to a cycle in K.

2° A cycle of L is homologous to zero in K if and only if it is the boundary of a cycle of K mod L.

3° A cycle of K is homologous to a cycle of L if and only if it is homologous to zero mod L.

(b) Show that assertions 1°–3° are equivalent to the exactness of the homology sequence of the pair (K, L).

(10.3) A connected simplicial complex K is called *unicoherent* if in each representation $K = K_1 \cup K_2$ as a union of two connected subcomplexes, the intersection $K_1 \cap K_2$ is also connected. Show: If K is a connected complex and $H_1(K) = 0$, then K is unicoherent.
[Use (10.1).]

(10.4) By a 2-*dimensional pseudomanifold* M is meant a 2-complex with the properties: (a) each 1-simplex is the face of exactly two 2-simplices and (b) M cannot be represented as the union of two nonempty subcomplexes having no 1-simplex in common. Let M be a 2-pseudomanifold, and let α_i denote the number of i-simplices, $i = 0, 1, 2$. Show: $\alpha_0 \geq (7 + \sqrt{49 - 24\chi})/2$, where $\chi = \alpha_0 - \alpha_1 + \alpha_2$.
[Because of property (a), we have $3\alpha_2 = 2\alpha_1$; moreover, $\alpha_1 \leq \binom{\alpha_0}{2}$.]

(10.5) Let K be a simplicial complex and L a subcomplex. Let N be the number of components of K that contain no elements of L. Show:
(a) $H_0(K, L)$ is the direct sum of N copies of Z.
(b) If $\dim K \leq n+1$ and K^n is the n-skeleton of K, then $H_{n+1}(K, K^n) \cong C_{n+1}(K)$.

(10.6) Prove: $\dim K \leq n$ if and only if $H_{n+1}(K, L) = 0$ for every subcomplex $L \subset K$.
[Take $L = K^n$.]

(10.7) Let K be a simplicial complex and $*$ a vertex of K. Show:

$$H_n(K, *) \cong H_n(K), \qquad n > 0,$$
$$\operatorname{rank} H_0(K, *) = -1 + \operatorname{rank} H_0(K), \qquad n = 0.$$

[Use the exact sequence for $(K, *)$.]

(10.8) Let K be a simplicial complex, L a subcomplex, and CL the cone over L. Show:

$$H_*(K, L) \cong H_*(K \cup CL, *).$$

[Use exactness to show that $H_*(K \cup CL, *) \cong H_*(K \cup CL, CL)$ and excise $CL - K$.]

(10.9) Let K_1, K_2 be subcomplexes of K, and P_1, P_2 subcomplexes of P. Assume

$$\tau : (C(K); C(K_1), C(K_2)) \to (C(P); C(P_1), C(P_2))$$

is a chain map such that both $\tau_* : H_*(K_1) \cong H_*(P_1)$ and $\tau_* : H_*(K_2) \cong H_*(P_2)$. Prove:

$$\tau_* : H_*(K_1 \cap K_2) \cong H_*(P_1 \cap P_2)$$

if and only if

$$\tau_* : H_*(K_1 \cup K_2) \cong H_*(P_1 \cup P_2).$$

[Use the exact sequences of $(K_1, K_1 \cap K_2)$ and $(P_1, P_1 \cap P_2)$, then the 5-lemma, excision, and the exact sequences of $(K_1 \cup K_2, K_2)$ and $(P_1 \cup P_2, P_2)$.]

(10.10) Let $K = K_1 \cup K_2$ with $K_1 \subset P_1$, $K_2 \subset P_2$ and $K = P_1 \cup P_2$. Assume that the inclusions $K_1 \hookrightarrow P_1$ and $K_2 \hookrightarrow P_2$ both induce isomorphisms in homology. Show:

$$i_* : H_*(K_1 \cap K_2) \cong H_*(P_1 \cap P_2).$$

(10.11) Let $K = K_1 \cup K_2$ and $L \subset K_1 \cap K_2$. Assume that $H_*(K_1, L) = H_*(K_2, L) = 0$. Show:

$$H_n(K, L) \cong H_{n-1}(K_1 \cap K_2, L) \quad \text{for all } n.$$

[Use the $(K_1, K_1 \cap K_2, L)$ exact sequence, excision, and the $(K_1 \cup K_2, K_2, L)$ exact sequence.]

11. Notes and Comments

Simplicial complexes

The study of 1- and 2-dimensional simplicial complexes appears already in the work of Euler and Cauchy. The early treatment of simplicial complexes of higher dimensions was given by J.B. Listing (*Der Census räumliche Complexe*, Göttingen, 1862); working on the topological ideas which had been suggested to him by his teacher C.F. Gauss, Listing was, in fact, the first to use the term "topology" in his *Vorstudien zur Topologie*, 1847.

Triangulations of differentiable manifolds and barycentric subdivisions were treated for the first time by Poincaré [1895] and used to establish duality results for manifolds.

Simplicial complexes can be generalized in various directions. A description of infinite simplicial complexes is given later on, in §13. By relaxing all "linearity" conditions, one arrives at the notion of CW-complex due to J.H.C. Whitehead (see, e.g., Maunder's book [1970]).

Simplicial maps and the simplicial approximation technique were developed by Brouwer. The version of the simplicial approximation theorem given in the text was first proved by J.W. Alexander. There exists an important relative version of this theorem, found independently by Jaworowski [1964] and Zeeman [1964]; a detailed proof of their result can be found in Maunder's book [1970]. We remark that the Jaworowski–Zeeman theorem can be used to justify the proof of the Brouwer fixed point theorem proposed by Hirsch [1963] (see the above book of Maunder, and Joshi [1999]) and also to give an elementary proof of the nonexistence of a continuous nonzero tangent field on S^{2n} (Jaworowski [1964]).

The fundamental concepts of the nerve of a covering and of the standard map into the nerve (which appear in Theorem (4.4)) were introduced by Alexandroff [1927] and Kuratowski [1933], respectively. The statement and proof of Theorem (4.4) were given (in the context of metric spaces) by Kuratowski [1933]; this proof made use for the first time of the technique of partitions of unity.

Simplicial homology

The simplicial homology theory presented in this chapter is the result of the work and influence of many mathematicians. Influenced by discussions and correspondence with B. Riemann, E. Betti published in 1871 a memoir ([1])

([1]) For details see A. Weil, *Riemann, Betti and the birth of topology*. Arch. History Exact Sci. 20 (1979), 91–96. It should be remarked that Riemann gained some background in topology from Listing, who was one of his professors in Göttingen between 1849 and 1851.

containing what is now called the "Betti numbers"; these were so named by H. Poincaré who was inspired to study topology through Betti's work on the subject. The concept of the (integral) homology group of a polyhedron, expressed in terms of numerical invariants (Betti numbers and torsion coefficients) was introduced by Poincaré [1895], [1900]; these invariants suffice for the classification of the 2-dimensional manifolds. The proof of the topological invariance of Betti numbers for polyhedra was first given by J.W. Alexander by applying the simplicial approximation technique of Brouwer.

Moscow, 1935. *Top row*: E. Čech, H. Whitney, K. Zarankiewicz, A. Tucker, S. Lefschetz, H. Freudenthal, F. Frankl, J. Nielsen, K. Borsuk, D. Sintsoff, L. Tumarkin, M. Nikolaenko, V. Stepanoff, E. van Kampen, A. Tychonoff. *Bottom row*: K. Kuratowski, J. Schauder, S. Cohn-Vossen, P. Heegard, J. Różańska, J.W. Alexander, H. Hopf, P. Alexandroff, P. Soloveff.

The introduction of group-theoretic methods circa 1925 led to the machinery of chain complexes, homology groups with various coefficients (which cannot be described by numerical invariants alone), and to cohomology. In the 1920s, J.W. Alexander, S. Lefschetz and H. Hopf developed simplicial homology theory attaining remarkable results, some of which are presented in this chapter. The term "homology group" appears for the first time in Vietoris [1927]. The reader interested in historical details is referred to the book of Dieudonné [1989] and also to J.-C. Pont's thesis, *La topologie algébrique des origines à Poincaré*, Presses Univ. France, 1974.

In connection with the proof of Theorem (7.5) see Schmidt [1974].

§9. The Lefschetz–Hopf Theorem and Brouwer Degree

Some topological properties of polyhedra can be determined directly from their homology: for example, the condition $H_0(P) = Z$ implies (in fact, is equivalent to) the connectedness of the polyhedron P In this paragraph we will obtain some algebraic conditions on a self-map of P that imply the existence of a fixed point.

We first prove the Lefschetz–Hopf fixed point theorem: this involves an alternating sum of the traces of the endomorphisms $f_n : H_n(P;Q) \to H_n(P;Q)$; the relevant facts about traces are presented separately before proceeding to the proof. After that, we show that the alternating sum of the number of the nonzero eigenvalues (counted with their multiplicities) of those endomorphisms also gives information about the map, this time on the existence of periodic points. Some simple applications of these two results are developed.

The use of homology is further illustrated by defining the degree of maps $S^n \to S^n$ and giving some of its uses, notably to determine the fixed point properties of such maps and also their behavior at antipodal pairs of points. We establish Hopf's theorem that the (algebraically defined) degree does, in fact, completely characterize the homotopy behavior of the map, and within this algebraic setting we prove Borsuk's theorem once again.

1. Algebraic Preliminaries

Let C denote the field of complex numbers. Given an $n \times n$ matrix $A = \|a_{ij}\|$ with entries from C, the *trace* $\mathrm{tr}(A)$ of A is the sum $\sum a_{ii}$ of its diagonal entries. We collect here various properties of the trace that will be needed.

Denote the determinant of any square matrix C by $|C|$. Letting λ denote a variable in C, the polynomial $p(\lambda) = |A - \lambda I|$ is called the *characteristic polynomial* of A. Clearly,

$$p(\lambda) = (-1)^n \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n$$

is a polynomial of the nth degree, and the constant term a_n equals $|A|$; more importantly, noting that the terms involving λ^n and λ^{n-1} can arise only from the product $(a_{11} - \lambda) \ldots (a_{nn} - \lambda)$ in the expansion of the determinant, we find that the coefficient of λ^{n-1} is $(-1)^{n-1}\,\mathrm{tr}(A)$. This immediately leads to

(1.1) PROPOSITION. *Let A be an $n \times n$ matrix over C. If B is any non-singular $n \times n$ matrix over C, then* $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$.

PROOF. Since the determinant of a product of $n \times n$ matrices is the product

of their determinants, we have

$$\begin{aligned}|BAB^{-1} - \lambda I| &= |B(A - \lambda I)B^{-1}| = |B| \cdot |A - \lambda I| \ \ |B^{-1}| \\ &= |A - \lambda I|\end{aligned}$$

so A and BAB^{-1} have the same characteristic polynomials, and therefore, in particular, the same trace. □

The roots of the characteristic polynomial of A are called the *eigenvalues* of A; because C is algebraically closed, every $n \times n$ matrix over C has exactly n eigenvalues, not necessarily all distinct and nonzero; however

(1.2) PROPOSITION. *Let $\lambda_1, \dots, \lambda_n$ be the n eigenvalues of the $n \times n$ matrix A. Then $\operatorname{tr}(A^s) = \sum_{j=1}^{n} \lambda_j^s$ for every integer $s \geq 1$.*

PROOF. Consider first the case $s = 1$. Writing

$$|A - \lambda I| = p(\lambda) = \prod_{j=1}^{n} (\lambda_j - \lambda)$$

shows that

$$p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \Big[\sum \lambda_j\Big] \lambda^{n-1} + \cdots;$$

since we know that the coefficient of λ^{n-1} is $(-1)^{n-1} \operatorname{tr}(A)$, the result for $s = 1$ is proved. Now let $s \geq 1$ be given; recall that if $\omega = e^{2\pi i/s}$ is an sth root of unity, then $a^s - \lambda^s = \prod_{j=0}^{s-1} (a - \omega^j \lambda)$ for any complex number a; numerical identities involving multiplication and addition being valid also for commuting matrices, we have

$$A^s - \lambda^s I = (A - \lambda I)(A - \omega\lambda I) \dots (A - \omega^{s-1}\lambda I),$$

so passing to determinants gives

$$\begin{aligned}|A^s - \lambda^s I| &= \prod_{j=0}^{s-1} |A - \omega^j \lambda I| \\ &= \prod_{j=1}^{n} (\lambda_j - \lambda) \cdot \prod_{j=1}^{n} (\lambda_j - \omega\lambda) \cdot \dots \ \prod_{j=1}^{n} (\lambda_j - \omega^{s-1}\lambda).\end{aligned}$$

Collecting terms and using the numerical identity repeatedly gives

$$|A^s - \lambda^s I| = \prod_{j=1}^{n} (\lambda_j^s - \lambda^s),$$

and hence, since λ^s is arbitrary, $|A^s - \mu I| = \prod_{j=1}^{n} (\lambda_j^s - \mu)$; the λ_j^s are therefore the eigenvalues of A^s, and as in the case $s = 1$, the proof is complete. □

In what follows we shall be dealing for the most part with vector spaces over the rationals Q, so we give here the facts and terminology we will use.

Let E be a finite-dimensional vector space over Q, and $\varphi : E \to E$ a linear transformation. The matrix $A = \|a_i^j\|$ of φ relative to a basis $\{u_1, \ldots, u_n\}$ in E is obtained by writing

$$\varphi(u_i) = \sum_{j=1}^{n} a_i^j u_j, \qquad i = 1, \ldots, n, \ a_i^j \in Q;$$

we define the *trace* $\operatorname{tr}(\varphi)$ of the endomorphism φ to be $\operatorname{tr}(A)$ and the eigenvalues of φ to be the eigenvalues of A. We point out explicitly that in talking about eigenvalues, we regard matrices as being over C, so that every $n \times n$ matrix over Q will have exactly n eigenvalues, some possibly complex.

(1.3) PROPOSITION. $\operatorname{tr}(\varphi)$, *and the eigenvalues of* φ, *depend only on* φ *and not on the particular basis* $\{u_1, \ldots, u_n\}$ *that is used.*

PROOF. If $\{v_1, \ldots, v_n\}$ is any other basis, we have $v_i = \sum_{j=1}^{n} c_i^j u_j$, $i = 1, \ldots, n$, where $C = \|c_i^j\|$ is a nonsingular matrix over Q; the matrix of φ relative to the basis $\{v_1, \ldots, v_n\}$ is easily calculated to be CAC^{-1}, so by (1.1), this will have the same characteristic polynomial as A, and in particular, the same trace as A. □

We remark that if $\varphi : E \to E$ has the matrix A relative to some basis, then the k-fold iterate $\varphi^k : E \to E$ will have the matrix A^k relative to that basis.

The following propositions express the two basic properties of the trace of an endomorphism:

(1.4) THEOREM (Commutativity). *If* $f : E' \to E''$ *and* $g : E'' \to E'$ *are linear maps of finite-dimensional vector spaces, then* $\operatorname{tr}(gf) = \operatorname{tr}(fg)$.

PROOF. Let $\{u_1, \ldots, u_n\}$ be a basis in E' and $\{v_1, \ldots, v_m\}$ be a basis in E'',

$$f(u_i) = \sum_{j=1}^{m} a_i^j v_j, \quad g(v_j) = \sum_{k=1}^{n} b_j^k u_k.$$

Then we have

$$gf(u_i) = \sum_{j=1}^{m} a_i^j g(v_j) = \sum_{k=1}^{n} \Big(\sum_{j=1}^{m} a_i^j b_j^k \Big) u_k,$$

$$fg(v_j) = \sum_{k=1}^{n} b_j^k f(u_k) = \sum_{l=1}^{m} \Big(\sum_{k=1}^{n} b_j^k a_k^l \Big) v_l.$$

and therefore

$$\operatorname{tr}(gf) = \sum_{k=1}^{n}\Big(\sum_{j=1}^{m} a^j_k b^k_j\Big) = \sum_{j=1}^{m}\Big(\sum_{k=1}^{n} b^k_j a^j_k\Big) = \operatorname{tr}(fg). \qquad \square$$

(1.5) THEOREM (Additivity). *Given a commutative diagram of finite-dimensional vector spaces with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r] \arrow[d, "\varphi'"'] & E \arrow[r] \arrow[d, "\varphi"'] & E'' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\ 0 \arrow[r] & E' \arrow[r] & E \arrow[r] & E'' \arrow[r] & 0 \end{tikzcd}$$

then

$$\operatorname{tr}(\varphi) = \operatorname{tr}(\varphi') + \operatorname{tr}(\varphi'').$$

PROOF. We may assume without loss of generality that $E' \subset E$ and $E'' = E/E'$. Let $\pi : E \to E''$ be the projection, let $\{u_1, \ldots, u_r\}$ be a basis of E', and let $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_n\}$ be its extension to a basis of E; we note that $\{\pi(u_{r+1}), \ldots, \pi(u_n)\}$ is a basis for E'' Let A' be the $r \times r$ matrix of φ' with respect to $\{u_1, \ldots, u_r\}$, and A'' the $(n-r) \times (n-r)$ matrix of φ'' with respect to $\{\pi(u_{r+1}), \ldots, \pi(u_m)\}$; then the matrix A of φ with respect to $\{u_1, \ldots, u_n\}$ is easily seen to be of the form $A = \begin{bmatrix} A' & 0 \\ B & A'' \end{bmatrix}$, and the conclusion follows. $\square$

2. The Lefschetz–Hopf Fixed Point Theorem

The Lefschetz–Hopf theorem, which we will establish in this section, includes many of the known facts about the fixed points for maps of polyhedra.

Let K be a finite simplicial complex. Taking chains over a field $\mathcal{F}$ and a chain transformation τ of $C_*(K; \mathcal{F})$ into itself, we recall that each $C_n(K; \mathcal{F})$ is a vector space over $\mathcal{F}$, each $\tau_n : C_n(K; \mathcal{F}) \to C_n(K; \mathcal{F})$ is a linear transformation, and each induced $\tau_{*n} : H_n(K; \mathcal{F}) \to H_n(K; \mathcal{F})$ is a linear transformation of a vector space. The following relation between the traces $\operatorname{tr}(\tau_i)$ and $\operatorname{tr}(\tau_{*i})$ is of fundamental importance:

(2.1) THEOREM (Hopf trace theorem). *Let* $\dim K = n$, *and let*

$$\tau_* : C_*(K; \mathcal{F}) \to C_*(K; \mathcal{F})$$

be any chain transformation. Then

$$\sum_{i=0}^{n} (-1)^i \operatorname{tr}(\tau_i) = \sum_{i=0}^{n} (-1)^i \operatorname{tr}(\tau_{*i}).$$

PROOF. We use repeatedly Theorem (1.5) on additivity of traces. Using the commutative diagram in (1.5) with the exact row

$$0 \to Z_r(K;\mathcal{F}) \xrightarrow{i} C_r(K;\mathcal{F}) \xrightarrow{\partial} B_{r-1}(K;\mathcal{F}) \to 0$$

and then the commutative diagram with exact row

$$0 \to B_r(K;\mathcal{F}) \xrightarrow{i} Z_r(K;\mathcal{F}) \xrightarrow{\beta} H_r(K;\mathcal{F}) \to 0$$

we get, for each $0 \le r \le n$,

$$\begin{aligned} \operatorname{tr}(\tau_r, C_r) &= \operatorname{tr}(\tau_r, Z_r) + \operatorname{tr}(\tau_{r-1}, B_{r-1}) \\ &= \operatorname{tr}(\tau_{*r}, H_r) + \operatorname{tr}(\tau_r, B_r) + \operatorname{tr}(\tau_{r-1}, B_{r-1}). \end{aligned}$$

Because $B_n = 0 = B_{-1}$, this gives

$$\sum_{i=0}^{n}(-1)^i \operatorname{tr}(\tau_i, C_i) = \sum_{i=0}^{n}(-1)^i \operatorname{tr}(\tau_{*i}, H_i),$$

and the proof is complete. □

Now let K be a finite polyhedron, and let $f : K \to K$ be a map. If we use the rationals Q as coefficients, then each induced homomorphism $f_{*n} : H_n(K;Q) \to H_n(K;Q)$, being an endomorphism of a finite-dimensional vector space, has a trace $\operatorname{tr}(f_{*n})$.

(2.2) DEFINITION. Let K be a finite polyhedron, $\dim K \le n$. The *Lefschetz number* $\lambda(f)$ of a map $f : K \to K$ is

$$\lambda(f) = \sum_{i=0}^{n}(-1)^i \operatorname{tr}[f_{*i}, H_i(K;Q)].$$

We first determine the nature of this number.

(2.3) THEOREM. *Let K be a finite polyhedron and $f : K \to K$ continuous. Then $\lambda(f)$ depends only on the homotopy class of f Moreover, $\lambda(f)$ is always an integer, and does not change if Q is replaced by any other field of characteristic 0, such as R or C.*

PROOF. The dependence on the homotopy class of f is immediate because homotopic maps induce identical homomorphisms. To see that $\lambda(f)$ is an integer, we will use the Hopf trace theorem. Letting $f_* = \varphi_* \circ \mathrm{Sd}_*^m$, where $\varphi : K^{(m)} \to K$ is a simplicial approximation to f, we find from (2.1) that

$$\lambda(f) = \sum_{i}^{n}(-1)^i \operatorname{tr}(\varphi \,\mathrm{Sd}^m, C_i(K;Q)).$$

Working now in any $C_r(K; Q)$ and using the basis $\{\sigma_i\}$ of oriented r-simplices of K, we have

$$\begin{aligned} \mathrm{Sd}^m \sigma_i &= \sum b_{ij}\check{\sigma}_j, \quad \text{all } b_{ij} = 0, \pm 1, \ \check{\sigma}_j \in K^{(m)}, \\ \varphi(\check{\sigma}_i) &= \sum c_{jk}\sigma_k, \quad \text{all } c_{jk} = 0 \text{ except possibly one, which is } \pm 1, \end{aligned}$$

so that

$$\varphi\,\mathrm{Sd}^m(\sigma_i) = \sum b_{ij}c_{jk}\sigma_k$$

is the expansion of the linear transformation $\varphi\,\mathrm{Sd}^m$ in terms of this basis. Being a matrix of integers, it has an integral trace, so each $\mathrm{tr}(\varphi\,\mathrm{Sd}^m, C_i)$ is an integer, and therefore $\lambda(f)$ is an integer. Observe that the expression of $\varphi\,\mathrm{Sd}^m$ in terms of the basis $\{\sigma_i\}$ remains the same regardless of the field $\mathcal{F}$ of coefficients used, and if the field has characteristic 0, then $\mathrm{tr}(\varphi\,\mathrm{Sd}^m, C_i)$ will also remain the same. This completes the proof. □

We now prove the main theorem of this section:

(2.4) THEOREM (Lefschetz–Hopf fixed point theorem). *Let K be a finite polyhedron, and let $f : K \to K$ be a map. If $\lambda(f) \neq 0$, then f has a fixed point.*

PROOF. Assume that f has no fixed point. Since K is compact, there is an $\varepsilon > 0$ such that $d(f(x), x) \geq \varepsilon$ for all $x \in K$. Using repeated barycentric subdivision, we start the proof by taking K with a fixed triangulation of mesh $< \varepsilon/3$.

Let $\varphi : K^{(m)} \to K$ be a simplicial approximation to f. By the Hopf trace theorem, it is enough to calculate the trace of $\varphi\,\mathrm{Sd}^m : C_r(K; Q) \to C_r(K; Q)$ for each r, and for this purpose, we take each $C_r(K, Q)$ with the basis $\{\sigma_i^r\}$ of all oriented r-simplices of K. Expressing $\varphi\,\mathrm{Sd}^m$ in terms of this basis we have

$$\mathrm{Sd}^m \sigma_i^r = \sum a_{ij}\tau_j^r, \qquad \tau_j^r \in K^{(m)}, \ \tau_j^r \subset \sigma_i^r,$$

and therefore

$$\varphi\mathrm{Sd}^m \sigma_i^r = \sum a_{ij}\varphi(\tau_j^r) = \sum b_{ij}\sigma_j^r.$$

In the last equation, no $\varphi(\tau_j^r) = \pm\sigma_i^r$: for if v is a vertex of any $\tau_j^r \subset \sigma_i^r$, then $f(v) \in f[\mathrm{St}(v)] \subset \mathrm{St}\,\varphi(v)$, so that $d(f(v), \varphi(v)) < \varepsilon/3$, and therefore

$$d(v, \varphi(v)) \geq d(v, f(v)) - d(f(v), \varphi(v)) \geq 2\varepsilon/3;$$

so if $\varphi(v)$ were also to belong to σ_i^r, we would have $\delta(\sigma_i^r) \geq 2\varepsilon/3$, contradicting that the mesh of the triangulation is $< \varepsilon/3$.

Thus, $\mathrm{tr}(\varphi\mathrm{Sd}^m, C_r(K; Q)) = 0$ for each r, and by the Hopf trace theorem, $\lambda(f) = 0$. This completes the proof. □

One immediate consequence is a broad generalization of Brouwer's theorem: A nullhomotopic map of any connected polyhedron always has a fixed point. For $f_{*i} : H_i(K;Q) \to H_i(K;Q)$ is the zero endomorphism for all $i > 0$ and the identity automorphism of the 1-dimensional vector space $H_0(K;Q)$, so $\lambda(f) = 1$, and f has a fixed point. More applications will be given later.

We remark that the converse of (2.4) is not true in general: for example, the identity map id : $S^1 \to S^1$ leaves every point fixed, and as the reader will easily verify, $\lambda(\mathrm{id}) = 0$.

3. The Euler Number of a Map. Periodic Points

Given $f : K \to K$, a point $x \in K$ is called a *periodic point* of f if $f^n(x) = x$ for some n; the minimal such n is called the *period* of the periodic point. For example, every fixed point of f itself has period 1, and for the antipodal map $\alpha : S^n \to S^n$, every point is a periodic point of period 2. In this section we introduce an algebraic measure that provides an effective tool in the study of periodic points.

(3.1) DEFINITION. Let K be a finite polyhedron, $\dim K \le n$. The *Euler number* $\chi(f)$ of a map $f : K \to K$ is

$$\chi(f) = \sum_{i=0}^{n} (-1)^i \dim \left[H_i(K;Q) / \bigcup_{m \ge 1} \operatorname{Ker} f_{*i}^m \right].$$

We first examine the nature of this number.

(3.2) THEOREM. *Let $f : K \to K$ be a map of a finite polyhedron. Then $\chi(f)$ is an integer depending only on the homotopy class of f. Moreover,*

$$\chi(f) = \sum_{i=0}^{n} (-1)^i \mu(i), \tag{$*$}$$

where $\mu(i)$ is the number of nonzero eigenvalues (counted with multiplicities) of the endomorphism

$$f_{*i} : H_i(K;Q) \to H_i(K;Q).$$

PROOF. Since homotopic maps induce the same homomorphism, the first assertion is obvious. To establish the second assertion observe that f_{*i} maps $Nf_{*i} \equiv \bigcup_{m \ge 1} \operatorname{Ker} f_{*i}^m$ into itself, and hence induces an endomorphism f''_{*i} on the factor space $\widetilde{H}_i(K) = H_i(K;Q)/Nf_{*i}$. From the commutative diagram

with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Nf_{*i} \arrow[r] \arrow[d, "f'_{*i}"] & H_i(K) \arrow[r] \arrow[d, "f_{*i}"] & \widetilde{H}_i(K) \arrow[r] \arrow[d, "f''_{*i}"] & 0 \\
0 \arrow[r] & Nf_{*i} \arrow[r] & H_i(K) \arrow[r] & \widetilde{H}_i(K) \arrow[r] & 0
\end{tikzcd}$$

we get as in (1.5) the relation between the characteristic polynomials $p_{f_{*i}}(\lambda) = p_{f'_{*i}}(\lambda)p_{f''_{*i}}(\lambda)$; since f'_{*i} is nilpotent, $p_{f'_{*i}}(\lambda) = \lambda^s$ ($s = \dim Nf_{*i}$), and hence f_{*i} and f''_{*i} have the same nonzero eigenvalues. This implies, because f''_{*i} is an isomorphism, that $\mu(i) = \dim \widetilde{H}_i(K)$, and our assertion follows. □

We now establish the main result of this section.

(3.3) THEOREM (Bowszyc). *Let K be a connected polyhedron, and let $f : K \to K$ be a map. If $\chi(f) \neq 0$, then f has a periodic point.*

PROOF. For each $q = 1, \dots, n = \dim K$, let $\{\lambda_{qi} \mid i = 1, \dots, n(q)\}$ be the nonzero eigenvalues of the endomorphism $f_{*q} : H_q(K; Q) \to H_q(K; Q)$, and consider the rational function

$$(*) \qquad L_f(z) = \frac{1}{1-z} + \sum_{q=1}^{n} (-1)^q \left[\sum_{i=1}^{n(q)} \frac{1}{1-\lambda_{qi} z} \right],$$

which is analytic around $z = 0$. We now get another expression for $L_f(z)$ by expanding each term in a power series around $z = 0$. This gives

$$\begin{aligned} L_f(z) = (1 + z + z^2 + \cdots) - [(1 + \lambda_{11} z + \lambda_{11}^2 z^2 + \cdots) \\ + (1 + \lambda_{12} z + \lambda_{12}^2 z^2 + \cdots) + \cdots] + \cdots \end{aligned}$$

If we collect like powers of z, the coefficient of z^k is the alternating sum of the sum of the kth powers of the eigenvalues in each dimension; according to (1.2), that coefficient is therefore $\lambda(f^k)$, the Lefschetz number of the kth iterate of f. As for the constant term, it is the alternating sum of the numbers of nonzero eigenvalues in each dimension, so the total, according to (3.2), is $\chi(f)$. Altogether then, we obtain a power series representation

$$(**) \qquad L_f(z) = \chi(f) + \sum_{n=1}^{\infty} \lambda(f^n) z^n$$

valid in a neighborhood of $z = 0$.

Observe next that letting $z \to \infty$ in $(*)$ gives $|L_f(z)| \to 0$, so if L_f is identically a constant, that constant must be zero. To prove the theorem, assume $\chi(f) \neq 0$. Since $\chi(f) = L_f(0) \neq 0$, the above observation assures that L_f is not identically constant, so some coefficient $\lambda(f^s)$ is nonzero, and f^s has a fixed point. The proof is complete. □

It is frequently of importance to determine, from the existence of a periodic point for an $f: K \to K$, whether or not f itself has a fixed point; the following relation between the Lefschetz number of f and that of its iterates has been useful in questions of this kind.

(3.4) THEOREM (The mod p theorem). *Let K be a polyhedron, let $f: K \to K$ be continuous, and let p be a prime. Then $\lambda(f^p) \equiv \lambda(f) \bmod p$. More generally, if $q = p^s$, where s is any positive integer, then $\lambda(f^q) \equiv \lambda(f) \bmod p$.*

PROOF. As seen in (2.3), the matrix of the linear transformation corresponding to $f_{r\#} = \varphi_{r\#}\mathrm{Sd}_r^m$ is a matrix A of integers; the matrix for $f_{r\#}^p$ is therefore A^p, so the problem (in the case $s = 1$) reduces to showing that for any matrix A of integers, $\mathrm{tr}(A) \equiv \mathrm{tr}(A^p) \bmod p$. We work with the characteristic polynomials. Let

$$|A - \lambda I| = \sum_{k=0}^{n} a_k \lambda^k \quad \text{and} \quad |A^p - \lambda I| = \sum_{k=0}^{n} b_k \lambda^k;$$

regarding these as elements of $Z_p[\lambda]$, we have

$$|A - \lambda^p I| = \sum a_k \lambda^{kp} \equiv \sum a_k^p \lambda^{pk} \bmod p,$$

because by Fermat's theorem, $a^p \equiv a \bmod p$ for every integer a. Thus, working mod p we have

$$\begin{aligned} |A - \lambda^p I| &\equiv \sum a_k^p \lambda^{kp} \equiv \left(\sum a_k \lambda^k\right)^p = |A - \lambda I|^p = |(A - \lambda I)^p| \\ &\equiv |A^p - \lambda^p I|, \end{aligned}$$

so that $\sum a_k \lambda^{kp} \equiv \sum b_k \lambda^{kp} \bmod p$; since $\mathrm{tr}(A) = (-1)^{n-1} a_{n-1}$ and $\mathrm{tr}(A^p) = (-1)^{n-1} b_{n-1}$, the proof for the case $s = 1$ is complete.

The general case follows from this by an obvious induction and the observation that if $q = p^s$ and $r = p^{s-1}$, then $f^q = (f^r)^p$ □

4. Applications

We give here some simple and direct uses of Lefschetz and Euler numbers, in fixed point problems and in some other areas of mathematics.

It has already been shown that any nullhomotopic map of any connected polyhedron has a fixed point. We generalize this in

(4.1) PROPOSITION. *Let K be a connected polyhedron and $f : K \to K$ such that f^n is nullhomotopic for some n. Then f has a fixed point.*

PROOF. Observe first that if f^n is nullhomotopic, then so also is f^{n+1}: for if $\Phi : K \times I \to K$ shows that f^n is nullhomotopic, then $(x, t) \mapsto \Phi[f(x), t]$

shows that f^{n+1} is nullhomotopic. We now start the proof. Choose a prime $p \geq n$; using our observation recursively, we find that f^p is nullhomotopic, therefore $f^p_{*i} : H_i(K;Q) \to H_i(K;Q)$ is the zero homomorphism for all $i > 0$, and consequently, $\lambda(f^p) = 1$. Since

$$\lambda(f) \equiv \lambda(f^p) \bmod p,$$

we have $\lambda(f) \neq 0$, and the proof is complete. □

Instead of restricting the map, we now restrict the spaces to get

(4.2) PROPOSITION. *Let K be a connected polyhedron with $H_i(K;Q) = 0$ for all $i > 0$. Then every continuous map $f : K \to K$ has a fixed point.*

PROOF. It is immediate from the definition of the Lefschetz number that $\lambda(f) = 1$ for every $f : K \to K$. □

We remark that the hypothesis of (4.2) is satisfied by the cone over any polyhedron (therefore, in particular, by a simplex) and also by any even-dimensional projective space. The result (4.2) can be extended.

(4.3) PROPOSITION. *Let K be a connected complex, and L a connected subcomplex such that $H_i(L;Q) = 0$ for all $i > 0$. Let $f : K \to K$ be such that $f^n(K) \subset L$ for some n. Then f has a fixed point.*

PROOF. Choose a prime $p \geq n$. Since $f^{n+1}(K) = f^n[f(K)] \subset f^n(K) \subset L$, we find, by proceeding recursively, that $f^p(K) \subset L$. We now factorize the map $f^p : K \to K$ as $K \xrightarrow{\varphi} L \xrightarrow{i} K$, where $\varphi : K \to L$ is the map $x \mapsto f^p(x)$ and $i : L \to K$ is the inclusion, so that $f^p = i\varphi$, and therefore $\lambda(f^p) = \lambda(i\varphi)$. By the commutativity property (1.4) of the trace, $\lambda(i\varphi) = \lambda(\varphi i)$, and since L is a connected complex with $H_i(L;Q) = 0$ for all $i > 0$, we get $\lambda(\varphi i) = 1$. Thus, $\lambda(f) \equiv \lambda(f^p) \equiv 1 \bmod p$, so $\lambda(f) \neq 0$, and f has a fixed point. □

Placing somewhat weaker restrictions on the space than those in (4.2), we get

(4.4) PROPOSITION. *Let K be a connected complex with odd-dimensional homology groups $H_{2i+1}(K;Q)$ all zero. Then every $f : K \to K$ has a periodic point.*

PROOF. The Euler number $\chi(f)$ is greater than or equal to 1, since there are no offsetting negative terms. □

By its classical definition, the *Euler characteristic* of a polyhedron K is $\chi(K) = \sum_{i\geq 0}(-1)^i \operatorname{rank} C_i(K;Z)$. This number is related to Euler and Lefschetz numbers in the following useful

(4.5) PROPOSITION. *Let* id : $K \to K$ *be the identity map. Then*

$$\chi(K) = \lambda(\mathrm{id}) = \chi(\mathrm{id}).$$

PROOF. If $\sigma_1, \ldots, \sigma_s$ are all the q-simplices of K, then $s = \operatorname{rank} C_q(K; Z)$; since the chains $1\sigma_1, \ldots, 1\sigma_s$ form a basis for $C_q(K; Q)$, it follows that the trace of id : $C_q(K; Q) \to C_q(K, Q)$ is exactly $\operatorname{rank} C_q(K; Z)$, so the Hopf trace theorem gives $\chi(K) = \lambda(\mathrm{id})$. Finally, the eigenvalues of the identity map $\mathrm{id}_* : H_i(K; Q) \to H_i(K; Q)$ all being equal to 1, we find $\lambda(\mathrm{id}) = \chi(\mathrm{id})$, and the proof is complete. □

All closed 2-manifolds except the torus and the Klein bottle have a nonzero Euler characteristic, as also does every even-dimensional sphere; these are therefore included among the polyhedra covered by

(4.6) PROPOSITION. *Let K be a connected polyhedron with Euler characteristic $\chi(K) \neq 0$. Then:*

(a) *any $f : K \to K$ homotopic to the identity map has a fixed point,*

(b) *any homeomorphism $h : K \to K$* (homotopic to the identity or not!) *has a periodic point.*

PROOF. Using (4.5), we find in case (a) that $0 \neq \chi(K) = \lambda(\mathrm{id}) = \lambda(f)$; in case (b), because h is a homeomorphism, we have $\chi(h) = \chi(\mathrm{id}) = \chi(K)$, and the proof is complete. □

The statement (4.6)(b) raises the question of the possible period of a homeomorphism; this is considered in

(4.7) PROPOSITION. *Let K be a connected complex and p a prime. If there is a fixed point free homeomorphism $h : K \to K$ with $h^q = \mathrm{id}$ for some $q = p^s$, then the Euler characteristic $\chi(K) \equiv 0 \bmod p$.*

PROOF. We have $\lambda(h^q) = \lambda(\mathrm{id}) = \chi(K)$, and therefore $\lambda(h) \equiv \chi(K) \bmod p$. If $\chi(K) \not\equiv 0 \bmod p$, then h would have a fixed point. □

This can be interpreted as saying that if a manifold admits a nontrivial fixed point free Z_{p^s}-action, then p must be a divisor of its Euler characteristic. Thus, for example, $\chi(S^{2n}) = 2$, so that an even-dimensional sphere can admit at most a Z_2-action, and the antipodal map is an example of such an action.

Our last application is to topological dynamics.

A *semiflow* on a polyhedron K is a continuous map $f : K \times R^+ \to K$ such that

$$f(x, t+\tau) = f[f(x,t), \tau], \qquad x \in K,\ t, \tau \in R^+,$$
$$f(x, 0) = x, \qquad x \in K.$$

A *fixed point* for the semiflow is a point x_0 such that $f(x_0, t) = x_0$ for all $t \in \boldsymbol{R}^+$ (i.e., f has a degenerate orbit).

(4.8) PROPOSITION. *If K is a finite polyhedron with $\chi(K) \neq 0$, then any semiflow on K has a fixed point.*

PROOF. For each $t_0 \in \boldsymbol{R}^+$, we denote the map $x \mapsto f(x, t_0)$ by f_{t_0}. Each f_{t_0} is homotopic to the identity: one need only set $(x, \varrho) \mapsto f(x, (1-\varrho)t_0)$ to get a homotopy of f_{t_0} to $f_0 = \mathrm{id}$. By (4.6)(a) each f_{t_0} has a fixed point. Let

$$A_n = \{x \mid f(x, 1/2^n) = x\};$$

each A_n is nonempty, closed, and therefore compact; moreover, $A_n \supset A_{n+1} \supset \cdots$, since $f(x, 1/2^{n+1}) = x$ gives

$$f\left(x, \frac{1}{2^n}\right) = f\left[f\left(x, \frac{1}{2^{n+1}}\right), \frac{1}{2^{n+1}}\right] = x.$$

By the finite intersection property, there is some $x_0 \in \bigcap_{n=1}^{\infty} A_n$; and since $f(x_0, 1/2^n) = x_0$ for each $n \in \boldsymbol{Z}$, we have $f(x_0, m/2^n) = x_0$ for all natural m and n. Because the set of dyadic rationals $\{m/2^n\}$ is dense in $\boldsymbol{R}^+$, continuity of f ensures $f(x_0, t) = x_0$ for all $t \geq 0$, and the proof is complete. □

5. The Brouwer Degree of Maps $S^n \to S^n$

As an illustration of the use of homological methods, directly related to fixed points, we study the properties of mappings of spheres into themselves. The study of such maps arises naturally in an attempt to determine whether a given map $f : (\boldsymbol{K}^{n+1}, \partial \boldsymbol{K}^{n+1}) \to (\boldsymbol{R}^{k+1}, \boldsymbol{R}^{k+1} - \{0\})$ necessarily has a zero; this question is reduced to the study of maps of spheres by considering the map $\Phi : \boldsymbol{S}^n \to \boldsymbol{S}^k$ given by $x \mapsto f(x)/\|f(x)\|$: if Φ is not nullhomotopic, then f must have a zero.

In the present section we study the case where $k = n$; if $k > n$, then Φ is always nullhomotopic (which implies that the given $f|S^n$ has an extension over $\boldsymbol{K}^{n+1}$ with no zero); if $k < n$, the study of Φ involves algebraic machinery much more involved than that considered in this book.

Let $n \geq 1$, let $f : \boldsymbol{S}^n \to \boldsymbol{S}^n$ be a continuous map, and let $f_* : H_n(\boldsymbol{S}^n) \to H_n(\boldsymbol{S}^n)$ be the induced homomorphism. Choose a generator $u \in H_n(\boldsymbol{S}^n) \cong \boldsymbol{Z}$; then $f_*(u) = d \cdot u$ for some integer d, positive, negative, or zero, and clearly d completely determines the homomorphism. Moreover, d depends only on f, and not on the generator selected: for if $-u$ were used, then $f_*(-u) = -f_*(u) = -du = d(-u)$. This justifies

(5.1) DEFINITION. The (*Brouwer*) *degree* of a map $f : \boldsymbol{S}^n \to \boldsymbol{S}^n$ is the unique integer $d(f)$ such that $f_*(u) = d(f) \cdot u$, where u is any generator of $H_n(\boldsymbol{S}^n)$.

The degree is a direct generalization of the winding number of analysis, with which it coincides when $n = 1$; we give a direct geometric interpretation of $d(f)$. The homomorphism induced by f being that of a simplicial approximation, it is enough to give its geometric interpretation for a simplicial map $h : S^n \to \Sigma^n$, different symbols being used for the n-spheres because they are taken with possibly different simplicial subdivisions. Let $\{\sigma_1, \ldots, \sigma_s\}$ be all the n-simplices of S^n and $\{\tau_1, \ldots, \tau_r\}$ those of Σ^n; recall that we can orient these simplices so that $c_n = \sum_{i=1}^s \sigma_i$, $\xi_n = \sum_{i=1}^r \tau_i$ are the basic cycles. Now choose any n-simplex τ^0. Let p be the number of oriented simplices such that $h(\sigma_i) = +\tau^0$, and k the number of those such that $h(\sigma_i) = -\tau^0$; we call $p - k$ the *algebraic number of times that* τ^0 *is covered*. If $d(h)$ is the degree of h, we have $h(c_n) = d(h)\xi_n = (p-k)\tau^0 + C$; because τ^0 does not occur in C, we must have $p - k = d(h)$. Since τ^0 is arbitrary, we conclude that $d(h)$ is the algebraic number of times that any given n-simplex of Σ^n is covered.

Using this interpretation it can be easily verified that the map $z \mapsto z^n : S^1 \to S^1$ has degree n, and moreover that if $f : S^n \to S^n$ has degree n, then so also does its suspension $Sf : S^{n+1} \to S^{n+1}$; it follows that for any degree d and any $n \geq 1$, one can therefore construct, by repeated suspension, a map $f : S^n \to S^n$ of degree d.

The degree has the following elementary properties:

(5.2) PROPOSITION. *Let* $n \geq 1$, *and let* $f, g : S^n \to S^n$ *Then:*
(a) $d(f \circ g) = d(f) \cdot d(g)$,
(b) *if* $f \simeq g$, *then* $d(f) = d(g)$,
(c) $\lambda(f) = 1 + (-1)^n d(f)$.

PROOF. (a) Since $(f \circ g)_* = f_* \circ g_*$, we find that $f_* g_*(u) = f_*(d(g)u) = d(g) f_*(u) = d(g) \cdot d(f)u$.

(b) is immediate because homotopic maps induce the same homomorphism, and (c) follows from the Lefschetz–Hopf theorem because $H_i(S^n) = 0$ for $i \neq 0, n$ and $H_n(S^n)$ is a one-dimensional vector space. □

We remark that (b) is in fact an equivalence. It was shown by H. Hopf that the degree characterizes the homotopy class of f. We shall give a proof of this important result in Section 8.

The degrees of some maps are easily calculated.

(5.3) PROPOSITION. (a) $\mathrm{id} : S^n \to S^n$ *has degree* $+1$.
(b) *A constant map* $S^n \to S^n$ *has degree* 0.
(c) *The antipodal map* $\alpha : S^n \to S^n$ *has degree* $(-1)^{n+1}$.

PROOF. (a) and (b) are obvious from the definition; for (c), note that α has no fixed points, so by the Lefschetz–Hopf theorem, $0 = \lambda(\alpha) = 1 + (-1)^n d(\alpha)$, completing the proof. □

The degree of a map gives information about its global behavior:

(5.4) THEOREM. *Let $f : S^n \to S^n$ be given.*
(a) *If $d(f) \neq (-1)^{n+1}$, then f has a fixed point.*
(b) *If $d(f) \neq 1$, then f sends some point to its antipode.*

PROOF. (a) Assume f has no fixed point; then $0 = \lambda(f) = 1 + (-1)^n d(f)$, and so $d(f) = (-1)^{n+1}$, which proves (a). To establish (b), assume that f sends no point to its antipode, i.e., $f(x) \neq \alpha(x)$ for all $x \in S^n$; this implies that $\alpha \circ f(x) \neq x$ for all $x \in S^n$, so αf has no fixed point. By (a), we must therefore have

$$(-1)^{n+1} = d(\alpha \circ f) = d(\alpha) \cdot d(f) = (-1)^{n+1} d(f),$$

so $d(f) = +1$, and the proof is complete. □

Because the sign of $(-1)^{n+1}$ depends on the parity of n, we shall see that there are fundamental differences in the behavior of self-maps of even- and of odd-dimensional spheres.

6. Theorem of Borsuk–Hirsch

The degree gives information about fixed points and points sent to antipodal points. The Borsuk–Hirsch theorem, which we will establish in this section, shows that the parity of the degree determines the behavior of the map at antipodal pairs of points.

Recall that a map $f : S^n \to S^n$ is *antipode-preserving* if $f\alpha = \alpha f$; we call f *antipode-collapsing* if $f\alpha = f$. We again begin by calculating the degree of some maps. The first part in the following theorem is the homological version of Borsuk's antipodal theorem:

(6.1) THEOREM. *Let $f : S^n \to S^n$.*
(a) *If f is antipode-preserving, then $d(f)$ is odd.*
(b) *If f is antipode-collapsing, then $d(f)$ is even.*

PROOF. In the proof of both (a) and (b), we shall take S^n with the triangulation used in the combinatorial lemma, so that for each simplex of the triangulation, $\alpha(\sigma)$ is also a simplex of the triangulation; clearly, every barycentric subdivision of S^n will also have this property.

(a) Let $h : S^n \to S^n$ be an antipode-preserving simplicial approximation to f. By the combinatorial lemma, h maps an odd number of simplices onto the main simplex of S^n, so by our description of degree, $d(h) = d(f)$ is odd.

(b) Let $h : S^n_+ \to S^n$ be an antipode-collapsing simplicial approximation to f. Each n-simplex τ of S^n will now be covered an even number of times: if $\sigma \subset S^n_+$ is mapped onto τ, then $\alpha(\sigma)$ is a simplex in S^n_- having the same

image as σ. Thus, the algebraic number of times a simplex of S^n is covered is even, and so $d(f)$ is even. This completes the proof. □

(6.2) THEOREM. *Let $f : S^n \to S^n$*

(a) (G. Hirsch) *If $d(f)$ is odd, then f must send some antipodal pair of points to an antipodal pair of points.*

(b) (Borsuk) *If $d(f)$ is even, then f must send some antipodal pair of points to the same point.*

PROOF. (a) Assume that antipodal pairs never have antipodal images, so that $f\alpha(x) \neq \alpha f(x)$ for each x. This means that $f\alpha(x)$ and $f(x)$ are never opposite, so that the segment joining them does not go through the origin. Projecting these segments from the origin onto the sphere gives a homotopy from $f\alpha$ to f, specifically,

$$F(x,t) = \frac{(1-t)f\alpha(x) + tf(x)}{\|(1-t)f\alpha(x) + tf(x)\|}.$$

In particular, f is homotopic to

$$F(x, \tfrac{1}{2}) = \frac{f\alpha(x) + f(x)}{\|f\alpha(x) + f(x)\|}.$$

Call this map Φ; it is clear from its definition that $\Phi(\alpha x) = \Phi(x)$, i.e., Φ is antipode-collapsing. Since $f \simeq \Phi$, we have $d(f)$ even, and (a) is proved.

To prove (b), assume that antipodal points never have the same image. Since $f\alpha(x) \neq f(x)$, it follows that $f\alpha(x)$ and $\alpha f(x)$ are never opposite, therefore are homotopic. As before,

$$F(x,t) = \frac{(1-t)f\alpha(x) + t\alpha f(x)}{\|(1-t)f\alpha(x) + t\alpha f(x)\|}$$

is a homotopy. Letting $\Phi(x) = F(x, \frac{1}{2})$ we find that αf is homotopic to

$$\Phi(x) = \frac{f\alpha(x) + \alpha f(x)}{\|f\alpha(x) + \alpha f(x)\|}.$$

Note now that $\alpha\Phi(x) = \Phi\alpha(x)$, so that Φ is antipode-preserving. Consequently, $\alpha\Phi$ is also antipode-preserving, and since $\alpha\Phi \simeq \alpha\alpha f = f$, we conclude that $d(f)$ is odd. This completes the proof. □

7. Maps of Even- and of Odd-Dimensional Spheres

In this section we apply (5.4) to obtain fundamental differences in the behavior of self-maps of spheres which depend on the dimension of the sphere being odd or even.

(7.1) THEOREM. *Every map $f : S^{2n} \to S^{2n}$ has a fixed point and/or sends some point to its antipode. Precisely:*

If $d(f) \neq \pm 1$, then f has a fixed point and sends some point to its antipode.
If $d(f) = +1$, then f has a fixed point.
If $d(f) = -1$, then f sends some point to its antipode.

PROOF. Because $(-1)^{2n+1} = -1$, every map has a degree differing from at least one of $+1$ and -1; the result follows from (5.4). $\square$

(7.2) THEOREM. *Every map $f : S^{2n+1} \to S^{2n+1}$ with degree $\neq +1$ has a fixed point and sends some point to its antipode. If $d(f) = +1$, it may have both, only one, or none of these features.*

PROOF. In this case, $(-1)^{(2n+1)+1} = +1$. We consider the case $d(f) = 1$. Here, id and α both have degree $+1$, and each has only one of the two properties. The map

$$R(x_1, x_2, \ldots, x_{2n+1}, x_{2n+2}) = (-x_2, x_1, \ldots, -x_{2n+2}, x_{2n+1})$$

has no fixed point, and sends no point to its antipode (therefore has degree 1). Finally, if we start with the map $\varphi : S^1 \to S^1$ of degree $+1$ given by

$$\varphi(e^{it}) = \begin{cases} e^{2ti}, & 0 \leq t \leq \pi, \\ 1, & \pi \leq t \leq 2\pi, \end{cases}$$

which has both a fixed point and sends some point to its antipode, repeated suspension of this map will give examples of maps having degree $+1$ and both properties. $\square$

This difference has two immediate consequences.

(7.3) COROLLARY. *Every rotation (i.e., orthogonal transformation of determinant $+1$) of S^{2n} has a fixed axis; but there are rotations of S^{2n+1} having no fixed axis.*

PROOF. A rotation is an antipode-preserving map that is homotopic to the identity, and therefore has degree $+1$. For S^{2n}, this must have a fixed point, so its antipode will also be fixed, and these provide a rotational axis. For S^{2n+1}, the map R in (7.2) is a rotation that has no fixed points. $\square$

Denote by $T(x)$ the n-dimensional hyperplane in R^{n+1} tangent to S^n at $x \in S^n$; translated to the origin, it is an n-plane L_x perpendicular to the vector x. A continuous, nonvanishing, tangent vector field on S^n is a continuous map assigning to each $x \in S^n$ a nonzero vector in L_x.

(7.4) COROLLARY (Poincaré Brouwer). *There is no continuous nonvanishing tangent vector field on S^{2n}; such fields exist on S^{2n+1}.*

PROOF. Assume that φ were such a field; the rule $f(x) = \varphi(x)/\|\varphi(x)\|$ would then give a map $f : S^n \to S^n$ that has no fixed points and sends no

point to its antipode. By (7.1) this is impossible for S^{2n} In the case of an S^{2n+1} however, the map sending

$$(x_1, x_2, \dots, x_{2n+1}, x_{2n+2})$$

to the vector with components

$$(-x_2, x_1, \dots, -x_{2n+2}, x_{2n+1})$$

defines a continuous nonvanishing tangent vector field. □

8. Degree and Homotopy. Theorem of Hopf

We have seen that homotopic maps have the same degree. Our objective in this section is to prove the converse, thereby attaining Hopf's theorem that two maps $S^n \to S^n$ are homotopic if and only if they have the same degree: their behavior in homology therefore completely determines their behavior in homotopy, so for example, the antipodal map of an odd-dimensional sphere, having degree +1, is homotopic to the identity map. The proof of Hopf's theorem given here will be inductive, based on first showing that for $n \geq 2$, each $f : S^n \to S^n$ is homotopic to the suspension of some $g : S^{n-1} \to S^{n-1}$

We use the symbols S^n, Σ^n for the n-sphere to distinguish it as domain and as range; S^n has n_+ (respectively n_-) and Σ^n has δ_+ (respectively δ_-) as its north (respectively south) pole; the northern (respectively southern) hemisphere of S^n is denoted by S^n_+ (respectively S^n_-) and its equatorial sphere by S^{n-1}; analogously for Σ^n Given a small disk $D \subset S^n_+$ centered at n^+, we will frequently use the following homeomorphism $h : S^n \to S^n$, called the *radial projection* identifying D with S^n_+: each arc of a great circle from n_+ to n_- meets ∂D at x and S^{n-1} at $\xi(x)$; h maps the arc n_+x linearly onto $n_+\xi(x)$, and xn_- linearly onto $\xi(x)n_-$. Since $h : (S^n, D) \to (S^n, S^n_+)$ is bijective and continuous, it is a homeomorphism; and because x and $h(x)$ are never antipodal, $h \simeq \text{id}$, as also $h^{-1} \simeq \text{id}$. The radial projection identifying a disk centered at n_- with S^n_- is defined in an entirely analogous manner.

The crucial result is

(8.1) LEMMA. *Let $n \geq 2$. Then each $f : S^n \to \Sigma^n$ is homotopic to a map φ such that $\varphi^{-1}(\delta_+)$ is either empty or a single point.*

PROOF. Triangulate Σ^n so that δ_+ and δ_- are interior points of n-simplices. Using a sufficiently fine barycentric subdivision on S^n, we can assume that $f : S^n \to \Sigma^n$ is simplicial. Only the case $f^{-1}(\delta_+) \neq \emptyset$ has to be considered, and because δ_+ is in the interior of an n-simplex, $f^{-1}(\delta_+)$ consists of finitely many points, $p_1, \dots, p_s$, at most one in each n-simplex of the triangulation of S^n. Similarly, $f^{-1}(\delta_-)$ is a finite set.

Because $n \geq 2$, the finite set $f^{-1}(\delta_-)$ does not disconnect S^n, so there is a polygonal path λ running from p_1 to p_2 to $p_3, \dots$, to p_s that avoids $f^{-1}(\delta_-)$;

each $x \in \lambda$ therefore has a spherical neighborhood V_x such that $f(\overline{V}_x) \subset \Sigma^n - \{\delta_-\}$. It follows that the compact set λ has a covering $\{V_1, \dots, V_T\}$ by finitely many open balls such that $p_1 \in V_1, p_s \in V_T, V_i \cap V_{i+1} \neq \emptyset, f(\overline{V}_i) \subset \Sigma^n - \{\delta_-\}$, and no p_j lies on any ∂V_i. Since $f(\bigcup \overline{V}_i)$ is a closed set avoiding δ_-, there is a closed disk Δ_- centered at δ_- that it avoids also; using a radial projection h identifying Δ_- with Σ^n_-, we can assume (since $h \circ f \simeq f$) that $f(\overline{V}_i) \subset$ interior of Σ^n_+ for each $i = 1, \dots, T$.

We now change f on V_1 to get a map $F_1 : S^n \to \Sigma^n$ homotopic to f and such that $V_1 \cap F_1^{-1}(\delta_+)$ is any single point $x_1 \in V_1 \cap V_2$: replace $f|\overline{V}_1$ by the map $r : \overline{V}_1 \to \Sigma^n$ that, for such $x \in \partial V_1$, sends the segment $x_1 x$ linearly onto $\delta_+ f(x)$. Then

$$F_1(x) = \begin{cases} f(x), & x \in S^n - V_1, \\ r(x), & x \in \overline{V}_1, \end{cases}$$

is continuous; moreover, for $x \in \overline{V}_1$, both $r(x)$ and $f(x)$ belong to the interior of Σ^n_+, so $F_1(x)$ and $f(x)$ are never opposite, and therefore $F_1 \simeq f$; finally, $V_1 \cap F_1^{-1}(\delta_+) = \{x_1\}$. This process is called *concentrating* $V_1 \cap f^{-1}(\delta_+)$ *at* x_1.

Now repeat this process with F_1 and V_2, concentrating $V_2 \cap F_1^{-1}(\delta_+)$ at an $x_2 \in V_2 \cap V_3$ to get $F_2 : S^n \to \Sigma^n$ homotopic to F_1, and therefore to f; and we have $F_2^{-1}(\delta_+) \subset V_2 \cup \cdots \cup V_T$. Proceed recursively in this manner: at the kth stage, $F_k^{-1}(\delta_+) \subset V_k \cup \cdots \cup V_T$; concentrating $V_k \cap F_k^{-1}(\delta_+)$ at an $x_{k+1} \in V_k \cap V_{k+1}$ yields an $F_{k+1} : S^n \to \Sigma^n$ homotopic to $F_k \simeq f$ such that $F_{k+1}^{-1}(\delta_+) \subset V_{k+1} \cup \cdots \cup V_T$. At the last stage, concentrating $V_T \cap F_{T-1}^{-1}(\delta_+)$ at p_s, we obtain $F_T \simeq f$ with $F_T^{-1}(\delta_+) = p_s$, and the proof is complete. □

As an immediate consequence, we have

(8.2) THEOREM. *Let $n \geq 2$. Then each $f : S^n \to \Sigma^n$ is homotopic to the suspension of some $g : S^{n-1} \to \Sigma^{n-1}$*

PROOF. If $f^{-1}(\delta_+) = \emptyset$, then f is nullhomotopic, and the conclusion is obvious. By Lemma (8.1), we can assume $f^{-1}(\delta_+) = n_+$, by a rotation if necessary. Choose a small closed disk Δ_- centered at δ_-; since f is continuous, there is a disk D centered at n_+ such that $f(\overline{D}) \subset \Sigma^n - \Delta_-$; on the compact $S^n - D$, no point goes to δ_+, so $f(S^n - D) \subset \Sigma^n - \Delta_+$ for some disk Δ_+ centered at δ_+.

We now use a "radial projection" $s : \Sigma^n \to \Sigma^n$ that pushes $\partial\Delta_+$ and $\partial\Delta_-$ onto Σ^{n-1}: for each $x \in \Sigma^{n-1}$, let $\delta_+ x \delta_-$ be the arc of a great circle through x; it meets $\partial\Delta_+$ at ξ_+ and $\partial\Delta_-$ at ξ_-; s maps $\delta_+\xi_+$ linearly onto $\delta_+ x$, $\xi_+\xi_-$ onto the point x, and $\xi_-\delta_-$ linearly onto $x\delta_-$. Then $sf(\overline{D}) \subset \Sigma^n_+$ and $sf(S^n - D) \subset \Sigma^n_-$; moreover $f(x)$ and $sf(x)$ are always on the same arc of a great circle, so are never antipodal, and therefore $f \simeq sf$. Taking a

radial projection h identifying D with S^n_+ we have $sfh^{-1} : (S^n, S^n_+, S^n_-) \to (\Sigma^n, \Sigma^n_+, \Sigma^n_-)$ and $sfh^{-1} \simeq sf$ because $h^{-1} \simeq \mathrm{id}$. Now let $g = sfh^{-1}|S^{n-1} : S^{n-1} \to \Sigma^{n-1}$; then the suspension $Sg : (S^n, S^n_+, S^n_-) \to (\Sigma^n, \Sigma^n_+, \Sigma^n_-)$; so since $Sg(x)$ and $s \circ f \circ h^{-1}(x)$ are never antipodal, they are homotopic, and the theorem is proved. □

We now prove the main result:

(8.3) THEOREM (Hopf). *Let $n \geq 1$. Then $f, g : S^n \to S^n$ are homotopic if and only if they have the same degree.*

PROOF. It is enough to show that if $\deg(f) = \deg(g)$, then $f \simeq g$, since the converse is already known. We consider first the case $n = 1$. Regarding S^1 as R/Z, the reals modulo 1, an $f : S^1 \to S^1$ is determined by a continuous $F : R \to R$ such that $F(x+1) - F(x) \equiv N$, where N is some integer, positive, negative, or zero: the function F determines the map $f : S^1 \to S^1$ given by $e^{2\pi it} \mapsto e^{2\pi iF(t)}$, and in the other direction, if $f : S^1 \to S^1$ is given, then F is a branch of $\frac{1}{2\pi i} \log f(e^{2\pi it})$ continuous on R. It is straightforward to verify that $f : S^1 \to S^1$ has degree N if and only if its representing function satisfies $F(x+1) - F(x) \equiv N$. Now let $g : S^1 \to S^1$ be $z \mapsto z^N$; it is represented by $G(x) = Nx$. Given any map $f : S^1 \to S^1$ of degree N, we show that it is homotopic to g: for if f is represented by F, then

$$\Phi(x,t) = \exp(2\pi i[tF(x) + (1-t)G(x)])$$

is a homotopy of g to f because $\Phi(x+1,t) - \Phi(x,t) \equiv N$, so that $\Phi(x,t)$ represents a map of S^1 into S^1 for each fixed t. Thus, all maps of degree N are homotopic to g, and therefore to each other, so the theorem is proved for $n = 1$.

We proceed by induction. Assume the theorem is true for $k = n-1$, and let $f, g : S^n \to S^n$ have the same degree. By the previous theorem, f and g are homotopic to the suspensions Sf', Sg' of maps $f', g' : S^{n-1} \to S^{n-1}$; since the suspension of a map and the map have the same degree, we have $\deg(f') = \deg(g')$, so by the induction hypothesis, $f' \simeq g'$. Suspending that homotopy shows that $Sf' \simeq Sg'$, and therefore $f \simeq g$. Thus, the theorem is true for $k = n$; this completes the induction, and the proof. □

9. Vector Fields on Spheres

Let $A \subset R^{n+1}$ A *vector field* on A is a continuous map f assigning to each $a \in A$ a vector at the origin in R^{n+1}; the vector field f is said to have a *singularity* at $a \in A$ if $f(a) = 0$. The study of vector fields is important in many areas of mathematics; for example, they arise in attempting to determine the global properties of solutions of ordinary differential equations.

A continuous vector field f on A determines a map $F : A \to \boldsymbol{R}^{n+1}$ by $x \mapsto x - f(x)$, and if the field is nonvanishing, no point of A is kept fixed by F Conversely, if $F : A \to \boldsymbol{R}^{n+1}$ is any continuous map, then the map $f(x) = x - F(x)$ is called the vector field *associated* with the continuous map F, and is nonvanishing only if F has no fixed point.

Take $A = \boldsymbol{K}^{n+1}$; if f is a vector field with no singular points on $\partial \boldsymbol{K}^{n+1} = \boldsymbol{S}^n$, then the map $c_f : x \mapsto f(x)/\|f(x)\| : \boldsymbol{S}^n \to \boldsymbol{S}^n$ is well defined; the degree $\varkappa(f)$ of c_f is called the *characteristic* of the vector field. Note that the characteristic is determined by $f|\boldsymbol{S}^n$ and depends only on the directions, and not on the lengths, of the vectors. Using these concepts, many results of the previous sections can be given additional geometric context. We list some of these interpretations; most are based on the technical results.

1. If f is a nonvanishing vector field on $\boldsymbol{K}^{n+1}$, then its characteristic is zero.

For c_f is then extendable to a map of $\boldsymbol{K}^{n+1}$ into $\boldsymbol{S}^n$.

2. The field of outward normals on $\boldsymbol{S}^n$ has characteristic $+1$, that of inward normals has characteristic $(-1)^{n+1}$.

For c_f is the identity map in the first case, and the antipodal map in the second.

3. (Poincaré–Bohl) If two nonvanishing vector fields on $\boldsymbol{S}^n$ have different characteristics, then there is at least one point at which the vectors are opposite.

For otherwise, the maps c_f, c_g would never be antipodal, and therefore would be homotopic.

4. Every nonvanishing vector field on $\boldsymbol{S}^n$ with characteristic $\neq +1$ has an inward normal; every nonvanishing vector field with characteristic $\neq (-1)^{n+1}$ has an outward normal.

This is a reinterpretation of (5.4). Similarly, from (6.2) follows

5. Let f be a nonsingular vector field on $\boldsymbol{S}^n$. Then there exists a pair of antipodal points at which the vectors are parallel.

For a reinterpretation of (6.2) we have

6. Let f be a nonsingular vector field on $\boldsymbol{K}^{n+1}$. Then there exists a pair of antipodes at which the vectors have the same direction.

For $d(c_f) = 0$; if the vectors at antipodal points never have the same direction, then c_f sends no antipodal pair to the same point, and therefore must have odd degree. This is a contradiction.

10. Miscellaneous Results and Examples

A. Degree and the Lefschetz-Hopf theorem

(A.1) (*Fundamental theorem of algebra*) Let $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a complex polynomial of degree $n > 1$. Show: $p(z_0) = 0$ for some $z_0 \in C$.

[Suppose that $p(z) \neq 0$ for all $z \in C$ with $c_0, c_n \neq 0$. Set

$$P(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n \qquad \text{for } z \in C,$$

and for $\zeta \in S^1$ and $t \in [0,1]$ let

$$f(\zeta, t) = \frac{p(t\zeta)}{P(t/\zeta)}.$$

To get a contradiction, show that $(\zeta, t) \mapsto f(\zeta,t)/|f(\zeta,t)|$ defines a homotopy $H_t : S^1 \to S^1$ joining a constant map H_0 to the function H_1 given by $\zeta \mapsto \zeta^n$, which is impossible, because $d(H_0) = 0$ and $d(H_1) = n$.]

(This proof is due to Burckel [1981].)

(A.2) Let $f, g : S^{2n} \to S^{2n}$. Prove: At least one of f, g, $g \circ f$ has a fixed point.

[Use $d(g \circ f) = dg \cdot df$.]

(A.3) Let $f : S^{2n} \to S^{2n}$ Show:

(a) Either f or f^2 has a fixed point.

(b) Either f has a fixed point, or else there is a point $x_0 \in S^{2n}$ such that both x_0 and $-x_0$ are fixed points of f^2

(A.4) Let $f : R^{2n+1} \to R^{2n+1}$. Show: There are $x \neq 0$ and $\lambda \in R$ such that $f(x) = \lambda x$.

(A.5) (*Generalized Borsuk-Ulam theorem*) Let K be a polyhedron and $T : K \to K$ periodic, of period 2. Assume that there is a $g : S^{n-1} \to K$ such that

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, "\alpha"'] & K \arrow[d, "T"] \\ S^{n-1} \arrow[r, "g"] & K \end{tikzcd}$$

commutes, where α is the antipodal map. Show: If $H_{n-1}(K)$ does not contain an infinite cyclic subgroup, then for each $f : K \to R^n$ there is a $y \in K$ with $f(y) = fT(y)$.

[Assume $f : K \to R^n$ with $f(y) \neq fTy$ for all y; then with

$$F(y) = \frac{f(y) - fTy}{\|f(y) - fTy\|}$$

we get a commutative diagram

$$\begin{tikzcd} K \arrow[r, "F"] \arrow[d, "T"'] & S^{n-1} \arrow[d, "\alpha"] \\ K \arrow[r, "F"] & S^{n-1} \end{tikzcd}$$

This gives us $\alpha F g = F g \alpha$, so that $F \circ g$ is antipode-preserving, so has odd degree. $F_* g_*(b_{n-1}) = (2k+1) b_{n-1}$ (where b_{n-1} is a generator of $H_{n-1}(S^{n-1})$). Now, $g_*(b_{n-1}) \in H_{n-1}(K)$, and for any integer s, $F_*[s g_*(b_{n-1})] = s F_* g_*(b_{n-1}) = (2k+1) s b_{n-1} \neq 0$. Therefore, $g_*(b_{n-1}) \neq 0$, and $\{g_*(b_{n-1})\}$ generates an infinite cyclic group.]

(A.6) (*Dugundji Fan theorem*) Let T be a triangulation of the boundary of an n-simplex $\sigma = (p_0, \dots, p_n)$. To each vertex $v \in T$ assign arbitrarily a number $\tau(v) \in \{0, 1, \dots, n\}$ and define $\varphi : T \to \dot{\sigma}$ by $v \mapsto p_{\tau(v)}$. This map is simplicial, and has degree d, obtained by counting the algebraic number of $(n-1)$-simplices of T that are mapped into a fixed $(n-1)$-face of $\dot{\sigma}$. Let ϱ_k be the algebraic number of simplices in T of dimension k that receive the same labels as the subscripts of their carriers in $\dot{\sigma}$. Show:

$$\sum_{k=0}^{n-1} (-1)^k \varrho_k = 1 + (-1)^{n-1} d.$$

[The Lefschetz number $\lambda(\varphi)$ is $1 + (-1)^{n-1} d$. Using the Hopf trace theorem, compute this by working with the traces of the chain map $\varphi \operatorname{Sd}_T$, and taking the k-faces of σ as basis for the k-chains. Observe that $\operatorname{Sd}_T s =$ the sum of all simplices of T carried by $s = (p_{i_0}, \dots, p_{i_k})$, so to say that $\varphi \operatorname{Sd}_T s = \cdots + as + \cdots$ means that a is the algebraic number of those simplices of T of dimension k carried by s that receive the labels $i_0, \dots, i_k$. Since $\lambda(\varphi) = \sum_{i=0}^{n-1} (-1)^k \operatorname{tr}(\varphi \operatorname{Sd}_T)$, this completes the proof.]

B. Vector fields

(B.1) Show: A vector field $f \cdot (K^{n+1}, S^n) \to (R^{n+1}, R^{n+1} - \{0\})$ is essential if and only if $\varkappa(f) \neq 0$.

(B.2) Prove: Two vector fields $f, g : (K^{n+1}, S^n) \to (R^{n+1}, R^{n+1} - \{0\})$ are homotopic if and only if $\varkappa(f) = \varkappa(g)$.

(B.3) Let $f : (K^{n+1}, S^n) \to (R^{n+1}, R^{n+1} - \{0\})$ be a vector field such that the vectors at antipodal points never have the same direction. Show: $\varkappa(f)$ is odd.

(B.4) Let $f : (K^{n+1}, S^n) \to (R^{n+1}, R^{n+1} - \{0\})$ be a vector field. Show:
- (i) If $\varkappa(f) \neq 1$, then f has an inward normal.
- (ii) If $\varkappa(f) \neq (-1)^{n+1}$, then f has an outward normal.
- (iii) If n is odd and $\varkappa(f) \neq 1$, then f has an outward normal and an inward normal.
- (iv) If n is even and $\varkappa(f) \neq \pm 1$, then f has an outward normal and an inward normal.
- (v) If n is even, then f has a normal.

(B.5) Let $f : (K^{n+1}, S^n) \to (R^{n+1}, R^{n+1} - \{0\})$ be a vector field. Show: Either f has singularities or $\varkappa(f) = 0$.

(B.6) A vector field $f : S^n \to R^{n+1} - \{0\}$ is *directed inward* (*outward*) at $x \in S^n$ if $f(x)$ lies on the same (opposite) side of the tangent plane to S^n at x as S^n itself. Clearly, if f is directed inward everywhere, then $-f$ is directed outward everywhere, and vice versa. Let $f \cdot (K^{n+1}, S^n) \to (R^{n+1}, R^{n+1} - \{0\})$ be a vector field. Show:
- (a) If $f|S^n$ is directed inward everywhere, then $\varkappa(f) = (-1)^{n+1}$ and $f(x_0) = 0$ for some $x_0 \in K^{n+1}$
- (b) If $f|S^n$ is directed outward everywhere, then $\varkappa(f) = 1$, and $f(x_0) = 0$ for some $x_0 \in K^{n+1}$

(B.7) Show: A vector field $f : K^{n+1} \to R^{n+1} - \{0\}$ is somewhere on ∂K^{n+1} inward normal and somewhere outward normal.

(B.8) Let $f : K^{n+1} \to R^{n+1} - \{0\}$. Show: There is a pair of antipodes on ∂K^{n+1} at which the vectors are parallel.

11. Notes and Comments

The Lefschetz–Hopf fixed point theorem

This is the basic fixed point result formulated in homological terms. It was first discovered for compact manifolds by Lefschetz [1923], [1926] and then extended by him (Lefschetz [1927]) to manifolds with boundary. Hopf [1928] gave a completely different, simple proof for arbitrary polyhedra; this is essentially the proof given in the text. Various extensions of the Lefschetz–Hopf theorem will be found in Chapter V.

Converse of the Lefschetz–Hopf theorem

Because $\lambda(f)$ is a homotopy class invariant, the converse of the Lefschetz–Hopf theorem is the statement: If $\lambda(f) = 0$, then there is some $g \simeq f$ having no fixed point. This is false, in general, even for self-maps of polyhedra. However, the converse is true for a broad class of polyhedra: Say that a polyhedron K is of type (λ) if $\dim K \geq 3$ and the star of each vertex has connected boundary (thus includes all the n-manifolds, $n \geq 3$). Then the converse is true for all simply connected polyhedra of type (λ).

The converse of (4.6)(a) is the statement: If the Euler characteristic $\chi(K) = 0$, then there is a fixed point free $f : K \to K$ that is homotopic to the identity. This is also false in general. However, it is true whenever K is either a manifold or any polyhedron of type (λ). For further details, see R.F. Brown [1966].

Existence of periodic points

The first result on the existence of periodic points was established by Fuller [1953], to whom (4.6)(b) is due; for (4.4) the reader is referred to Hájek [1964]. These results are special cases of Theorem (3.3) proved by Bowszyc [1969]. Bowszyc introduced the Euler number and the periodicity number of a map, and using these invariants, established a more general result. These notions will be treated in more detail in Chapter V. For other closely related results on periodic points see Kelley–Spanier [1968], Halpern [1968], and also the survey by Fadell [1970].

Applications of the Lefschetz–Hopf theorem

Theorem (3.4) is a special case of a more general result due to Zabreĭko–Krasnosel'skiĭ [1971], and independently, Steinlein (see Steinlein [1980]). The first direct proof of (3.4) was given by Peitgen [1976]. Theorems (4.1) and (4.3) are the simplest results in the so-called asymptotic fixed point theory. Theorem (4.3) was found by Bourgin [1956a] and independently, by Berstein

[1957]. Theorem (4.7) is due to Floyd [1952] and Theorem (4.8) to Myshkis [1954]. G. Hirsch [1943] observed that Borsuk's antipodal theorem can be obtained using the Lefschetz–Hopf theorem.

The Brouwer degree

The general notion of topological degree of a map between orientable manifolds is due to Brouwer [1912]; he gave some of its properties and used it to settle some significant questions such as domain invariance in $\boldsymbol{R}^n$. The related notion of the characteristic $\varkappa(f)$ of a "vector field" $f : S^n \to \boldsymbol{R}^{n+1} - \{0\}$ ($\varkappa(f) = d(\widehat{f})$, $\widetilde{f} = f/\|f\|$) was developed by Kronecker in a series of papers in 1869–1878, studying existence of common zeros for a given system of real-valued functions. In the differentiable context, he expressed it by an integral, which he showed to be integer-valued; its value is, in fact, equal to $\varkappa(f)$ as defined above. Kronecker's ideas were elaborated by Hadamard [1910], who also gave numerous geometric applications.

Theorem (6.1)(a), proved by Hopf (see Alexandroff–Hopf [1935], p. 483), represents a homological version of Borsuk's antipodal theorem. Part (a) of (6.2) is due to G. Hirsch [1946]; part (b) of (6.2) was established by Borsuk [1937].

Degree and homotopy theorem of Hopf

Let $[S^n, S^n]$ denote the set of homotopy classes of maps of S^n into itself. The bijectivity of deg : $[S^n, S^n] \to \boldsymbol{Z}$ for $n = 2$ is due to Brouwer [1912], and for arbitrary n to Hopf [1927]. The proof of the theorem of Hopf (8.3) is an adaptation of that given in Dugundji's book [1965]. Theorem (8.2) in a different form was established by Borsuk [1933b].

Vector fields on spheres

The nonexistence of a continuous nonzero tangent vector field on S^2 was proved by Poincaré; the fact that S^n has such a field if and only if n is odd was established by Brouwer [1912]. For elementary proofs of (7.3) (without using the degree) see Jaworowski [1964] and Milnor [1978]. One can generalize the problem and seek the maximal number of nonvanishing linearly independent tangent fields on S^n (i.e., the number of vector fields $v_1, \dots, v_k$ on a given S^n such that for each x the vectors $v_1(x), \dots, v_k(x)$ are linearly independent in $\boldsymbol{R}^{n+1}$). Write $n + 1 = 2^b s$, where s is odd and $b = 4d + c$ with $0 \le c \le 3$, and let $\varrho(n) = 2^c + 8d - 1$. Hurwitz, and independently Radon, showed in 1923 that there are at least $\varrho(n)$ such fields on a given S^n; the much more difficult part that there are exactly $\varrho(n)$ such fields was proved in 1962 by F Adams.

The genus function γ

Let $(X;T)$ be a pair, where X is a paracompact space and $T : X \to X$ a fixed point free involution on X, i.e., $T^2 = \mathrm{id}$. The *genus* $\gamma(X;T)$ of X (by abuse of notation denoted simply by $\gamma(X)$) is the smallest positive integer n such that there exists an equivariant map $f : X \to S^n$, i.e., $f(Tx) = -f(x)$.

This invariant (which was introduced by Yang ([1954], [1955]) and Krasnosel'skiĭ [1955]) is closely related to the notion of the Lusternik–Schnirelmann category and to various extensions of the Borsuk antipodal theorem.

The following statements (Yang [1954]) are equivalent:

(i) $\gamma(X;T) \geq n$;

(ii) *if X is covered by $n+1$ closed sets, then at least one of them contains a pair x, Tx;*

(iii) *if $f : X \to \boldsymbol{R}^n$, then there is a point $x \in X$ such that $f(x) = f(Tx)$;*

(iv) *there are no n closed sets $M_1, \dots, M_n$ such that $M_i \cap TM_i = \emptyset$ and $X = \bigcup_{i=1}^n (M_i \cup TM_i)$.*

Note that in case $X = S^n$ and $T =$ antipodal map, (ii) is the Lusternik-Schnirelmann–Borsuk theorem and (iii) is the Borsuk–Ulam theorem.

The function γ can be used to establish the following extension (due to Bourgin [1955] and Yang [1955]) of the Borsuk–Ulam theorem: *If (X,T) has $\gamma(X) \geq n$ and $f : X \to \boldsymbol{R}^k$ is any map, then the set $A_f = \{x \in X \mid f(x) = f(Tx)\}$ is closed and T-invariant, and $\gamma(A_f) \geq n-k$.* A consequence is that the (covering) dimension of A_f is at least $n-k$.

The basic properties of the genus function γ are the following:

(i) (*Functoriality*) If $f : X \to Y$ is an equivariant map between spaces with involution, then $\gamma(X) \leq \gamma(Y)$.

(ii) (*Continuity*) If $A \subset X$ is closed and invariant, then $\gamma(A) = \gamma(U)$ for some closed invariant nbd U of A.

(iii) (*Normalization*) $\gamma(S^n) = n$.

Other numerical-valued functions satisfying the above properties have been studied by several authors and are referred to as *numerical-valued indices*. One such theory based on cohomology was introduced by Yang [1954] and Conner-Floyd [1960]. Let $(X;T)$ be a space with a fixed point free involution and for the double covering $X \to X/T$ denote by $c \in H^1(X/T)$ (Čech cohomology with Z_2 coefficients) its characteristic class. Yang [1954] and Conner–Floyd [1960] defined the cohomology index γ_c of (X,T) by $\gamma_c(X) = \sup\{n \mid c^n \neq 0\}$ and proved that γ_c has the properties of the genus function.

For related and more general results, see Conner–Floyd [1962], Holm Spanier [1970], Fadell–Husseini [1987], and also Fadell [1989] and Jaworowski [1989], where further references can be found.

The genus function Γ

The *genus* $\Gamma(X;T)$ of a pair $(X;T)$ as above is the smallest positive integer n for which one can find n closed sets $M_1, \dots, M_n$ such that $M_i \cap TM_i = \emptyset$ and $X = \bigcup_{i=1}^n M_i \cup TM_i$. Clearly for any $(X;T)$ we have $\Gamma(X;T) = \gamma(X;T) + 1$.

For every involution T on S^n we have $\Gamma(S^n;T) \geq n+1$ (Fet [1954]). More generally, if X is a compactum with Čech homology $H_k(X;Z_2) \cong H_k(S^n;Z_2)$ for $k \leq n$, then for every T, $\Gamma(X;T) \geq n+1$ (Jaworowski [1955]). Hence, for such spaces the generalizations of Theorems (5.2)1, (5.2)2 and (5.2)4 of §5 are valid. More general or related results can be found in Yang [1954], [1955], Jaworowski [1956], and Davies [1956]; see also Hopf [1944], G. Hirsch [1944], Geraghty [1961], Bacon [1966]. For uses of the genus in critical point theory see Krasnosel'skiĭ's monograph [1956].

Let X be a paracompact space on which a finite group G acts without fixed points. The *genus* $\Gamma(X;G)$ of X with respect to G is the smallest positive integer n for which one can find n closed sets $M_1,\ldots,M_n$ such that (i) $M_i \cap g(M_i) = \emptyset$ unless $g = 1$, and (ii) $X = \bigcup\{g(M_i) \mid i = 1,\ldots,n,\ g \in G\}$. This invariant was considered for $G = Z_p$ by Krasnosel'skiĭ [1955] and in full generality by Schwarz [1957].

The notion of genus can be considered in the following more general sense. Let $\pi : E \to X$ be a fibration (or more generally a surjective map). The *Schwarz genus* $\Gamma(\pi)$ of π is the minimal number of open sets U_i needed to cover X such that there are continuous $s_i : U_i \to E$ with $\pi \circ s_i = \mathrm{id}_{U_i}$. For more details or generalizations, see Schwarz [1961], [1962], and Steinlein [1980].

A surprising application of the Schwarz genus to problems in computer science was given by Smale [1987].

Genus and critical point theory

Let E be a Banach space and $\mathscr{G}$ be the family of all closed symmetric sets $A \subset E - \{0\}$ (each equipped with the involution $a \mapsto -a$). If $\Phi : E \to \boldsymbol{R}$ is an even C^1 function, one way of finding critical points of Φ is the Lusternik–Schnirelmann method of minimaxing Φ over certain subfamilies of $\mathscr{G}$. Specifically, letting $S = \{x \in E \mid \|x\| = 1\}$, consider for each $j \geq 1$ the family

$$G_j = \{A \in \mathscr{G} \mid A \subset S,\ \gamma(A) \geq j\}$$

and define

$$c_j = \inf_{A \in G_j} \sup_{a \in A} \Phi(a).$$

Under suitable conditions on Φ, it can be shown that each c_j is a critical value of Φ, and if $c_j = c_{j+1} = \cdots = c_{j+p} = c$ for some $p \geq 0$, then $\gamma(K_c) \geq p+1$, where $K_c = \{x \in E \mid \Phi(x) = c,\ \Phi'(x) = 0\}$; this implies that either Φ has $p+1$ distinct critical points or K_c is uncountable. It can also be shown that if $\Phi_c = \{x \in S \mid \Phi(x) = c,\ \Phi'(x) = 0\}$, then $c_j = \inf\{c \in \boldsymbol{R} \mid \gamma(\Phi_c) \geq j\}$. For details, the reader is referred to the book of Mawhin–Willem [1989], Rabinowitz's lecture notes [1984], and Szulkin [1989].

IV.

Leray–Schauder Degree and Fixed Point Index

This chapter is devoted to the concept of the topological degree and the fixed point index. With the aid of some fairly elementary facts from linear algebra and simplicial topology, we develop first the theory in the simple setting of Euclidean space. Then, using some of the techniques developed in Chapter II, we extend the index in $\boldsymbol{R}^n$ to infinite dimensions, and establish the fixed point index theory for compact maps in arbitrary metric ANRs. As a special case we also obtain the Leray–Schauder degree for compact fields in normed linear spaces. The chapter ends with a number of applications.

§10. Topological Degree in $\boldsymbol{R}^n$

Our aim in this paragraph is to establish in an elementary manner the definition and main properties of the topological degree in $\boldsymbol{R}^n$.

Let U be a bounded open set in $\boldsymbol{R}^n$, and let $C(\overline{U}, \boldsymbol{R}^n)$ be the space of all continuous maps $f : \overline{U} \to \boldsymbol{R}^n$ taken with the sup metric. In general terms, the (Brouwer) *degree problem* is to assign an integer $d(f, U)$ to each $f \in C(\overline{U}, \boldsymbol{R}^n)$ which indicates the minimal number of zeros that f, and all functions sufficiently close to f, must have.

Clearly, local constancy of $f \mapsto d(f, U)$ is not possible if the functions are allowed to have zeros on the boundary ∂U of U, so attention must be restricted to the (open) subspace $C_0(\overline{U}, \boldsymbol{R}^n) = \{f \mid f : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})\}$. The development then starts by choosing a dense set $\mathscr{A} \subset C_0(\overline{U}, \boldsymbol{R}^n)$ such that each $\phi \in \mathscr{A}$ has only finitely many zeros in U. Counting these zeros algebraically (examples such as $x \mapsto x^2$ on $(-1, 1)$ show that a simple count need not be locally constant) gives a degree function that turns out to be locally constant on the subspace $\mathscr{A}$. Defining the degree $d(f, U)$ for each $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ to be the degree of any sufficiently close approximation $\phi \in \mathscr{A}$ to f then yields the required degree function.

In the analytic approach, $\mathscr{A}$ is usually taken to be the C^∞ functions having 0 as a regular value, and this requires some knowledge about C^∞ approximation of continuous functions, and of Sard's theorem. In our approach, we take $\mathscr{A}$ to consist of piecewise linear maps; the development then requires little more than fairly simple linear algebra, and generally known facts from PL topology.

1. PL Maps of Polyhedra

In this section we describe the fundamental notions and obtain the basic approximation theorem. We recall that given an n-simplex $\sigma^n = (p_0, \dots, p_n) = \{\sum_{i=0}^n \lambda_i p_i \mid \sum_{i=0}^n \lambda_i = 1,\ 0 \le \lambda_i \le 1,\ i = 0, \dots, n\}$ in some $\boldsymbol{R}^s$, a map $\phi : \sigma^n \to \boldsymbol{R}^k$ is called *affine* if $\phi(\sum_{i=0}^n \lambda_i p_i) = \sum_{i=0}^n \lambda_i \phi(p_i)$.

A *polyhedron* $K \subset \boldsymbol{R}^s$ is a union of a finite number of simplices, where the intersection of any two simplices is either a common face or is empty; we let K^q denote the *q-skeleton* of K, i.e., the subpolyhedron consisting of all the simplices of dimension $\le q$. By a *pair* (K, Q) *of polyhedra* in $\boldsymbol{R}^s$ is meant a polyhedron $K \subset \boldsymbol{R}^s$ together with its subpolyhedron $Q \subset K$. We say that such a pair (K, Q) is *special* if for any simplex $\sigma^m = (p_0, p_1, \dots, p_m)$ of K with all p_i in Q^0 we have $\sigma^m \in Q$; note that for any pair (K, Q) its barycentric subdivision $(\widehat{K}, \widehat{Q})$ is always special.

(1.1) DEFINITION. Let $K \subset \boldsymbol{R}^s$ be a polyhedron. A map $\phi : K \to \boldsymbol{R}^n$ is called PL if it is affine on each simplex, and PL^n if it is also a homeomorphism (onto its image) on each simplex of dimension $\le n$.

By slightly moving the images of vertices and placing them in general position, any PL map can be approximated by a PL^n map; in fact, we need a more precise version of this result.

(1.2) LEMMA. *Let (K, Q) be a special pair of polyhedra in $\boldsymbol{R}^s$, and let $f : K \to \boldsymbol{R}^n$ be a* PL *map that is* PL^n *on Q. Then, for each $\varepsilon > 0$, there is a* PL^n *map $\phi : K \to \boldsymbol{R}^n$ with $\phi|Q = f|Q$ and $\|f - \phi\| < \varepsilon$.*

PROOF. For any finite set $S \subset \boldsymbol{R}^n$, let $\mathcal{F}(S)$ be the union of the flats spanned by all subsets of $k \le n$ elements of S; by Baire's theorem, $\mathcal{F}(S)$ has an empty interior (everything being $\le n-1$ dimensional). Consider the vertices $p_1, \dots, p_t$ of K not in Q. Choose $y_1 \notin \mathcal{F}[f(Q^0)]$ with $\|y_1 - f(p_1)\| < \varepsilon$, and proceeding recursively, choose

$$y_k \notin \mathcal{F}[f(Q^0) \cup \{y_1, \dots, y_{k-1}\}] \quad \text{with } \|y_k - f(p_k)\| < \varepsilon,\ k = 2, \dots, t.$$

Defining

$$\begin{aligned} \phi(q_j) &= f(q_j), & q_j &\in Q^0, \\ \phi(p_i) &= y_i, & i &= 1, \dots, t, \end{aligned}$$

and extending affinely over each simplex gives the desired map ϕ: indeed, for any σ^l with $l \leq n$, the images of the vertices do not lie on any k-flat with $k < l$, so $\phi|\sigma^l$ is a homeomorphism; and if $x = \sum_{i=0}^{k} \lambda_i r_i$ is any point of $\sigma^k = (r_0, \ldots, r_k) \in K$, then $\|f(x) - \phi(x)\| = \|\sum_{i=0}^{k} \lambda_i [f(r_i) - \phi(r_i)]\| < \varepsilon$.□

More important for our purposes is the PL^n approximation of arbitrary continuous maps:

(1.3) THEOREM. *Let (K, Q) be a pair of polyhedra in $\boldsymbol{R}^s$, and $f : K \to \boldsymbol{R}^n$ a continuous map that is* PL^n *on Q. Then, for each $\varepsilon > 0$, there is a subdivision $\widehat{K}$ of K and a* PL^n *map $\psi : \widehat{K} \to \boldsymbol{R}^n$ with $\psi|Q = f|Q$ and $\|\psi - f\| < \varepsilon$.*

PROOF. Since K is compact, uniform continuity of f gives a $\delta > 0$ with $\|f(x) - f(y)\| < \varepsilon/2$ whenever $\|x - y\| < \delta$. Subdivide K barycentrically sufficiently many times to get $\widehat{K}$ with mesh $\widehat{K} < \delta$, set $\phi(\widehat{p}) = f(\widehat{p})$ for each vertex $\widehat{p}$ of $\widehat{K}$, and extend affinely over each simplex of $\widehat{K}$ to get a PL map $\phi : \widehat{K} \to \boldsymbol{R}^n$. If $\overline{\sigma}$ is a simplex of Q and $\widehat{\sigma} \subset \overline{\sigma}$ a simplex of $\widehat{K}$, the maps ϕ, f are both affine on $\overline{\sigma}$ and coincide on the vertices of $\widehat{\sigma}$, so $f|\widehat{\sigma} = \phi|\widehat{\sigma}$, and the $\widehat{\sigma}$ match up to give $\phi|Q = f|Q$. Finally, if $x \in \widehat{K}$ is in the interior of some simplex $(\widehat{p}_0, \ldots, \widehat{p}_l) \in \widehat{K}$, then $\|x - \widehat{p}_i\| < \delta$ for $i = 0, \ldots, l$, so $\|f(x) - \phi(x)\| = \|f(x) - \sum_{i=0}^{l} \lambda_i f(\widehat{p}_i)\| < \varepsilon/2$. Applying now Lemma (1.2) to ϕ gives a PL^n map ψ with $\|\psi - f\| < \varepsilon$ and $\psi|Q = f|Q$. □

(1.4) DEFINITION. Let K be a polyhedron and K^{n-1} its $(n-1)$-skeleton. For any continuous $f : K \to \boldsymbol{R}^n$, we call $f(K^{n-1})$ its set of *critical values*, and all the points in $\boldsymbol{R}^n - f(K^{n-1})$ its *regular values*.

With this terminology, we have the following PL-analog of Sard's theorem:

(1.5) THEOREM. *Let $K \subset \boldsymbol{R}^s$ be a polyhedron and $\phi : K \to \boldsymbol{R}^n$ a* PL *map. Then the set of regular values of ϕ is open and dense in $\boldsymbol{R}^n$.*

PROOF. The critical values of ϕ are contained in the union of the finitely many sets $\phi(\sigma^{n-1})$, $\sigma^{n-1} \in K$; since each has no interior, their union has no interior, and the proof is complete. □

In what follows, we will call a PL^n map $\phi : K \to \boldsymbol{R}^n$ *special* PL^n if 0 is a regular value of ϕ. Thus, a PL map $\phi : (K, K^{n-1}) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ is special PL^n if ϕ is a homeomorphism on each simplex of K^n

2. Polyhedral Domains in $\boldsymbol{R}^n$. Degree for Generic Maps

An open set $U \subset \boldsymbol{R}^s$ is called a *polyhedral domain* if its closure $\overline{U}$ is a finite polyhedron; we shall assume each polyhedral domain endowed with a fixed simplicial subdivision. We now define our main approximation class $\mathscr{A}$:

(2.1) DEFINITION. Let $U \subset \boldsymbol{R}^n$ be a polyhedral domain. A continuous map $f : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ is called *PL-generic* (or simply *generic*) if it is special PL^n on some subdivision of $\overline{U}$. We let

$$\mathscr{A}(\overline{U}, \boldsymbol{R}^n) = \{f \in C_0(\overline{U}, \boldsymbol{R}^n) \mid f \text{ is PL-generic}\}.$$

Theorem (1.3) leads to the basic approximation result:

(2.2) THEOREM. *If U is a polyhedral domain in $\boldsymbol{R}^n$, then the set $\mathscr{A}(\overline{U}, \boldsymbol{R}^n)$ of generic maps is dense in $C_0(\overline{U}, \boldsymbol{R}^n)$.*

PROOF. Starting with $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ and any $\varepsilon > 0$, apply (1.3) to get a PL^n map ϕ on some subdivision $\widehat{K}$ of $\overline{U}$ with $\|\phi - f\| < \varepsilon/2$; since the regular values of ϕ are open dense, there is a regular value y_0 with $\|y_0\| < \varepsilon/2$. Then $\phi - y_0$ is generic and $\|(\phi - y_0) - f\| < \varepsilon$; the proof is complete. □

We now consider the zeros of a generic map $\phi \in \mathscr{A}(\overline{U}, \boldsymbol{R}^n)$. Since $\dim \overline{U} = n$ and ϕ is special PL^n on some subdivision $\widehat{K}$ of $\overline{U}$, ϕ has only finitely many zeros, and at most one in the interior of each n-simplex $\sigma^n \in \widehat{K}$. We will count these algebraically, by determining for each $p \in \phi^{-1}(0)$ whether or not ϕ reverses the orientation of the n-simplex containing p in its interior.

To motivate the formula, recall that if $(u_1, \ldots, u_n)$ is an ordered basis determining an orientation of $\boldsymbol{R}^n$, an ordered n-simplex $\sigma^n = (p_0, \ldots, p_n) \subset \boldsymbol{R}^n$ is, by definition, oriented as $\boldsymbol{R}^n$ or in the opposite way, according as the nonsingular linear transformation

$$L : (u_1, \ldots, u_n) \mapsto (p_1 - p_0, \ldots, p_n - p_0)$$

has a positive or negative determinant. Let $((p_i)_1, \ldots, (p_i)_n) \in \boldsymbol{R}^n$ be the coordinates of the p_i. Then the determinant of L is

$$\det L = \begin{vmatrix} (p_1)_1 - (p_0)_1 & \cdots & (p_n)_1 - (p_0)_1 \\ \cdots & \cdots & \cdots \\ (p_1)_n - (p_0)_n & \cdots & (p_n)_n - (p_0)_n \end{vmatrix} = \begin{vmatrix} 1 & \cdots & 1 \\ (p_0)_1 & \cdots & (p_n)_1 \\ \cdots & \cdots & \cdots \\ (p_0)_n & \cdots & (p_n)_n \end{vmatrix}.$$

We abbreviate $\det L / |\det L|$ by $[\![p_0, \ldots, p_n]\!]$.

Now let $\phi : \sigma^n \to \boldsymbol{R}^n$ be an affine homeomorphism of the simplex σ^n; the product $[\![p_0, \ldots, p_n]\!] \cdot [\![\phi(p_0), \ldots, \phi(p_n)]\!]$ expresses then whether or not σ^n and $\phi(\sigma^n)$ have the same orientation. Since a transposition of the p_i changes the sign of both determinants, the product is clearly independent of the particular orientation chosen for $\boldsymbol{R}^n$, and of the order in which the vertices of the simplex are written.

We are now prepared to define the degree $d(\phi, U)$ for maps in the approximating class $\mathscr{A}$:

(2.3) DEFINITION. Let U be a polyhedral domain in $\boldsymbol{R}^n$ and $\phi : \overline{U} \to \boldsymbol{R}^n$ be generic. If $p \in \phi^{-1}(0)$ belongs to the interior of the n-simplex $(p_0, \dots, p_n)$, then the *local index* $J(\phi, p)$ of ϕ at p is

$$J(\phi,p) = [\![p_0,\dots,p_n]\!] \quad [\![\phi(p_0),\dots,\phi(p_n)]\!],$$

and the *degree* of ϕ on U is

$$d(\phi,U) = \sum\{J(\phi,p) \mid p \in \phi^{-1}(0)\}.$$

The above formulas allow explicit calculation of $J(\phi, p)$ and $d(\phi, U)$ for any generic ϕ; another equally direct description is given in the following

(2.4) THEOREM. *Let $U \subset \boldsymbol{R}^n$ be polyhedral and $\phi \in \mathscr{A}(\overline{U}, \boldsymbol{R}^n)$. Assume that $p \in \phi^{-1}(0)$ belongs to the interior of $\sigma^n = (p_0, \dots, p_n)$, and let T be the linear part of the affine homeomorphism $\phi|\sigma^n$. Then*

$$J(\phi,p) = \operatorname{sgn}\det T = (-1)^{\lambda},$$

where λ is the number of negative eigenvalues of T, each counted with its multiplicity.

PROOF. Let $(u_1, \dots, u_n)$ be a basis for $\boldsymbol{R}^n$. If L is the linear transformation sending $(u_1, \dots, u_n)$ to $(p_1 - p_0, \dots, p_n - p_0)$, then $T \circ L$ is linear, sending $(u_1, \dots, u_n)$ to $(\phi(p_1) - \phi(p_0), \dots, \phi(p_n) - \phi(p_0))$, so that

$$[\![\phi(p_0),\dots,\phi(p_n)]\!] = \operatorname{sgn}\det(T\circ L) = \operatorname{sgn}\det T \operatorname{sgn}\det L,$$

and therefore

$$\begin{aligned} J(\phi,p) &= [\![p_0,\dots,p_n]\!] \cdot [\![\phi(p_0),\dots,\phi(p_n)]\!] \\ &= (\operatorname{sgn}\det L)^2 \operatorname{sgn}\det T = \operatorname{sgn}\det T. \end{aligned}$$

Since T is nonsingular and has n nonzero eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicities) with $\det T = \prod_i \lambda_i$, and since complex eigenvalues occur in conjugate pairs, we find $\operatorname{sgn}\det T = (-1)^{\lambda}$. □

It is an important consequence of (2.4) that $J(\phi, p)$ depends only on the affine map $\phi|\sigma^n$, and not on any special simplex $(p_0, \dots, p_n)$ containing p in its interior that is used for the explicit calculation. Consequently, the degree of a generic map depends only on the local behavior of ϕ, and is independent of the particular subdivision used to calculate it. Moreover, the degree of a generic map is invariant under all sufficiently small translations.

(2.5) PROPOSITION. *Let $U \subset \boldsymbol{R}^n$ be polyhedral, and let $\phi : \overline{U} \to \boldsymbol{R}^n$ be generic. Then there is a neighborhood $V(0)$ of the origin in $\boldsymbol{R}^n$ such that for every $y \in V(0)$ the map $\phi - y$ is generic.*

PROOF. If K is a subdivision of $\overline{U}$ on which ϕ is special PL^n, let σ_i^n, $i = 1, \dots, N$, be the set of n-simplices of K having a zero of ϕ in their interior.

Noting that $\varepsilon_i = \operatorname{dist}(0, \phi(\partial\sigma_i^n)) > 0$ for $i = 1, \ldots, N$ and that

$$\eta = \operatorname{dist}\left[0, \phi\left(\overline{U} - \bigcup_{i=1}^{N} \sigma_i^n\right)\right] > 0,$$

we can take for $V(0)$ a ball of radius $\varepsilon < \min\{\varepsilon_1, \ldots, \varepsilon_N, \eta\}$. □

3. Local Constancy and Homotopy Invariance

Our next objective is to show that $d(\phi, U)$ is locally constant on $\mathscr{A}(\overline{U}, \boldsymbol{R}^n)$. This will follow easily after we show that $d(\phi, U)$ is invariant under homotopies $\phi : (\overline{U}, \partial U) \times \boldsymbol{I} \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$. The discussion of homotopy invariance is somewhat lengthy, and requires some attention to detail; but the effort gives, in fact, a more general version of homotopy invariance.

We start with the simple

(3.1) LEMMA. *Let $W \subset \boldsymbol{R}^{n+1}$ be a polyhedral domain, and $\phi : W \to \boldsymbol{R}^n$ be special* PL^n. *Then each component of $\phi^{-1}(0)$ is homeomorphic either to the unit interval $\boldsymbol{I}$ or to a circle; and in the first case, the component has both endpoints in ∂W*

PROOF. Assume that $\sigma^{n+1} \cap \phi^{-1}(0) \neq \emptyset$; because $\phi|\sigma^{n+1}$ is affine and $\dim \phi(\sigma^{n+1}) = n$, we find that $\dim(\phi|\sigma^{n+1})^{-1}(0) = 1$, so $(\phi|\sigma^{n+1})^{-1}(0)$ is a line segment. It cannot be contained in any n-face σ^n of σ^{n+1} because $\phi|\sigma^n$ is a homeomorphism, so it must join points in the interiors of two distinct n-faces of σ^{n+1}. If any of these n-faces is the face of some other σ_1^{n+1}, the segment continues into σ_1^{n+1}. Thus, the component will consist of a finite string of segments, each meeting the next at a point in the interior of some σ^n. Since $\phi^{-1}(0) \cap \sigma^{n+1}$ consists of at most one segment, the string can never cross itself. Thus, if the string returns to a previous point, the component is homeomorphic to a circle; otherwise, it is homeomorphic to $\boldsymbol{I}$. In the latter case, the component cannot have an endpoint c in the interior of W: for then c would be in the interior of some n-simplex σ^n and also the center of an $(n+1)$-ball B contained in W; if B^+, B^- are the intersections of B with the two open half-spaces determined by the linear span of σ^n, then both are nonempty $(n+1)$-dimensional subsets of W, which must therefore be contained in two distinct simplices having σ^n as face; and as we have seen, the component will continue across σ^n. □

We next develop a formula that traces the evolution of the local index as one moves along $\phi^{-1}(0)$.

Let $\sigma^n = (p_0, \ldots, p_n)$ be an ordered n-simplex in oriented $\boldsymbol{R}^{n+1}$; then, for each ξ not in the n-flat spanned by σ^n, the ordered simplex has the relative orientation $[p_0, \ldots, p_n, \xi]$, and we have the following simple

(3.2) PROPOSITION.

(a) $[\![p_0, \dots, p_n, \xi]\!] = \pm[\![p_0, \dots, p_n, \eta]\!]$ *according as* $\xi, \eta \in \boldsymbol{R}^{n+1}$ *are on the same side, or on different sides, of the n-flat* Σ *spanned by* $\sigma^n = (p_0, \dots, p_n)$.

(b) *If* $q_0, q_1, \dots, q_n$ *are all in a common hyperplane* $\boldsymbol{R}^n \times \{t\}$ *of* $\boldsymbol{R}^{n+1}$ *i.e.,* $q_i = (\widehat{q}_i, t)$, *and* $(\nu, \tau) \in \boldsymbol{R}^{n+1}$, $\tau \neq t$, *then*

$$[\![q_0, \dots, q_n, (\nu, \tau)]\!] = [\![\widehat{q}_0, \dots, \widehat{q}_n]\!] \cdot \operatorname{sgn}[\tau - t].$$

PROOF. (a) Let $(1-t)\xi + t\eta$, $0 < t < 1$, be the open line segment having ξ, η as endpoints. Then letting $((p_i)_1, \dots, (p_i)_{n+1}) \in \boldsymbol{R}^{n+1}$ be the coordinates of p_i for $i = 0, 1, \dots, n+1$, and $\det[p_0, \dots, p_n, p_{n+1}]$ be the determinant of the $(n+2) \times (n+2)$ matrix

$$\begin{vmatrix} 1 & \cdots & 1 \\ (p_0)_1 & \cdots & (p_{n+1})_1 \\ \cdots & \cdots & \cdots \\ (p_0)_n & \cdots & (p_{n+1})_n \\ (p_0)_{n+1} & \cdots & (p_{n+1})_{n+1} \end{vmatrix},$$

we have

$$\det[p_0, \dots, p_n, (1-t)\xi + t\eta] = (1-t)\det[p_0, \dots, p_n, \xi] + t\det[p_0, \dots, p_n, \eta],$$

so the values form an interval $(\widehat{\xi}, \widehat{\eta})$ in $\boldsymbol{R}$. Now, $\det[p_0, \dots, p_n, \alpha] = 0$ if and only if $p_0, \dots, p_n, \alpha$ are affinely dependent, i.e., if and only if $\alpha \in \Sigma$. Thus, if the line segment does not meet Σ, then $0 \notin (\widehat{\xi}, \widehat{\eta})$, so the signs at the endpoints are the same. If $(\xi, \eta) \cap \Sigma \neq \emptyset$, then $0 \in (\widehat{\xi}, \widehat{\eta})$, so that the endpoints have different signs.

(b) For $\det[(\widehat{q}_0, t), \dots, (\widehat{q}_n, t), (\nu, \tau)]$ we have

$$\begin{vmatrix} 1 & \cdots & 1 & 1 \\ (\widehat{q}_0)_1 & \cdots & (\widehat{q}_n)_1 & \nu_1 \\ \cdots & \cdots & \cdots & \cdots \\ (\widehat{q}_0)_n & \cdots & (\widehat{q}_n)_n & \nu_n \\ t & \cdots & t & \tau \end{vmatrix} = \begin{vmatrix} 1 & \cdots & 1 & 1 \\ (\widehat{q}_0)_1 & \cdots & (\widehat{q}_n)_1 & \nu_1 \\ \cdots & \cdots & \cdots & \cdots \\ (\widehat{q}_0)_n & \cdots & (\widehat{q}_n)_n & \nu_n \\ 0 & \cdots & 0 & \tau - t \end{vmatrix}$$
$$= (\tau - t)\det[\widehat{q}_0, \dots, \widehat{q}_n],$$

and the assertion (b) follows. □

Let $\sigma^n = (p_0, \dots, p_n) \subset \boldsymbol{R}^{n+1}$, and let $\phi : \sigma^n \to \boldsymbol{R}^n$ be an affine homeomorphism. Choosing a ξ not in the flat determined by σ^n, we introduce a measure that compares the orientation of $(p_0, \dots, p_n, \xi)$ in $\boldsymbol{R}^{n+1}$ to that of $(\phi(p_0), \dots, \phi(p_n))$ in $\boldsymbol{R}^n$ by means of

$$\Delta(\phi, \sigma^n; \xi) = [\![p_0, \dots, p_n, \xi]\!] \cdot [\![\phi(p_0), \dots, \phi(p_n)]\!],$$

a quantity independent of the order in which the vertices $(p_0, \ldots, p_n)$ of σ^n are written [1].

The main burden of the proof of the homotopy invariance of the degree will be carried by the following

(3.3) LEMMA. *Let $(p_0, \ldots, p_{n+1}) = \sigma^{n+1} \subset \boldsymbol{R}^{n+1}$, and let*

$$\phi : (\sigma^{n+1}, (\sigma^{n+1})^{n-1}) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$$

be PL^n *such that $\phi^{-1}(0) \neq \emptyset$. Then $\phi^{-1}(0)$ is a line segment joining points in 2 distinct n-faces, say*

$$a_0 \in \sigma_0 = (p_0, \ldots, p_n) \quad \textit{to} \quad a_1 \in \sigma_1 = (p_1, \ldots, p_{n+1}),$$

and

$$\Delta(\phi, \sigma_0; a_1) = -\Delta(\phi, \sigma_1; a_0).$$

PROOF. The points a_1 and p_{n+1} are in the same half-space determined by $(p_0, \ldots, p_n)$ because $a_1 \in (p_1, \ldots, p_{n+1})$ can be joined to p_{n+1} by a line segment in σ_1; and similarly for a_0 and p_0; so by (3.2)(a),

$$\begin{aligned}
\Delta(\phi, \sigma_0; a_1) &= [\![p_0, \ldots, p_n, a_1]\!] \cdot [\![\phi(p_0), \ldots, \phi(p_n)]\!] \\
&= [\![p_0, \ldots, p_n, p_{n+1}]\!] \ [\![\phi(p_0), \ldots, \phi(p_n)]\!] \\
&= (-1)^{n+1}[\![p_1, \ldots, p_{n+1}, p_0]\!](-1)^n [\![\phi(p_1), \ldots, \phi(p_n), \phi(p_0)]\!] \\
&= -[\![p_1, \ldots, p_{n+1}, a_0]\!] \ [\![\phi(p_1), \ldots, \phi(p_n), \phi(p_0)]\!].
\end{aligned}$$

Now, the $(n-1)$-flat Σ spanned by $(\phi(p_1), \ldots, \phi(p_n)) \subset \boldsymbol{R}^n$ does not contain 0; but both $(\phi(p_1), \ldots, \phi(p_n), \phi(p_0))$ and $(\phi(p_1), \ldots, \phi(p_n), \phi(p_{n+1}))$ do; therefore, $\phi(p_0)$ and $\phi(p_{n+1})$ are in the same half-space determined by Σ, which shows that

$$\begin{aligned}
\Delta(\phi, \sigma; a_1) &= -[\![p_1, \ldots, p_{n+1}, a_0]\!] \ [\![\phi(p_1), \ldots, \phi(p_n), \phi(p_{n+1})]\!] \\
&= -\Delta(\phi, \sigma_1; a_0).
\end{aligned}$$

□

Let W be a polyhedral domain in $\boldsymbol{R}^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}$ with $\overline{W} \subset \boldsymbol{R}^n \times \boldsymbol{I}$. We will assume that

$$\begin{aligned}
\overline{W}_0 &= \overline{W} \cap (\boldsymbol{R}^n \times \{0\}), \\
\overline{W}_1 &= \overline{W} \cap (\boldsymbol{R}^n \times \{1\})
\end{aligned}$$

are both nonempty, and call the closure of $\partial W - (\overline{W}_0 \cup \overline{W}_1)$ the *vertical boundary* $\widehat{\partial} W$ of W. With this notation, the basic result is:

(3.4) THEOREM. *Let $\Phi : (\overline{W}, \widehat{\partial} W) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ be continuous and special* PL^n *on $\overline{W}_0 \cup \overline{W}_1$. Then $d(\Phi|\overline{W}_0, W_0) = d(\Phi|\overline{W}_1, W_1)$.*

---

(1) For an interpretation of $\Delta(\phi, \sigma^n; \xi)$ as the intersection number of the simplex and the oriented line $[\sigma^n]\xi$ (where $[\sigma^n]$ is the barycenter of σ^n) see the monograph by Alexandroff and Hopf [1935].

PROOF. We can assume that Φ is PL^n with regular value 0. Indeed, define $\varepsilon = \operatorname{dist}(0, \Phi(\widehat{\partial} W)) > 0$ and observe that $(\overline{W}, \overline{W}_0 \cup \overline{W}_1)$ is an n-homogeneous pair of polyhedra. Consequently, by our approximation theorem (1.3), there is a PL^n map $H : \overline{W} \to \boldsymbol{R}^n$ with $\|H - \Phi\| < \varepsilon/4$ and $H|\overline{W}_0 = \Phi|\overline{W}_0 \equiv \phi_0$, $H|\overline{W}_1 = \phi|\overline{W}_1 \equiv \phi_1$. Now, by (2.5), there is a neighborhood $V(0)$ such that $d(\phi_i - y, W_i) = d(\phi_i, W_i)$, $i = 0, 1$, for all $y \in V(0)$, and since the regular values of H are dense, $V(0)$ contains a regular value $\widehat{y}$ of H with $\|\widehat{y}\| < \varepsilon/4$. Then $H - \widehat{y}$ is PL^n with regular value 0, $d(\phi_i - \widehat{y}, W_i) = d(\phi_i, W_i)$, $i = 0, 1$, and because $\|H - \widehat{y} - \Phi\| < \varepsilon/2$, the map $H - \widehat{y}$ has no zeros on $\widehat{\partial} W$ Thus, we can indeed assume that Φ is PL^n with regular value 0.

To count zeros, we need to consider only the components of $\Phi^{-1}(0)$ that meet $\overline{W}_0 \cup \overline{W}_1$: any component that does not meet these sets will not give points affecting the calculation of $d(\phi_0, W_0)$ or $d(\phi_1, W_1)$.

Let then L be a component of $\Phi^{-1}(0)$ containing $a_0 \in \operatorname{Int} \sigma_0^n \subset \overline{W}_0$. In view of (3.1), this component cannot be homeomorphic to a circle since the $(n+1)$-simplex having σ_0^n as a face would then contain two distinct line segments of L. Thus, L must be homeomorphic to $\boldsymbol{I}$, having one endpoint a_0 and the other on ∂W; since Φ has no zero on $\widehat{\partial} W$, the latter endpoint must therefore be either on $\overline{W}_0$ or $\overline{W}_1$.

Suppose L starts at $a_0 \in W_0$, where a_0 is in the interior of the n-simplex σ_0^n, and meets successively the n-simplices $\sigma_1^n, \dots, \sigma_k^n$ at points $a_i \in \operatorname{Int} \sigma_i^n$, where $\sigma_i^n, \sigma_{i+1}^n$ are faces of some $(n+1)$-simplex $\sigma_{i,i+1}^{n+1}$. The quantity

$$\Delta(\Phi, \sigma_i^n; a_{i+1}) = [\![p_0^i, \dots, p_n^i, a_{i+1}]\!] \cdot [\![\Phi(p_0^i), \dots, \Phi(p_n^i)]\!]$$

is constant along L, since by (3.3),

$$\Delta(\Phi, \sigma_i^n; a_{i+1}) = -\Delta(\Phi, \sigma_{i+1}^n; a_i) = \Delta(\Phi, \sigma_{i+1}^n; a_{i+2})$$

because a_i, a_{i+2} are in opposite half-spaces determined by σ_{i+1}^n.

At the starting point, we have

$$\begin{aligned} \Delta(\Phi, \sigma_0^n; a_1) &= [\![(p_0, 0), \dots, (p_n, 0), a_1]\!] \cdot [\![\phi_0(p_0), \dots, \phi_0(p_n)]\!] \\ &= [\![p_0, \dots, p_n]\!] \cdot [\![\phi_0(p_0), \dots, \phi_0(p_n)]\!] = J(\phi_0, a_0) \end{aligned}$$

because a_1 is above the hyperplane $\boldsymbol{R}^n \times \{0\}$ carrying the vertices of σ_0. We examine the other endpoint of L:

CASE 1: L ends on $\boldsymbol{R}^n \times \{1\}$. Then

$$J(\phi_0, a_0) = \Delta(\Phi, \sigma_0^n; a_1) = \cdots = \Delta(\Phi, \sigma_{k-1}^n; a_k) = -\Delta(\Phi, \sigma_k^n; a_{k-1});$$

as a_{k-1} is below the hyperplane carrying the vertices of σ_k, (3.2)(b) yields

$$-\Delta(\Phi, \sigma_k^n; a_{k-1}) = -(-J(\phi_1, a_k)) = J(\phi_1, a_k).$$

the same as the index of ϕ_0 at a_0.

CASE 2: L ends on $\boldsymbol{R}^n \times \{0\}$. Then

$$J(\phi_0, a_0) = \Delta(\Phi, \sigma^n_{k-1}; a_k) = -\Delta(\Phi, \sigma^n_k; a_{k-1});$$

this time a_{k-1} is above the hyperplane carrying the vertices of σ_k, so

$$J(\phi_0, a_0) = -J(\phi_0, a_k),$$

canceling the contribution to the index of ϕ_0 made by a_0.

The argument is clearly reversible, starting from $\overline{W}_1$. Thus, only the zeros on components L running from $\overline{W}_0$ to $\overline{W}_1$ will contribute to the degrees of ϕ_0, ϕ_1, and each gives the same amount; therefore, $d(\phi_0, W_0) = d(\phi_1, W_1)$, and the proof is complete. □

4. Degree for Continuous Maps

We are now ready to define the degree $d(f, U)$ for each $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ by approximating f with elements of $\mathscr{A}(\overline{U}, \boldsymbol{R}^n)$. However, before proceeding, it is convenient to state explicitly a simple general result that is frequently used:

(4.1) PROPOSITION. *Let $f : X \to \boldsymbol{R}^n$ be a continuous map, and let $A \subset X$ be such that* $\operatorname{dist}(0, f(A)) = \alpha > 0$. *If $g : X \to \boldsymbol{R}^n$ satisfies $\|g - f\| < \alpha/2$, then:*
 (i) *$g(A)$ does not contain* 0,
 (ii) *the maps $f, g : (X, A) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ are homotopic.*

PROOF. (i) This follows at once by noting that for $a \in A$, we have $\|g(a)\| \geq \|f(a)\| - \|f(a) - g(a)\| \geq \alpha/2$.

(ii) For any $a \in A$, we have $g(a) \in B(f(a), \alpha/2)$; putting

$$h_t(x) = (1 - t)f(x) + tg(x)$$

we have $h_t(a) \in B(f(a), \alpha/2)$ for any $a \in A$, and because $0 \notin B(f(A), \alpha/2)$ we conclude that $h_t : (X, A) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ is a homotopy joining f and g. □

(4.2) DEFINITION. Let U be a polyhedral domain in $\boldsymbol{R}^n$, and let the map $f : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ be continuous. Choose a generic $\phi \in \mathscr{A}(\overline{U}, \boldsymbol{R}^n)$ such that $\|f - \phi\| < \frac{1}{2} \operatorname{dist}(0, f(\partial U))$. The *degree* of f on U is $d(f, U) = d(\phi, U)$.

By (2.2) such ϕ exist. We must show that $d(f, U)$ is independent of the particular ϕ selected. Choose another $\psi \in \mathscr{A}(\overline{U}, \boldsymbol{R}^n)$ satisfying the condition, and let K, K' be subdivisions of $\overline{U}$ on which ϕ, ψ are special PL^n. It is known that there is a simplicial subdivision of the standard polyhedron $\overline{U} \times \boldsymbol{I}$ that coincides with K on $\overline{U} \times \{0\}$ and with K' on $\overline{U} \times \{1\}$: working with the simplices of $\overline{U} \times \boldsymbol{I}$ that do not lie in $\overline{U} \times \{0\}$ or in $\overline{U} \times \{1\}$,

proceed by induction on dimension, subdividing each such σ^{k+1} into the simplices $(b, y_0, \dots, y_k)$, where b is the barycenter of σ^{k+1} and $(y_0, \dots, y_k)$ runs through all the k-simplices found on the boundary of σ^{k+1}. Taking $\overline{U} \times \boldsymbol{I}$ with this subdivision, and setting $\Phi(x,t) = (1-t)\phi(x) + t\psi(x)$, we find that Φ is continuous on $\overline{U} \times \boldsymbol{I}$ and special PL^n on $U \times \{0\} \cup U \times \{1\}$; since

$$\begin{aligned}\|f(x) - \Phi(x,t)\| &\le (1-t)\|f(x) - \phi(x)\| + t\|f(x) - \psi(x)\| \\ &< \frac{1}{2}\,\mathrm{dist}(0, f(\partial U)),\end{aligned}$$

it follows from (4.1) that Φ has no zeros on $\partial U \times \boldsymbol{I}$, and therefore, by (3.4), that $d(\phi, U) = d(\psi, U)$ as required.

If $\alpha = d(0, f(\partial U))$, we note that the same generic ϕ can be used to approximate all maps in the ball $B(f, \alpha/8)$; this leads to the main theorem on the topological degree:

(4.3) THEOREM. *Let U be a polyhedral domain in $\boldsymbol{R}^n$. Then $f \mapsto d(f,U)$ defines a locally constant (hence continuous) function $d : C_0(\overline{U}, \boldsymbol{R}^n) \to \boldsymbol{Z}$ satisfying:*

(1) (Strict normalization) *If $T : \overline{U} \to \boldsymbol{R}^n$ is an affine homeomorphism and $T(u) = 0$ for some $u \in U$, then $d(T,U) = (-1)^\lambda$, where λ is the number of negative eigenvalues of the linear part of T, each counted with its multiplicity.*

(2) (Additivity) *For any pair of disjoint polyhedral domains $U_1, U_2 \subset U$ and for any $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ having all its zeros in $U_1 \cup U_2$ we have*

$$d(f,U) = d(f,U_1) + d(f,U_2).$$

(3) (Homotopy) *If $H : (\overline{U}, \partial U) \times \boldsymbol{I} \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ is a homotopy, then $d(H|\overline{U} \times \{0\}, U) = d(H|\overline{U} \times \{1\}, U)$.*

PROOF. We have already proved (1).

For (2), let $f_i = f|\overline{U}_i$, $i = 1, 2$, and

$$\alpha = \mathrm{dist}(0, f(\overline{U} - (U_1 \cup U_2))).$$

Let ϕ be a generic map with $\|\phi - f\| < \alpha/2$. Since the $\overline{U}_i$ are subpolyhedra, the maps $\phi_i = \phi|\overline{U}_i$ are generic and approximate f_i, and since all the zeros are contained in $U_1 \cup U_2$, we have $d(\phi, U) = d(\phi_1, U_1) + d(\phi_2, U_2)$, which gives the result.

(3) is immediate from (3.4): using sufficiently close generic approximations ϕ, ψ of $h_0 = H|\overline{U} \times \{0\}$, $h_1 = H|\overline{U} \times \{1\}$, respectively, we have $\phi \simeq h_0$, $\psi \simeq h_1$, and since $\phi \simeq \psi$, our assertion follows.

REMARK. In fact, a more general version of the homotopy invariance is valid: $d(f, U)$ is invariant under simultaneous continuous deformations of both f and U, provided that no zeros appear on the boundaries of the regions deformed.

It is a simple matter to see that the properties in (4.3) characterize the degree function:

(4.4) THEOREM. *Let U be a polyhedral domain in $\boldsymbol{R}^n$ If $\widehat{d} : C_0(\overline{U}, \boldsymbol{R}^n) \to \boldsymbol{Z}$ is any function with the strict normalization, additivity, and homotopy properties, then $\widehat{d}(f, U) = d(f, U)$.*

PROOF. We use the conditions to calculate $\widehat{d}(f, U)$. By (2.2) and (4.1), a linear homotopy shows that $\widehat{d}(f, U) = \widehat{d}(\phi, U)$ for some special PL^n map $\phi \in \mathscr{A}(\overline{U}, \boldsymbol{R}^n)$. Then $\phi^{-1}(0)$ is a finite set $\{u_1, \ldots, u_k\}$ and choosing disjoint open simplices with $u_k \in \operatorname{Int} \sigma_i^n$, $i = 1, \ldots, k$, gives by additivity that

$$\widehat{d}(\phi, U) = \sum_{i=1}^{k} \widehat{d}(\phi, \operatorname{Int} \sigma_i^n);$$

strict normalization shows

$$\widehat{d}(\phi, \operatorname{Int} \sigma_i^n) = d(\phi, \operatorname{Int} \sigma_i^n). \qquad \square$$

It is an immediate consequence that our $d(f, U)$, as defined above, is independent of the simplicial subdivision with which U is taken, i.e., it is a topological, rather than combinatorial, invariant.

REMARK. Note that the set of conditions in (4.3) is not an independent set of axioms for the degree function: strict normalization can be replaced simply by $d(\mathrm{id}, U) = +1$ if $0 \in U$. A proof that the resulting axioms are independent, and imply (4.3)(1), will be given later.

5. Some Properties of Degree

In this section we present some further features of the degree. They are all immediate consequences of Theorem (4.3), so that they follow once the properties given there are established in some way. We will give proofs by formal deduction from (4.3), to emphasize this dependence, even when there are more direct proofs (even trivial) from the way we have defined $d(f, U)$.

By taking $U = U_1 = U_2 = \emptyset$, the additivity property implies formally first that $d(f, \emptyset) = 0$ and then the very useful

(5.1) THEOREM (Excision). *If $V \subset U \subset \boldsymbol{R}^n$ are polyhedral domains and $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ has all its zeros in V, then $d(f, U) = d(f, V)$.*

Proof. The zeros of f all being in the disjoint sets V and $\emptyset$, additivity gives the result. □

It is an immediate consequence of the excision property that if $f(U) \subset \boldsymbol{R}^n - \{0\}$, then $d(f,U) = d(f,\emptyset) = 0$; this leads to the important result that underlies many applications of degree theory in analysis.

(5.2) Theorem (Existence property). *Let $U \subset \boldsymbol{R}^n$ be a polyhedral domain and $f \in C_0(\overline{U}, \boldsymbol{R}^n)$. If $d(f,U) \neq 0$, then $f(U)$ contains a neighborhood of* 0.

Proof. From the preceding remarks, if $d(f,U) \neq 0$, then the equation $f(u) = 0$ has a solution in U. By local constancy, $d(g,U) \neq 0$ for all g sufficiently close to f, so for some $\varepsilon > 0$, all the functions $g = f - y$ with $\|y\| < \varepsilon$ have zeros in U, and this proves the theorem. □

Another important property of the degree is given in the following

(5.3) Theorem (Multiplicativity). *Let $U_1 \subset \boldsymbol{R}^n$ and $U_2 \subset \boldsymbol{R}^k$ be polyhedral domains, and let $f_1 \in C_0(\overline{U}_1, \boldsymbol{R}^n)$, $f_2 \in C_0(\overline{U}_2, \boldsymbol{R}^k)$. Then $f_1 \times f_2 \in C_0(\overline{U}_1 \times \overline{U}_2, \boldsymbol{R}^{n+k})$ and*

$$d(f_1 \times f_2, U_1 \times U_2) = d(f_1, U_1) \cdot d(f_2, U_2).$$

Proof. By local constancy of the degree function, we can take a sufficiently close special PL^n (respectively special PL^k) approximation ϕ_i of f_i with $d(f_i, U_i) = d(\phi_i, U_i)$, $i = 1, 2$, and with $\phi_1 \times \phi_2$ having the same degree as $f_1 \times f_2$. If Z_i is the zero set of ϕ_i, then $Z_1 \times Z_2$ is the zero set of $\phi_1 \times \phi_2$; we will calculate the local index of $\phi_1 \times \phi_2$ at each zero $x_i \times y_j$.

Because $\phi_1 \times \phi_2$ is affine whenever the ϕ_i are both affine, if we let L_1 be the linear part of ϕ_i on the simplex σ_i^n containing x_i in its interior, and L_2 that of ϕ_2 on the simplex σ_j^k containing y_j, then by (2.4) the index is determined by the sign of $\det(L_1 \times L_2)$. Choosing a basis $(e_1, \ldots, e_n, \ldots, e_{n+k})$ for $\boldsymbol{R}^{n+k}$ such that $(e_1, \ldots, e_n)$ is a basis for $\boldsymbol{R}^n$, and $(e_{n+1}, \ldots, e_{n+k})$ one for $\boldsymbol{R}^k$, we find that $L_1 \times L_2$ is expressible in the form $\left(\begin{smallmatrix} L_1 & 0 \\ 0 & L_2 \end{smallmatrix}\right)$, so

$$\det(L_1 \times L_2) = \det L_1 \cdot \det L_2,$$

and therefore

$$J(\phi_1 \times \phi_2, x_i \times y_j) = J(\phi_1, x_i) \cdot J(\phi_2, y_j).$$

Summing over all $x_i \in Z_1$ gives, for each y_j,

$$\sum_{x_i \in Z_1} J(\phi_1 \times \phi_2, x_i \times y_j) = \Big[\sum_{x_i \in Z_1} J(\phi_1, x_i)\Big] J(\phi_2, y_j)$$
$$= d(\phi_1, U_1) \cdot J(\phi_2, y_j),$$

and now summing over all y_j gives the required formula. □

6. Extension to Arbitrary Open Sets

We are now going to remove the requirement that the open set U be polyhedral; indeed, we will not require U to be bounded, or even that the functions considered be defined on ∂U.

The starting point is to note that given a polyhedral domain U and $f \in C_0(\overline{U}, \boldsymbol{R}^n)$, the zero set $Z(f) = f^{-1}(0)$ (being closed and bounded by continuity of f) is a compact subset of the open set U. This description makes no explicit reference to $f|\partial U$. Guided by this, we make the following

(6.1) DEFINITION. Let $W \subset \boldsymbol{R}^n$ be any open set. A continuous map $f : W \to \boldsymbol{R}^n$ is called *compactly rooted* in W if $Z(f)$ is compact. The set of all compactly rooted maps on W is denoted by $C_0(W, \boldsymbol{R}^n)$.

Even if W is a polyhedral domain, the set $C_0(W, \boldsymbol{R}^n)$ is somewhat larger than $C_0(\overline{W}, \boldsymbol{R}^n)$, since continuity on ∂W is not required. We are going to define a degree function $d : C_0(W, \boldsymbol{R}^n) \to \boldsymbol{Z}$.

Since $Z(f) \subset W$ is compact, there is always a polyhedral domain U with $Z(f) \subset U \subset \overline{U} \subset W$: the distance between the compact $Z(f)$ and the closed $\boldsymbol{R}^n - W$ is greater than some positive α, so taking a cubical grating of $\boldsymbol{R}^n$ with mesh $< \alpha/2$ and keeping only the n-cubes that meet $Z(f)$ provides such a domain.

(6.2) DEFINITION. Let $W \subset \boldsymbol{R}^n$ be any open set. Given an $f \in C_0(W, \boldsymbol{R}^n)$, choose any polyhedral domain U with $Z(f) \subset U \subset \overline{U} \subset W$ and define the degree of f on W by $d(f, W) = d(f, U)$.

The value $d(f, W)$ is independent of the polyhedral domain selected: if V is another such polyhedral domain, then since the intersection of two such domains is also a polyhedral domain (the intersections $\sigma_U^n \cap \sigma_V^n$ are all convex, so the corresponding complex has a simplicial subdivision), excision gives $d(f, U) = d(f, U \cap V) = d(f, V)$.

The requirement that each H_t be compactly rooted is too weak to give homotopy invariance of the degree: for example, with $W = (0, 1) \subset \boldsymbol{R}^1$ and $H(w, t) = w - (t + 1/2)$, each H_t is compactly rooted in W, but $d(H_0, W) = 1$, whereas $d(H_1, W) = 0$. The appropriate requirement is that the homotopy H be compactly rooted in $W \times \boldsymbol{I}$. This can be stated in various equivalent ways:

(6.3) PROPOSITION. *Let $H : W \times \boldsymbol{I} \to \boldsymbol{R}^n$ be continuous. The following statements are equivalent:*

(1) *$H^{-1}(0)$ is compact (i.e., H is compactly rooted in $W \times \boldsymbol{I}$).*

(2) *$P_W H^{-1}(0)$ is compact, where $P_W : W \times \boldsymbol{I} \to W$ is the projection.*

(3) *$\bigcup \{Z(H_t) \mid 0 \leq t \leq 1\}$ is compact.*

PROOF. That (1)⇒(2) is obvious; since the set in (3) is exactly $P_W H^{-1}(0)$, it is also clear that (2)⇔(3). It remains to show that (2)⇒(1). For this, note that $H^{-1}(0)$ is closed in the product of compact sets $P_W H^{-1}(0)$ and $\boldsymbol{I}$. □

Note that if $H : W \times \boldsymbol{I} \to \boldsymbol{R}^n$ is compactly rooted, then because each $Z(H_t)$ is closed, it follows from (6.3)(3) that each H_t is compactly rooted in W.

We can now establish the general version of Theorem (4.3). Note also that if W is bounded and $H : \overline{W} \times \boldsymbol{I} \to \boldsymbol{R}^n$, then H is compactly rooted in $W \times \boldsymbol{I}$ if and only if $H|\partial W \times \boldsymbol{I}$ has no zeros.

(6.4) THEOREM. *Let W be any open set in $\boldsymbol{R}^n$. Then $f \mapsto d(f, W)$ defines a function $d : C_0(W, \boldsymbol{R}^n) \to \boldsymbol{Z}$ satisfying:*
(1) Strict normalization
(2) Additivity
(3) (Homotopy) *If $H : W \times \boldsymbol{I} \to \boldsymbol{R}^n$ is a compactly rooted homotopy, then $d(H|W \times \{0\}, W) = d(H|W \times \{1\}, W)$.*

PROOF. Properties (1) and (2) are obvious from the definitions. For (3), choose a polyhedral domain U with $H^{-1}(0) \subset U \times \boldsymbol{I}$ and $U \subset \overline{U} \subset W$; apply (4.3). □

7. Axiomatics

We have seen in (4.4) that the strict normalization, additivity, and homotopy properties characterize the degree; and it was pointed out that these conditions are not independent. Specifically, we are going to show that using the additivity and homotopy properties, one can deduce the strict normalization condition (4.3)(1) from the following simpler axiom:

(1′) (Normalization) *If $0 \in U$ and* $\mathrm{id} : U \to U$, *then* $d(\mathrm{id}, U) = 1$.

Let $\boldsymbol{R}^s = A \oplus B$ be a direct sum decomposition, and let π_A, π_B be the projections. Let $U \subset \boldsymbol{R}^s$ be an open set containing the origin; we are going to calculate $d(-\pi_A + \pi_B, U)$ using (1′) and the remaining properties; (4.3)(1) will then follow easily.

(7.1) PROPOSITION. *If* $\dim A$ *is even, then*

$$d(-\pi_A + \pi_B, U) = +1.$$

PROOF. Choose a basis $e_1, \dots, e_{2n}$ for A and extend it to a basis for $\boldsymbol{R}^s$ Define $J : A \to A$ by $J(e_{2i-1}) = e_{2i}$, $J(e_{2i}) = -e_{2i-1}$; then J is a nonsingular linear transformation with $J^2 = -\mathrm{id}_A$. We now show that each of the

homotopies

$$H_0(x,t) = (1-t)[-\pi_A(x)] + tJ\pi_A(x) + \pi_B(x),$$
$$H_1(x,t) = (1-t)J\pi_A(x) + t\pi_A(x) + \pi_B(x),$$

of $-\pi_A + \pi_B$ to $J\pi_A + \pi_B$ to $\pi_A + \pi_B$ is compactly rooted in $U \times \boldsymbol{I}$. Only the proof for H_0 is given, the argument for H_1 being similar.

Assume that $H_0(x,t) = 0$; then $\pi_B(x) = 0$, and it is enough to show that $\pi_A(x) = 0$ (so that $x = 0$). We have $(1-t)[-\pi_A(x)] + tJ\pi_A(x) = 0$. If $t = 0$, then $-\pi_A(x) = 0$; if $t = 1$, then $J\pi_A(x) = 0$ and since J is nonsingular, we get $\pi_A(x) = 0$. Finally, if $0 < t < 1$, then $\pi_A(x) = \frac{t}{1-t}J\pi_A(x)$, so $J\pi_A(x) = \frac{t}{1-t}(-\pi_A x) = -(\frac{t}{1-t})^2 J\pi_A(x)$, showing once again that $\pi_A(x) = 0$. The two homotopies show that $-\pi_A + \pi_B$ is homotopic to $\pi_A + \pi_B = \mathrm{id}$, and axiom (1′) above gives our result. □

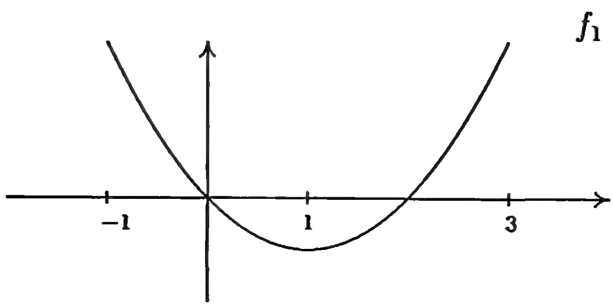
(7.2) PROPOSITION. *If* $\dim A$ *is odd, then*

$$d(-\pi_A + \pi_B, U) = -1.$$

PROOF. It is enough to consider the case where $\dim A = 1$: for we can write $A = A_1 \oplus A_2$, where $\dim A_1 = 1$ and $\dim A_2 = 2n$, and by (7.1) we find

$$\begin{aligned} d(-\pi_A + \pi_B, U) &= d(-\pi_{A_1} - \pi_{A_2} + \pi_B, U) \\ &= d(-\pi_{A_1} + \pi_{A_2} + \pi_B, U) = d(-\pi_{A_1} + \pi_{A_2+B}, U). \end{aligned}$$

We take $A = \boldsymbol{R}$, and define $f_0(x)$ to be the constant function 1 and $f_1(x) = x(x-2)$. We can assume that $U \subset \boldsymbol{R}$ contains the open interval $(-1,3)$ in its interior; and since the zeros of the functions considered are all in $(-1,3)$, excision shows we need operate only on this interval.



We now proceed in several steps:

(a) $f_0 \simeq f_1$ on $(-1,3)$, since the homotopy $(1-t) + tx(x-2)$ is never zero for any $t \in \boldsymbol{I}$ when $x = -1, 3$.

(b) On the first half of the interval:

$$f_1 \simeq -\mathrm{id} \quad \text{on } (-1,1),$$

because $(1-t)x(x-2) + t(-x)$ is not zero when $t \in \boldsymbol{I}$ and $x = \pm 1$.

(c) On the second half of the interval:

$$f_1 \simeq \mathrm{id} - 2 \quad \text{on } (1, 3),$$

again by using a linear homotopy.

(d) On the entire interval:

$$\mathrm{id} - 2 \simeq \mathrm{id} \quad \text{on } (-1, 3)$$

for reasons as above.

Now, $f_0(x) \neq 0$ on $(-1, 3)$ and $f_1(1) \neq 0$, so from additivity and the homotopies above we get

$$\begin{aligned} 0 = d[f_0, (-1,3)] = d[f_1, (-1,3)] &= d[f_1, (-1,1)] + d[f_1, (1.3)] \\ &= d[-\mathrm{id}, (-1,1)] + d[\mathrm{id} - 2, (1,3)] \\ &= d[-\mathrm{id}, (-1,1)] + d[\mathrm{id} - 2, (-1,3)]; \end{aligned}$$

because id -2 has no zeros in $(-1, 1)$, it follows that

$$\begin{aligned} 0 &= d[-\mathrm{id}, (-1,1)] + d[\mathrm{id}, (-1,3)] \\ &= d[-\mathrm{id}, (-1,1)] + 1. \end{aligned}$$

By additivity, we conclude that $d(-\mathrm{id}, U) = -1$, and as we have seen, this proves our assertion. □

We now show

(7.3) THEOREM. *Let $U \subset \boldsymbol{R}^n$ be open and $T : U \to \boldsymbol{R}^n$ a nonsingular linear map. If U contains 0, then $d(T, U) = (-1)^s$, where s is the number of negative eigenvalues of T, each counted with its multiplicity.*

PROOF. Let $\lambda_1, \ldots, \lambda_k$ be all the distinct negative eigenvalues of T. From linear algebra, it is known that there is a direct sum decomposition

$$E = N_1 \oplus \cdots \oplus N_k \oplus M,$$

where M and all the N_i are invariant under T, and λ_i is the only eigenvalue of $T|N_i$ with $\dim N_i =$ multiplicity of λ_i. The eigenvalues of $T|M$ are all the remaining eigenvalues. Let $N = N_1 \oplus \cdots \oplus N_k$ and define

$$T_1 = T|N, \qquad T_2 = T|M,$$
$$H(x, t) = (1 - t)(T_1 \oplus T_2) + t(-\mathrm{id}_N \oplus \mathrm{id}_M).$$

It is easy to see that $H|\partial U \times \boldsymbol{I}$ is never zero (since T_1 has no positive eigenvalues and T_2 has no negative ones). By (7.1), (7.2), and multiplicativity we have

$$d(T, U) = d(-\mathrm{id}_N, U \cap N) = (-1)^s.$$

For $s = 0$, the result follows from normalization, $d(\mathrm{id}, U) = 1$.

Thus, in our Theorem (1.3) condition (1) (strict normalization) follows from (2), (3), and the simpler version (normalization). The last three axioms are, in fact, independent:

(a) $d(f,U) \equiv 0$ satisfies axioms (2), (3).

(b) If d satisfies (1′), (2), (3), define $\hat{d}(f,U) = d(f,U)^3$; then $\hat{d}$ satisfies (1) and (3), but not, in general, (2).

(c) Define $d(f,U) = 1$ if $0 \in U$ and there is some nbd V of 0 with $f|V = \mathrm{id}$; otherwise, $d(f,U) = 0$. Then normalization and (2) are satisfied, but (3) is not.

8. The Main Theorem on the Brouwer Degree in $\boldsymbol{R}^n$

We now summarize the preceding discussion by stating the fundamental theorem on existence and uniqueness of the topological degree in $\boldsymbol{R}^n$

(8.1) THEOREM. *Let $\mathcal{M}$ denote the class of all maps $f \in C_0(\overline{U}, \boldsymbol{R}^n)$, where U is open and bounded in some $\boldsymbol{R}^n$ Then:*

(A) *There exists a function $d : \mathcal{M} \to \boldsymbol{Z}$ assigning a degree $d(f) = d(f,U)$ to each $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ and satisfying:*

(I) (Normalization) *If $0 \in U$ and $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ is the identity $\mathrm{id}_{\overline{U}}$, then $d(f,U) = 1$.*

(II) (Additivity) *If $U \supset V_1 \cup V_2$ with V_1, V_2 open and disjoint, $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ and $Z(f) \subset V_1 \cup V_2$, then*

$$d(f,U) = d(f,V_1) + d(f,V_2).$$

(III) (Homotopy) *If $h_t : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ is a homotopy, then $d(h_0) = d(h_1)$.*

(IV) (Existence) *If $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ and $d(f) \neq 0$, then $f(U)$ contains a neighborhood of zero.*

(V) (Excision) *If $V \subset U$ are open in $\boldsymbol{R}^n$ and $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ has all its zeros in V, then $d(f,U) = d(f,V)$.*

(B) *The degree function $d : \mathcal{M} \to \boldsymbol{Z}$ is uniquely determined by the normalization, additivity, and homotopy properties* (I) (III). □

We remark that properties (IV) and (V) follow from (I) (III) by the same argument as in the proof of (5.1) and (5.2).

One of the striking consequences of the homotopy property in (8.1) is that $d(f,U)$ depends only on the boundary values $f|\partial U$ Precisely:

(8.2) THEOREM (Dependence on boundary values). *If $U \subset \boldsymbol{R}^n$ is open and bounded, and the maps $f, g \in C_0(\overline{U}, \boldsymbol{R}^n)$ satisfy $f|\partial U = g|\partial U$, then $d(f,U) = d(g,U)$.*

PROOF. Define $H : \overline{U} \times \boldsymbol{I} \to \boldsymbol{R}^n$ by $H(x,t) = (1-t)f(x) + tg(x)$. Since $H_t|\partial U = f|\partial U = g|\partial U$ for all t, we conclude that $f,g : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ are homotopic, and therefore $d(f,U) = d(g,U)$. □

The mechanism of this dependence on boundary values, and the determination of $d(f,U)$ from the knowledge of $f|\partial U$ alone, will be discussed later.

Another frequently used property relating to boundary behavior is

(8.3) THEOREM (Rouché). *If* $f,g \in C_0(\overline{U}, \boldsymbol{R}^n)$ *satisfy*

$$\|f(x) - g(x)\| < \|f(x)\| \quad \textit{for } x \in \partial U,$$

then f *and* g *are homotopic in* $C_0(\overline{U}, \boldsymbol{R}^n)$ *and*

$$d(f,U) = d(g,U).$$

PROOF. Observe that $h_t(x) = (1-t)f(x) + tg(x) \neq 0$ on ∂U, since otherwise we would have $t\|f(x) - g(x)\| = \|f(x)\|$ for some $x \in \partial U$ and $t \in (0,1)$; thus f,g are homotopic, and our conclusion follows from (8.1). □

On the basis of (8.3), we show how to calculate the degree of a differentiable map at its regular zero.

(8.4) THEOREM. *Let* $f : U \to \boldsymbol{R}^n$ *be differentiable, and let* $a \in U$ *be a regular zero of* f, *i.e.,* $f'(a)$ *is nonsingular.*

(a) *For a sufficiently small ball* $B(a,r) = V \subset U$, *the map* $f|\overline{V}$ *is homotopic to the affine map* $x \mapsto f'(a)(x-a)$ *by a homotopy* $H_t : (\overline{V}, \partial V) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$.

(b) $d(f,V) = \det f'(a)/|\det f'(a)| = \operatorname{sgn}\det f'(a)$.

PROOF. Since $f'(a)$ is invertible, we have $\|f'(a)(x-a)\| \geq m\|x-a\|$ for all $x-a$ and suitable $m > 0$. Choose $B(a,r) = V$ so small that

$$\|f(x) - f'(a)(x-a)\| \leq \frac{1}{2} m\|x-a\| \quad \text{on } \overline{V}.$$

Then $f : (\overline{V}, \partial V) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ and $f'(a)(x-a) \neq 0$ on ∂V (since $\|x-a\| = r$ on ∂V). Moreover,

$$\|f(x) - f'(a)(x-a)\| < \|f'(a)(x-a)\| \quad \text{on } \partial V.$$

So (8.3) gives (a) and

$$d(f,V) = d(f'(a),V) = \frac{\det f'(a)}{|\det f'(a)|}.$$

and the proof is complete. □

We now formulate the main theorem on the topological degree in $\boldsymbol{R}^n$ in an equivalent form which will serve as the basis for the fixed point index theory that we shall develop in §12.

(8.5) THEOREM (Degree for compactly rooted maps). *Let $\mathcal{M}^*$ be the class of all compactly rooted maps $f : U \to \boldsymbol{R}^n$, where U is open in some $\boldsymbol{R}^n$. Then:*

(A) *There exists a function $D : \mathcal{M}^* \to \boldsymbol{Z}$ assigning a degree $D(f) = D(f, U)$ to each $f : U \to \boldsymbol{R}^n$ in $\mathcal{M}^*$ and satisfying:*

(I) (Normalization) *If $f = \mathrm{id}_U$ and $0 \in U$, then $D(f) = 1$.*

(II) (Additivity) *For every pair V_1, V_2 of disjoint open subsets of U, if $Z(f) \subset V_1 \cup V_2$, then $D(f, U) = D(f, V_1) + D(f, V_2)$.*

(III) (Homotopy) *Let $H_t : W \to \boldsymbol{R}^n$ be a compactly rooted homotopy. Then $D(H_0, W) = D(H_1, W)$.*

(IV) (Existence) *If $D(f, U) \neq 0$, then $Z(f) \neq \emptyset$.*

(V) (Excision) *If $V \subset U$ is open and $Z(f) \subset V$, then $D(f, U) = D(f, V)$.*

(VI) (Multiplicativity) *Let $U \subset \boldsymbol{R}^n$ and $V \subset \boldsymbol{R}^m$ be open and $f \in C_0(U, \boldsymbol{R}^n)$, $g \in C_0(V, \boldsymbol{R}^m)$. Then $f \times g \in C_0(U \times V, \boldsymbol{R}^{n+m})$ and $D(f \times g, U \times V) = D(f, U) \cdot D(g, V)$.*

(B) *The degree function $D : \mathcal{M}^* \to \boldsymbol{Z}$ is uniquely determined by the normalization, additivity, and homotopy properties* (I)–(III). □

9. Extension of the Antipodal Theorem

Using the degree we now establish another generalization of the Borsuk antipodal theorem. Let $\overline{U} \subset \boldsymbol{R}^n$ be *centrally symmetric* (i.e., $-\overline{U} = \overline{U}$). We recall that a map $f : \overline{U} \to \boldsymbol{R}^n$ is *odd* if $f(x) = -f(-x)$ for all $x \in \overline{U}$.

(9.1) THEOREM. *Let $U \subset \boldsymbol{R}^n$ be a centrally symmetric polyhedral domain, and let $f : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ be continuous and odd on $\overline{U}$. If $0 \notin U$, then $d(f, U)$ is even.*

PROOF. Let $\varepsilon = \mathrm{dist}[f(\partial U), 0] > 0$. There exists a triangulation T of $\overline{U}$ and a PL-generic approximation φ of f such that:

(i) $\sigma \in T \Rightarrow -\sigma \in T$,

(ii) φ is odd.

The first assertion can be proved by induction: $n = 1$ is obvious; then, given a decomposed $\{(x_1, \ldots, x_n) \in \boldsymbol{R}^n \mid x_n = 0\} \cap \overline{U}$, we triangulate $\{(x_1, \ldots, x_n) \in \boldsymbol{R}^n \mid x_n \geq 0\} \cap \overline{U}$; the antipode of this offers no contradiction. Then we extend φ simplicially over the northern part, and reflect it for the antipodal values. Therefore $d(f, U) = d(\varphi, U)$.

Now,

$$d(\varphi, U) = \sum_{x \in \varphi^{-1}(0)} J(\varphi, x),$$

where

$$\varphi^{-1}(0) = \{x_1, -x_1, x_2, -x_2, \ldots, x_k, -x_k\}$$

and

$$J(\varphi, x_i) = [\![p_0, \ldots, p_n]\!] \cdot [\![\varphi(p_0), \ldots, \varphi(p_n)]\!].$$

Since φ is odd,

$$\begin{aligned} J(\varphi, -x_i) &= [\![-p_0, \ldots, -p_n]\!] \ [\![\varphi(-p_0), \ldots, \varphi(-p_n)]\!] \\ &= (-1)^n(-1)^n J(\varphi, x_i) = J(\varphi, x_i). \end{aligned}$$

Therefore,

$$d(\varphi, U) = 2 \sum_{\varphi(x_i)=0} J(\varphi, x_i)$$

is even. □

We now state the desired extension of the Borsuk theorem:

(9.2) THEOREM (Antipodal theorem). *Let U be a centrally symmetric polyhedral domain and $f : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ a continuous map such that:*
(i) *$f|\partial U$ is an odd function,*
(ii) *$0 \in U$.*
Then $d(f, U)$ is odd.

PROOF. Choose a closed ball $\overline{U}_0 \subset U$ around the origin. Define $\varphi : \overline{U}_0 \cup \partial U \to \boldsymbol{R}^n$ by $\varphi|\overline{U}_0 = \mathrm{id}$, $\varphi|\partial U = f|\partial U$. Tietze's theorem gives an extension $\Phi : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ of φ over $\overline{U}$. Defining

$$\Psi(x) = \frac{1}{2}[\Phi(x) - \Phi(-x)]$$

we find:
(a) $\Psi|\partial U = f|\partial U$ because f is odd on ∂U.
(b) $\Psi|\overline{U}_0 = \mathrm{id}$.
(c) $\Psi : \overline{U} \to \boldsymbol{R}^n$ is odd.
Now, by (a), $d(f, U) = d(\Psi, U)$, because of the boundary value theorem (8.2). Next, by additivity,

$$d(\Psi, U) = d(\Psi, U_0) + d(\Psi, U - \overline{U}_0).$$

By normalization, $d(\Psi, U_0) = +1$. Since Ψ is odd on $U - \overline{U}_0$, and $0 \notin U - \overline{U}_0$, we conclude by (9.1) that $d(\Psi, U - \overline{U}_0)$ is even. Thus, $d(f, U)$ is odd. □

10. Miscellaneous Results and Examples

A. Properties of the topological degree in $\boldsymbol{R}^n$

Given an open bounded $U \subset \boldsymbol{R}^n$ and $f \in C_0(\overline{U}, \boldsymbol{R}^n)$, we let $Z(f) = \{x \in U \mid f(x) = 0\}$.

(A.1) Let $V \subset \boldsymbol{R}^n$ be open and bounded, $U \subset \boldsymbol{R}^k$ an open bounded nbd of 0, and $f \in C_0(\overline{V}, \boldsymbol{R}^n)$. Let $W = V \times U$, and let $F = (f, \mathrm{id}_U)$ be given by $(x, y) \mapsto (f(x), y)$. Show: The map F sends $(\overline{W}, \partial W)$ into $(\boldsymbol{R}^{n+k}, \boldsymbol{R}^{n+k} - \{0\})$ and $d(F, W) = d(f, U)$.

(A.2) Let $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ and assume that $V(x_0)$ is a nbd of $x_0 \in Z(f)$ such that $Z(f) \cap V(x_0) = \{x_0\}$. We let $i_0(f, x_0) = d(f, V(x_0))$ be the *local index* of x_0 for f. Show: $i_0(f, x_0)$ depends only on f and x_0.

(A.3) Let $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ with $Z(f) = \{x_1, \ldots, x_k\}$. Show: $d(f, U) = \sum_{i=1}^{k} i_0(f, x_i)$.

(A.4) Let $U \subset \boldsymbol{R}^n$ be open and bounded, and $f, g \in C_0(\overline{U}, \boldsymbol{R}^n)$. Show:
(a) If $\mu f(x) = g(x)$ for all $x \in \partial U$ and $\mu > 0$, then $d(f, U) = d(g, U)$.
(b) If $\mu f(x) = -g(x)$ for all $x \in \partial U$ and $\mu > 0$, then $d(f, U) = (-1)^n d(g, U)$.
(c) $d(f, U) = (-1)^n d(-f, U)$.

(A.5) Let $U \subset \boldsymbol{R}^{2n+1}$ be an open bounded nbd of 0, and let $f : \partial U \to \boldsymbol{R}^{2n+1} - \{0\}$ be continuous. Show: There is an $(x_0, \lambda_0) \in \partial U \times (\boldsymbol{R} - \{0\})$ such that $x_0 = \lambda_0 f(x_0)$.

(A.6) Let $U \subset \boldsymbol{R}^n$ be open and bounded, and $f \in C_0(\overline{U}, \boldsymbol{R}^n)$. Show:
(a) If either (i) $0 \in U$ and $d(f, U) \neq 1$, or (ii) $0 \notin \overline{U}$ and $d(f, U) \neq 0$, then there are $x_0 \in \partial U$ and $\lambda_0 < 0$ such that $x_0 = \lambda_0 f(x_0)$.
(b) If either (i) $0 \in U$ and $d(f, U) \neq (-1)^n$, or (ii) $0 \notin \overline{U}$ and $d(f, U) \neq 0$, then there are $y_0 \in \partial U$ and $\lambda_0 > 0$ such that $y_0 = \lambda_0 f(y_0)$.

(A.7) For any bounded open $U \subset \boldsymbol{R}^n$ and $b \in \boldsymbol{R}^n$, let $C_b(\overline{U}, \boldsymbol{R}^n)$ be the set of all continuous $f : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{b\})$. Given an $f \in C_b(\overline{U}, \boldsymbol{R}^n)$, its *degree with respect to b* is the integer $d(f, U, b) = d(f - b, U)$, where $d(f - b, U)$ is the Brouwer degree of the map $f - b : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$. Establish the following properties of the degree:
(1) (*Normalization*) $d(\mathrm{id}, U, b) = 1$ or 0 depending on whether or not $b \in U$.
(2) (*Additivity*) If $U \supset U_1 \cup U_2$ with U_1, U_2 open disjoint and $b \notin f(\overline{U} - (U_1 \cup U_2))$, then

$$d(f, U, b) = d(f, U_1, b) + d(f, U_2, b).$$

(3) (*Homotopy*) If $H_t : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{b\})$ is a homotopy, then $d(H_0, U, b) = d(H_1, U, b)$.
(4) (*Existence property*) If $d(f, U, b) \neq 0$, then $f(U)$ is a nbd of b in $\boldsymbol{R}^n$
(5) (*Excision*) If U_1 is an open subset of U and $f \in C_b(\overline{U}, \boldsymbol{R}^n)$ satisfies $b \notin f(U - U_1)$, then $d(f, U_1, 0) = d(f, U, 0)$.
(6) (*Dependence on boundary values*) For any $f, g \in C_b(\overline{U}, \boldsymbol{R}^n)$ with $f|\partial U = g|\partial U$, we have $d(f, U, b) = d(g, U, b)$.
(7) (*Component dependence*) If $f \in C_b(\overline{U}, \boldsymbol{R}^n)$, then $d(f, U, \cdot)$ is constant on each connected component of $\boldsymbol{R}^n - f(\partial U)$.

(A.8) Let $f \in C_0(\overline{U}, \boldsymbol{R}^n)$ be C^1 on U, and let x_0 be a unique regular zero for f (i.e., $f(x_0) = 0$ and the Jacobian $Jf(x_0)$ is nonzero). Show: $d(f, U) = i_0(f, x_0) = \pm 1$, according to whether the Jacobian $Jf(x_0)$ at x_0 is positive or negative.

(A.9) Let $f \in C_0(\overline{U}, R^n)$ be a C^1 map such that $Z(f)$ consists of a finite number of points $x_1, \dots, x_k$, all regular. Show:

$$d(f,U) = \sum_{j=1}^{k} \operatorname{sgn} Jf(x_j) = \sum_{j=1}^{k} Jf(x_j)/|Jf(x_j)| = \sum_{j=1}^{k} i_0(f,x_j).$$

(A.10)* (*Leray composition theorem*) Let $U \subset R^n$ be an open bounded subset, and let $f : \overline{U} \to R^n$, $g : R^n \to R^n$ be two maps. Let $K_1, K_2, \dots$ denote the bounded components of $R^n - f(\partial U)$, and let $b \in R^n - gf(\partial U)$. Show:

$$d(gf, U, b) = \sum_j d(f, U, K_j) d(g, K_j, b),$$

where $d(f, U, K_j)$ is the value of the constant function defined by $a \mapsto d(f, U, a)$, $a \in K_j$ (Leray [1935]).

[Let K_∞ denote the unbounded component of $R^n - f(\partial U)$, $K = \bigcup_j K_j$, $L = f^{-1}(K)$, $L_j = f^{-1}(K_j)$, and $L_\infty = f^{-1}(K_\infty)$; note that

$$d(gf, U, b) = d(gf, L_\infty, b) + d(gf, L, b) = d(gf, L, b) = \sum_j d(gf, L_j, b)$$

(since $d(gf, L_\infty, b) = 0$), and then show that $d(gf, L_j, b) = d(f, U, K_j) d(g, K_j, b)$, using the fact that $d(f, U, K_j) = d(f, U, b_j)$, where $b_j \in K_j \cap g^{-1}(b)$.]

(A.11)* (*Degree and critical points*) Let $f \; R^n \to R$ be a C^2 function with a finite number of critical points such that $f(x) \to \infty$ as $\|x\| \to \infty$. Show:

(a) For a sufficiently large ball $B_r = B(0, r)$ we have $d(\operatorname{grad} f, B_r) = 1$.

(b) If 0 is a nondegenerate critical point of f (i.e., $\det |\partial^2 f/\partial x_i \partial x_j(0)| \neq 0$) and f attains its minimum at a point $x \neq 0$, then f has at least three stationary points (Castro–Lazer [1979]).

B. Degree in R^n and Brouwer's degree for maps $S^n \to S^n$

Let S^n be the unit sphere in R^{n+1} and $\widehat{R}^n = R^n \cup \{\infty\}$ be the one-point compactification of R^n We let $\sigma_n : R^n \to S^n$ denote the inverse of the stereographic projection, given by

$$\sigma_n(x) = \Big(\frac{\|x\|^2 - 1}{\|x\|^2 + 1}, \frac{2x}{\|x\|^2 + 1} \Big) \in R \times R^n \qquad \text{for } x \in R^n$$

We extend σ_n in a natural way to a homeomorphism $\widehat{\sigma}_n : \widehat{R} \to S^n$ by sending ∞ to the "north pole" $(1, 0, \dots, 0) \in S^n$; using $\widehat{\sigma}_n$ we identify $\widehat{R}^n$ with S^n

(B.1) Let $U \subset R^n$ be open and bounded, let $f : (\overline{U}, \partial U) \to (R^n, R^n - \{0\})$ be a map, and $f_0 = f|\partial U : \partial U \to R^n - \{0\}$.

(a) Show: There exists a map $\widehat{f}_0 : \widehat{R}^n - U \to \widehat{R}^n - \{0\}$ such that $\widehat{f}_0|\partial U = f_0$.

[Use the following facts: (i) ∂U is the common boundary of $\widehat{R}^n - U$ and U; (ii) the values of f_0 are in $R^n - \{0\} \subset \widehat{R}^n - \{0\}$; (ii) $\widehat{R}^n - \{0\}$ is an absolute retract.]

(b) Define $\widehat{f} : \widehat{R}^n \to \widehat{R}^n$ by

$$\widehat{f}(x) = \begin{cases} \widehat{f}_0(x) & \text{if } x \in \widehat{R}^n - U, \\ f(x) & \text{if } x \in \overline{U}, \end{cases}$$

and call $\widehat{f}$ an *allowable extension* of f over $\widehat{R}^n$. Show: Any two allowable extensions of f over $\widehat{R}^n$ are homotopic.

(B.2) Let $U \subset R^n$ be open and bounded, and $f \in C_0(\overline{U}, R^n)$. Let $\hat{f} : \hat{R}^n \to \hat{R}^n$ be an allowable extension of f. Set $\deg(f) = d_B(\hat{f})$, where $d_B(\hat{f})$ is the Brouwer degree of $\hat{f} : S^n \to S^n$. Show: $\deg(f)$ does not depend on the choice of $\hat{f}$.

(B.3) Let $U \subset R^n$ be open and bounded, and let $h_t : (\overline{U}, \partial U) \to (R^n, R^n - \{0\})$ be a homotopy. Show: $t \mapsto \deg(h_t)$ is constant.

(B.4) Let $U \subset R^n$ be open and bounded, $f : (\overline{U}, \partial U) \to (R^n, R^n - \{0\})$, and U_1, U_2 open subsets of U such that $Z(f) \subset U_1 \cup U_2$. Show: $\deg(f) = \deg(f, U_1) + \deg(f, U_2)$.

(B.5) Let $U \subset R^n$ be open and bounded, and $f \in C_0(\overline{U}, R^n)$. Show: $\deg(f) = d(f)$.

(In connection with the above results, see Bers's lecture notes [1957] and Gęba Massabò Vignoli [1986].)

C. Isolating neighborhoods and complementing functions

Throughout this section we assume $f : R^n \times R \to R^n$ to be a C^1 function such that: (i) the set $\Lambda_f = \{(0, \lambda) \mid \det D_x f(0, \lambda) = 0\}$ of its *singular points* is discrete, (ii) $f(0, \lambda) = 0$ for all $\lambda \in R$.

(C.1) Let $(0, \lambda_0) \in \Lambda_f$. An open $U \subset R^n \times R$ is called an $(\varepsilon; r)$-*isolating nbd* of $(0, \lambda_0)$ provided:

(i) $U = B(\lambda_0; r, \varepsilon) = \{(x, \lambda) \in R^n \times R \mid \|x\| < r,\ |\lambda_0 - \lambda| < \varepsilon\}$ is a boxlike nbd of $(0, \lambda_0)$ determined by $\varepsilon > 0$ and $r > 0$,

(ii) $(0, \lambda_0)$ is the unique singular point of f in $\overline{U}$,

(iii) $f(x, \lambda) \neq 0$ for $|\lambda_0 - \lambda| = \varepsilon$ and all x with $0 < \|x\| < r$.

Prove: If $\varepsilon > 0$ and $r > 0$ are sufficiently small, then $(0, \lambda_0)$ admits an $(\varepsilon; r)$-isolating nbd.

(C.2) If U is an $(\varepsilon; r)$-isolating nbd of $(0, \lambda_0) \in \Lambda_f$, then a continuous $\theta : \overline{U} \to R$ is a *complementing function* for f, written $\theta \in \mathscr{C}_{\lambda_0}(U)$, provided:

(i) $\theta(0, \lambda_0 + \varepsilon) > 0$, $\theta(0, \lambda_0 - \varepsilon) > 0$.

(ii) $\theta(x, \lambda) < 0$ for $\|x\| = r$.

Prove:

(a) The function $\theta : \overline{U} \to R$ given by

$$\theta(x, \lambda) = |\lambda_0 - \lambda|(r - \|x\|)/r - \|x\|$$

is a complementing function for f.

(b) For any η with $0 < \eta < \varepsilon$, the function $\theta : \overline{U} \to R$ given by

$$\theta(x, \lambda) = [(\lambda - \lambda_0)^2 - \eta^2](r^2 - \|x\|^2) - \|x\|^2$$

is a complementing function for f.

(c) For any η with $0 < \eta < r$ consider the "tube" $J_\eta = S(0, \eta) \times [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ and $U = B(\lambda_0; r, \varepsilon)$. Let $\theta : \overline{U} \to R$ be a continuous function such that (i) $\theta | J_\eta = 0$, (ii) θ is negative outside J_η, (iii) θ is positive inside J_η. Then $\theta \in \mathscr{C}_{\lambda_0}(U)$.

(d) If $\theta_1, \theta_2 \in \mathscr{C}_{\lambda_0}(U)$, then $(1 - t)\theta_1 + t\theta_2 \in \mathscr{C}_{\lambda_0}(U)$ for any $t \in [0, 1]$.

(e) If $\theta \in \mathscr{C}_{\lambda_0}(U)$ and $\theta(x, \lambda) = 0$ for some $(x, \lambda) \in \overline{U}$ with $\lambda \neq \pm\varepsilon$ then $(x, \lambda) \in U$.

(C.3) If U is an $(\varepsilon; r)$-isolating nbd of $(0, \lambda_0)$ and $\theta \in \mathscr{C}_{\lambda_0}(U)$, let $F = (f, \theta) : \overline{U} \to R^{n+1}$ be given by $(x, \lambda) \mapsto (f(x, \lambda), \theta(x, \lambda))$ for $(x, \lambda) \in \overline{U}$. Prove:

(a) $F \in C_0(\overline{U}, R^{n+1})$.

(b) If $F_1 = (f, \theta_1)$, $F_2 = (f, \theta_2)$, where $\theta_1, \theta_2 \in \mathscr{C}_{\lambda_0}(U)$, then $F_1, F_2 \in C_0(\overline{U}, R^{n+1})$ are homotopic and $d(F_1, U) = d(F_2, U)$.

(C.4) Let $0 < \varepsilon_1 < \varepsilon_2$, $0 < r_1 < r_2$ and $U_1 = U(\varepsilon_1, r_1) \subset U_2 = U(\varepsilon_2, r_2)$ be two isolating nbds of $(0, \lambda_0)$. For $i = 1, 2$ let $F_i \in C_0(U_i, R^{n+1})$ be given by $F_i = (f, \theta_i)$, where $\theta_i \in \mathscr{C}_{\lambda_0}(U_i)$. Show: $d(F_1, U_1) = d(F_2, U_2)$.

(C.5) If U is an $(\varepsilon; r)$-isolating nbd of $(0, \lambda_0) \in \Lambda_f$, we let

$$\Gamma(\lambda_0) = \operatorname{sgn}\det D_x f(0, \lambda_0 + \varepsilon) - \operatorname{sgn}\det D_x f(0.\lambda_0 - \varepsilon).$$

Prove:

(a) $\Gamma(\lambda_0)$ does not depend on the choice of U.

(b) For any $F = (f, \theta) \in C_0(\overline{U}, R^{n+1})$, where $\theta \in \mathscr{C}_{\lambda_0}(U)$, we have $\Gamma(\lambda_0) = d(F, U)$.

(c) If $\Gamma(\lambda_0) \neq 0$, then $f(x, \lambda) = 0$ for some $(x, \lambda) \in U$ with $x \neq 0$.

[For (b): consider $G = (g, \theta) \in C_0(\overline{U}, R^{n+1})$, where $g(x, \lambda) = D_x f(0, \lambda)(x)$ and $\theta \in \mathscr{C}_{\lambda_0}(U)$ is as in (C.2)(b); observe that $(0, \lambda_0 + \eta)$, $(0, \lambda_0 - \eta)$ are the only solutions in U of $G(x, \lambda) = 0$. To conclude, calculate $D_x G(0, \lambda_0 + \eta)$, $D_x G(0, \lambda_0 - \eta)$, and then using (B.2) show that $\Gamma(\lambda_0) = d(G, U) = d(F, U)$. For (c) use an appropriate complementing function.]

D. Bifurcation results in R^n

Consider the equation

$$(*) \qquad f(x, \lambda) = 0.$$

We let $\mathscr{B}_f = \mathscr{T}_f \cap \mathscr{S}_f$, where $\mathscr{T}_f = \{0\} \times R$ is the set of *trivial solutions* of $(*)$ and $\mathscr{S}_f$ is the closure in $R^n \times R$ of the set of nontrivial solutions of $(*)$. Any $(0, \lambda) \in \mathscr{B}_f$ is called a *bifurcation point* of $(*)$.

(D.1) Show: The set $\mathscr{B}_f$ of bifurcation points is contained in Λ_f.

(D.2) (*Krasnosel'skiĭ theorem*) Let $J \subset R$ be an open interval with $\Lambda_f \cap (\{0\} \times J) = \{(0, \lambda_0)\}$. Given any $\lambda_+, \lambda_- \in J$ with $\lambda_- < \lambda_0 < \lambda_+$, let

$$\gamma(\lambda_0) = \operatorname{sgn}\det D_x f(0, \lambda_+) - \operatorname{sgn}\det D_x f(0, \lambda_-).$$

Show: If $\gamma(\lambda_0) \neq 0$, then $(0, \lambda_0) \in \mathscr{B}_f$ (cf. Krasnosel'skiĭ's book [1964]).

(D.3) Assume that $\lambda \mapsto D_x f(0, \lambda)$ is of the form $I - \lambda T$ for some $T \in \mathscr{L}(R^n)$. Prove: If $1/\lambda_0$ is an eigenvalue of T of odd multiplicity, then $(0, \lambda_0)$ is a bifurcation point for $(*)$.

(D.4) For $\lambda_0 < \lambda_1 < \cdots < \lambda_k$, set $[\lambda_0, \lambda_1, \ldots, \lambda_k] = \{(0, \lambda_0), (0, \lambda_1), \ldots, (0, \lambda_k)\}$. Assume $[\lambda_0, \lambda_1, \ldots, \lambda_k] \subset \mathscr{B}_f$. Call $\{U_0, U_1, \ldots, U_k\}$ an *$(\varepsilon; r)$-isolating system* of $[\lambda_0, \lambda_1, \ldots, \lambda_k]$ if:

(i) each $U_i = B(\lambda_i; r, \varepsilon)$ is an $(\varepsilon; r)$-isolating nbd of $(0, \lambda_i)$,

(ii) $\overline{U}_i \cap \overline{U}_{i+1} = \emptyset$ for $i = 0, \ldots, k-1$.

Prove: Any $[\lambda_0, \lambda_1, \ldots, \lambda_k] \subset \mathscr{B}_f$ admits an isolating system.

(D.5) Let C be a bounded component of $\mathscr{S}_f$ such that $C \cap \mathscr{B}_f = [\lambda_0, \lambda_1, \ldots, \lambda_k]$. Show: There exists an open bounded set $W \subset R^{n+1}$ such that: (i) $C \subset W$; (ii) $\partial W \cap \mathscr{S}_f = \emptyset$; (iii) $\overline{W} \cap \Lambda_f = [\lambda_0, \lambda_1, \ldots, \lambda_k]$.

[Take a sufficiently large ϱ so that $C \subset B(0, \varrho)$, and let $X = \mathscr{S}_f \cup \Lambda_f \cup (R^{n+1} - B(0, \varrho))$, $K = \Lambda_f - [\lambda_0, \lambda_1, \ldots, \lambda_k]$; observe that $X \subset R^{n+1}$ is closed and K is a closed subset of $X - C$. Prove that there exists an open bounded $W \subset R^{n+1}$ such that $C \subset W$, $\partial W \cap X = \emptyset$ and $\overline{W} \cap K = \emptyset$.]

(D.6) (*Rabinowitz theorem*) Let $(0,\lambda) \in \mathscr{B}_f$, and let C be a bounded component of $\mathscr{S}_f$ containing $(0,\lambda)$. Prove:

(a) There exists $[\lambda_0, \lambda_1, \ldots, \lambda_k] \subset \mathscr{B}_f$ such that $C \cap \mathscr{B}_f = [\lambda_0, \lambda_1, \ldots, \lambda_k]$.

(b) $\sum_{i=0}^{k} \Gamma(\lambda_i) = 0$, where $\Gamma(\lambda_i)$ is defined in (C.5) (Rabinowitz [1971]).

[For (b), take an $(\varepsilon; r)$-isolating system $\{U_i\}$ of $[\lambda_0, \lambda_1, \ldots, \lambda_k]$ and W as in (D.5), so that $U = \bigcup U_i = W \cap (\{x \in \boldsymbol{R}^n \mid \|x\| < r\} \times \boldsymbol{R})$. Define $\theta : \overline{W} \to \boldsymbol{R}$ by $\theta(x,\lambda) = r^2 - 2\|x\|^2$ and prove that $F = (f,\theta) \in C_0(\overline{W}, \boldsymbol{R}^{n+1})$ is homotopic in $C_0(\overline{W}, \boldsymbol{R}^{n+1})$ to a constant map, and thus $d(F,W) = 0$.]

(The proofs of (D.2) and (D.4), based on the technique of complementing functions introduced by Ize [1976], are due to K. Gęba).

11. Notes and Comments

Topological degree in $\boldsymbol{R}^n$

Let U be a bounded open subset of $\boldsymbol{R}^n$ and $b \in \boldsymbol{R}^n$ Let $C_b(\overline{U}, \boldsymbol{R}^n)$ denote the set of all continuous $f : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{b\})$. Given any $f \in C_b(\overline{U}, \boldsymbol{R}^n)$, its *degree with respect to* b is defined to be the integer $\deg(f, U, b) = d(f - b, U)$. The properties of the degree function given in Theorem (8.1) imply at once the corresponding properties of $\deg(f, U, b)$. The degree $\deg(f, U, b)$ can be regarded as the "localization" of Brouwer's original theory for maps between n-dimensional manifolds. This localization was, in fact, first formulated and carried out explicitly by Leray and Schauder in their classical memoir of [1934] (as the first step towards constructing a degree for compact fields in Banach spaces). For some further comments concerning this matter, the reader is referred to Dieudonné's book [1989]; see also Leray's survey [1936].

There are essentially three general methods to handle the theory: (i) geometric; (ii) analytic; and (iii) algebraic.

The first one, presented in this paragraph, uses only some elementary facts from linear algebra and simplicial topology and is based on PL-approximations of continuous maps. This type of approach was used, in fact, for the first time by Leray in his unpublished 1948 1949 lectures at the Collège de France (for a brief outline see Leray [1950]), but the details and the form of the presentation given here are due to Dugundji [1985]; the reader may consult also Peitgen–Siegberg [1981], where the PL-approach is studied from the viewpoint of finding algorithms for computing fixed points.

The second method, based on using various tools of analysis, has its roots in the work of Kronecker [1878], who already developed, in the setting of C^1 maps, what could be called a degree theory. Various analytic approaches for continuous maps were given (cf. for example Nagumo [1951] and Heinz [1959]), and are described in several books and lecture notes.

We recommend the following references for the analytic approach:

(a) lecture notes: Schwartz [1965], Nirenberg [1974], Rabinowitz [1975] (in French), Dold [1986] (in Spanish),

(b) monographs: Krasnosel'skiĭ–Zabreĭko [1975], Berger [1977], Eisenack–Fenske [1978] (in German), Lloyd [1978], Deimling [1985], Zeidler [1985], Rothe [1986], and also Milnor [1965].

The third method, based on tools of algebraic topology, is briefly described in §15; for a complete presentation, the interested reader is referred to Dold's monograph [1972]. In Section 7 it is shown that the normalization, additivity, and homotopy properties characterize the degree. This fact (which was first proved by Führer [1971] and later but independently by Zeidler [1972] and Amann–Weiss [1973]) implies that each of the above approaches yields the same notion of degree.

Theorem (9.1) extending Borsuk's antipodal theorem was proved, using the analytic approach, in Schwartz's lecture notes [1969]; the proof in the text is close to that given by Gęba–Granas [1983].

Historical comments

The general degree theory for maps between manifolds was created by Brouwer in 1910–1912. Using the newly introduced technique of simplicial approximation, Brouwer was able to establish in a rigorous way a number of remarkable results:

1° the "invariance of dimension" theorem (already conjectured by Dedekind) stating that $\boldsymbol{R}^n$ is not homeomorphic to $\boldsymbol{R}^m$ when $n \neq m$;

2° the "invariance of domain" theorem asserting that if $U \subset \boldsymbol{R}^n$ is open and $f : U \to \boldsymbol{R}^n$ is continuous and injective, then $f(U)$ is open in $\boldsymbol{R}^n$;

3° the proof that the suitably modified definition of "dimension" of compact subsets of $\boldsymbol{R}^n$ given by Poincaré and the definition given by Lebesgue are equivalent;

4° the "Jordan theorem" in $\boldsymbol{R}^n$, stating that if $\Sigma^{n-1} \subset \boldsymbol{R}^n$ is homeomorphic to $\boldsymbol{S}^{n-1}$, then $\boldsymbol{R}^n - \Sigma^{n-1}$ has exactly two components;

5° "fixed points theorems":

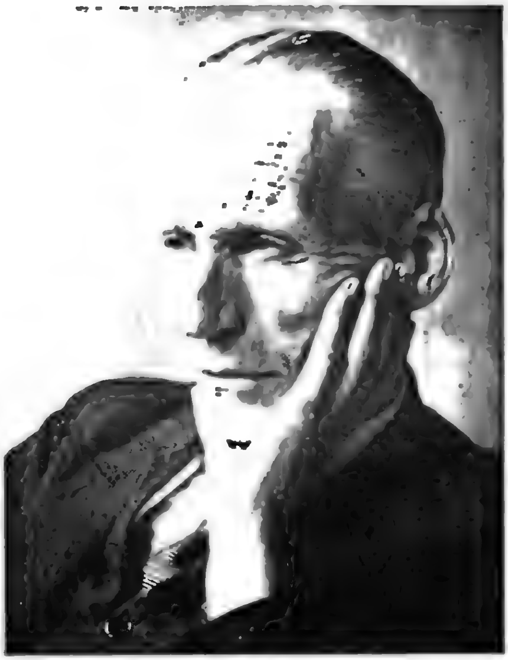
(a) If $f : \boldsymbol{S}^n \to \boldsymbol{S}^n$ has $d(f) \neq (-1)^{n+1}$, then f has a fixed point. [For if not, the consideration of the great circle on $\boldsymbol{S}^n$ joining $-x$ and $f(x)$ provides a homotopy of f to the antipodal map that has degree $(-1)^{n+1}$; cf. Dieudonné's book [1989], pp. 454–472.]

(b) If f maps a closed ball in $\boldsymbol{R}^n$ into itself, then f has a fixed point.

[We may assume that $f : K \to K$, where $K = \boldsymbol{S}^n_+$, the upper hemisphere of $\boldsymbol{S}^n$; let $g : \boldsymbol{S}^n \to K$ be given by

$$g(x) = \begin{cases} f(x), & x \in \boldsymbol{S}^n_+, \\ f(s(x)), & x \in \boldsymbol{S}^n_-, \end{cases}$$

where s is the symmetry with respect to the equator $S^{n-1} \subset S^n$; clearly, $g|S^n_+ = f$, and since g is not surjective, $\deg(g)$ must be zero; by (a), $g(x_0) = x_0$ for some $x_0 \in S^n_+$, and therefore $x_0 = f(x_0)$.]



L.E.J. Brouwer, 1943

Brouwer's approach to degree was based on his fundamental idea that given a map $f : M \to N$ between n-dimensional manifolds, much information about the map can be obtained from the study of the inverse images of a single point in N. He defined the degree of f (by first taking a simplicial approximation φ of f) as "number of inverse images of a point" (counted with appropriate multiplicities) and then proved that the degree is a homotopy invariant of f.

We also remark that while the intuitive notion of deformation was used (without definition) by many mathematicians in the 19th century, it was Brouwer who gave first the modern definition of homotopy (cf. Dieudonné's book [1989], p. 462).

In the special case where $M = N = S^2$, Brouwer succeeded in showing that two maps $f, g : S^2 \to S^2$ are homotopic if and only if $d(f) = d(g)$. The proof of the extension of this result to arbitrary n, due to Hopf [1929], involved restating Brouwer's degree theory in terms of homology. This theorem of Hopf (i.e., Theorem (9.8.3)), together with the Lefschetz–Hopf theorem and the Alexander duality, were the earliest and most complete successes of homology theory.

Very soon after proving Theorem (9.8.3), Hopf found an example of a nontrivial map $f : S^3 \to S^2$, showing that the induced homomorphism in homology cannot detect whether the homotopy class of f is nontrivial; the method used in the construction of the above example was again Brouwer's method of inverse images; the invariant that can distinguish two nonhomotopic maps from S^3 to S^2 is the linking coefficient of the inverse images of two points.

We insert some comments about Brouwer's work due to Freudenthal:

> Seen in historical perspective, Brouwer's performance at that time, in particular his first great paper, on the invariance of dimension, looks like witchcraft. As his magic wand he used the seemingly simple theorem: *If a continuous map f of the n-dimensional cube $K^n = [-1,1]^n$ into $\boldsymbol{R}^n$ displaces every point less than* $1/2$, *then the image $f(K)$ contains an interior point.* He proved this theorem by simplicial approximation of the given mapping, a method which due to Brouwer now seems rather obvious but which was revolutionary in 1911. It is the first example of mixing up combinatorial and set theory methods, which later would lead to the most profound results in topology. Because of this discovery we may rightly claim that, notwithstanding all precursors, modern topology started with Brouwer.
>
> It is no exaggeration to refer to Brouwer's new tools as a magic wand; the results indeed look like witchcraft, the invariance of dimension, the mapping degree, the singularities of spherical vector fields, which factually were his starting point, fixed point theorems, the invariance of domain, with applications to automorphic functions, the Jordan theorem for arbitrary dimension, the invariance of the closed curve and its connectivity, which in germ contains the homology theory of compacta. This indeed is a staggering series of old conjectures come true, and of new theorems and concepts which in due course were to be fundamental in topology, all produced in less than two years.

In the process of preparing the publication of Brouwer's *Collected Works*, Freudenthal unexpectedly discovered "The exercise book" of Brouwer that sheds a new light on the origin and development of Brouwer's discoveries (cf. Freudenthal [1975] and also his article in Brouwer's *Collected Works*). This document (written in Dutch) contains a draft or copy of Brouwer's letter to Hadamard dated January 4, 1910. We insert a fragment of this letter showing that Brouwer, already in January 1910, had a proof of his fixed point theorem in $\boldsymbol{R}^n$, before the publication of Hadamard [1910]:

> Je puis à présent vous communiquer quelques extensions du théorème du point invariant dans les transformations biunivoques et continues de la sphère. Elles se rapportent aux transformations univoques et continues de la sphère. A une telle transformation on peut attribuer un nombre fini n comme son *degré*.
>
> (...) Revenons à une transformation univoque et continue générale. Elle peut être approximée par une série de transformations polynomiales; on démontre que ces dernières ont toutes le même degré: c'est encore le degré de la transformation-limite. Le degré est toujours un nombre entier fini, positif ou négatif. Le degré d'une transformation *bi*univoque est de $+1$, si l'indicatrice reste la même, et de -1, si elle est renversée.

Maintenant le théorème du point invariant généralisé devient le suivant: Toute transformation univoque et continue de la sphère en elle-même, dont le degré n'est pas -1, possède au moins un point invariant.

J'ai encore étendu ce théorème aux sphères à m dimensions. Il s'énonce alors de la manière suivante: Toute transformation univoque et continue de la sphère m-dimensionale en elle-même possède au moins un point invariant, excepté a) quand m est impair et si le degré n est $+1$, b) quand m est pair et si le degré n est -1.

(...) Pour le volume d'une sphère m-dimensionale dans l'espace à $m+1$ dimensions (si nous y comprenons la sphère elle-même) j'ai réussi dernièrement à établir un résultat plus général encore, à savoir: Toute transformation univoque (pas nécessairement biunivoque) et continue du volume d'une sphère m-dimensionale en lui-même possède au moins un point invariant.

Analytical approach

Theorem (8.4) is the basis for the approach to degree theory using only analytical methods. This starts with the observation that given an open bounded $U \subset \boldsymbol{R}^n$, the set $C^\infty(\overline{U}, \boldsymbol{R}^k)$ is dense in $C(\overline{U}, \boldsymbol{R}^k)$; here $f \in C^\infty(\overline{U}, \boldsymbol{R}^k)$ if for some open $V \subset \boldsymbol{R}^n$ with $\overline{U} \subset V$, there is a C^∞ map $g : V \to \boldsymbol{R}^k$ such that $g|\overline{U} = f$. Next, recall that $y_0 \in \boldsymbol{R}^k$ is called a critical value of f if there is some $x_0 \in f^{-1}(y_0)$ for which $f'(x_0) \in \mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^k)$ is not surjective; otherwise, y_0 is called a regular value. A necessary condition for having regular values in $f(\overline{U})$ is $n \geq k$. We now take $n = k$. The Sard theorem asserts that if $f \in C^\infty(\overline{U}, \boldsymbol{R}^k)$, then the set of critical values of f has Lebesgue measure zero. If $C_{\mathrm{r}}^\infty = C_{\mathrm{r}}^\infty(\overline{U}, \boldsymbol{R}^n)$ is the set of all $f \in C^\infty(\overline{U}, \boldsymbol{R}^n)$ for which 0 is a regular value, then C_{r}^∞ is dense in $C(\overline{U}, \boldsymbol{R}^n)$: given a continuous f, approximate it first by a C^∞ map and then translate the latter slightly if 0 is not a regular value.

For $f \in C_{\mathrm{r}}^\infty$ the inverse mapping theorem assures that each $x \in f^{-1}(0)$ has a nbd mapped diffeomorphically onto a nbd of 0, so if U is bounded and $0 \notin f(\partial U)$, then the set $f^{-1}(0)$ is finite. One then defines

$$d(f, U) = \sum \left\{ \frac{\det f'(x_i)}{|\det f'(x_i)|} \,\middle|\, f(x_i) = 0 \right\}$$

for $f \in C_{\mathrm{r}}^\infty$ and proves local constancy (essentially by following the sign of $\det D_x h(x(s), \lambda(s))$ along the curve $s \mapsto (x(s), \lambda(s))$ of the zeros in $U \times \boldsymbol{I}$) and then extends to all $C_0(\overline{U}, \boldsymbol{R}^n)$ exactly as we have done. This satisfies all the axioms, so by uniqueness, it coincides with the degree we have obtained. Observe that this approach and the PL-approach both proceed by "linearizing" maps and both are based on Brouwer's method of inverse images.

§11. Absolute Neighborhood Retracts

A natural topological framework for the concept of the fixed point index, which we shall develop in this chapter, is provided by the theory of ANRs. This paragraph contains a concise presentation of the basic notions and facts of the theory. The last section is concerned with some general fixed point results for compact maps in ANRs.

1. General Properties

(1.1) DEFINITION. A space Y is an *absolute neighborhood retract* if (i) Y is metrizable and (ii) for each metrizable X and every closed $A \subset X$, each continuous $f : A \to Y$ is extendable over some nbd U of A. The class of all absolute neighborhood retracts is denoted by ANR.

It is evident that if Y is an ANR, then every space homeomorphic to Y is also an ANR. We now consider subsets.

(1.2) PROPOSITION. *Let Y be an ANR. Then:*
(i) *any open subset of Y is an ANR,*
(ii) *any nbd retract of Y is an ANR.*

PROOF. (i) Let $W \subset Y$ be open, and let $f : A \to W$ be defined on a closed subset of a metrizable space X. Then f extends over a nbd $U \supset A$ to an $F : U \to Y$; since $A \subset F^{-1}(W)$ and $F^{-1}(W)$ is open, the map $F|F^{-1}(W) : F^{-1}(W) \to W$ is the desired extension of f.

(ii) Let B be a nbd retract of Y, and let $f : A \to B$ be defined on a closed subset of a metrizable space X. Let $r : W \to B$ be a retraction of some nbd W of B in Y. Since W is open, it is an ANR, and hence f has an extension $F : U \to W$ for some nbd $U \supset A$; then $r \circ F$ is the required extension for f. □

For cartesian products, we have

(1.3) PROPOSITION. *A finite cartesian product $\prod_{i=1}^{n} Y_i$ is an ANR if and only if each Y_i is an ANR.*

PROOF. If $\prod Y_i$ is an ANR, then because each factor Y_i is homeomorphic to a cross-section in $\prod Y_i$, and each cross-section is evidently a retract of $\prod Y_i$, it follows that each Y_i is an ANR. Conversely, if each Y_i is an ANR, and if $f : A \to \prod Y_i$ is given, where $A \subset X$ is closed in a metrizable X, we can extend each coordinate function f_i to an $F_i : U_i \to Y_i$ defined on some nbd $U_i \supset A$; then $\bigcap U_i$ is a nbd of A on which all the F_i are defined, and $x \mapsto (F_1(x), \dots, F_n(x))$ is the required extension over that nbd. □

REMARK. (1.3) is not true for infinite cartesian products: although $Y = \prod_{i=1}^{\infty} Y_i$ being an ANR implies that each Y_i is an ANR, the converse is not true.

The term ANR arises because the ANRs are characterized by a very strong retraction property. It is clear that if $i, j : A \to X$ are two embeddings of a space A into X, one of them may send A onto a retract of X, whereas the other may not. To ask that the image of every embedding of a given space into another space be a retract is a strong requirement, nevertheless we have

(1.4) THEOREM. *A metrizable space is an ANR if and only if it is a nbd retract of every metrizable space in which it is embedded as a closed set.*

PROOF. Suppose Y is an ANR, and that $Y \subset Z$ is a closed subset. Consider now the identity $1_Y : Y \to Y$; since $Y \subset Z$ is closed and Y is an ANR, the map 1_Y extends over a nbd U of Y in Z to an $r : U \to Y$, and r is clearly a retraction. Conversely, suppose Y has the property of the theorem. Embed Y as a closed subset of a normed linear space E. It is a nbd retract of E, which is an AR; therefore, by (1.2), Y is an ANR. □

We say that a space X is *r-dominated* by a space Y if there exists a pair of maps $X \xrightarrow{s} Y \xrightarrow{r} X$ such that $rs = 1_X$; for example, if $X \subset Y$ and $r : Y \to X$ is a retraction, then X is r-dominated by Y

With this terminology we give another convenient characterization of ANRs.

(1.5) THEOREM. *A metrizable space X is an ANR if and only if one of the following equivalent conditions holds:*
(a) *X is a nbd retract of some normed linear space.*
(b) *X is r-dominated by an open subset of some normed linear space.*

PROOF. This is a consequence of (1.4) and the Arens–Eells embedding theorem (see Appendix). □

2. ARs and ANRs

Recall that a metrizable Y is an *absolute retract* whenever for any metrizable X and closed $A \subset X$ each $f : A \to Y$ is extendable over X; the class of absolute retracts is denoted by AR.

We clearly have AR $\subset$ ANR; the question of determining which of the ANR are AR has a particularly simple resolution.

(2.1) THEOREM. *Y is an AR if and only if it is a contractible ANR.*

PROOF. Let Y be an AR. Then it is in particular an ANR. Moreover, it must be contractible: for consider the closed subspace $S = Y \times \{0\} \cup Y \times \{1\} \subset Y \times \boldsymbol{I}$ and the map $f : S \to Y$ given by $f(y,0) = y$, $f(y,1) = y_0 \in Y$; because Y is an AR, the map is extendable to an $F : Y \times \boldsymbol{I} \to Y$, and F is precisely a homotopy $\mathrm{id} \simeq y_0$.

Conversely, suppose Y is an ANR and contractible; let $H : Y \times \boldsymbol{I} \to Y$ be a homotopy with $H(y,0) = y$, $H(y,1) = y_0$. Let X be metrizable, $A \subset X$ closed, and $f : A \to Y$. Since Y is an ANR, there is an extension $\widehat{f} : U \to Y$, where $U \supset A$ is open in X. Choose a nbd V with $A \subset V \subset \overline{V} \subset U$ and a continuous $\lambda : X \to \boldsymbol{I}$ with $\lambda|A = 0$, $\lambda|(X - V) = 1$ and define $F : X \to Y$ by

$$F(x) = \begin{cases} H(\widehat{f}(x), \lambda(x)), & x \in \overline{V}, \\ y_0, & x \in X - V \end{cases}$$

This function is continuous, for if $x \in \overline{V} \cap (X - V)$, then $\lambda(x) = 1$ and both definitions agree. Since $F(a) = H(\widehat{f}(a), 0) = \widehat{f}(a) = f(a)$ for $a \in A$, we find that F is the required extension of f. □

3. Local Properties

We first state a simple result that it will be convenient to have at our disposal.

(3.1) LEMMA (Tube lemma). *Let X be a metric space, $A \subset X$ closed, and V an open subset of $X \times \boldsymbol{I}$ with $A \times \boldsymbol{I} \subset V$. Then there exists an open set $U \supset A$ in X such that $U \times \boldsymbol{I} \subset V$.*

PROOF. For each $(a,t) \in A \times \boldsymbol{I}$ take an open set of the form $U_{a,t} \times I_{a,t} \subset V$, where $U_{a,t}$ is a nbd of a in X and $I_{a,t}$ is a nbd of t in $\boldsymbol{I}$. By compactness, a finite number of such nbds $U_{a,t_i} \times I_{a,t_i}$, $i \in [k]$, cover $\{a\} \times \boldsymbol{I}$. Now let $U_a = \bigcap_{i=1}^{k} U_{a,t_i}$ and finally $U = \bigcup_{a \in A} U_a$. □

We are now going to show that the ANRs have a number of nice internal properties. One of these asserts that sufficiently close points can be connected by arcs that vary continuously with the endpoints; i.e., each point has a nbd U and a family of maps $\varphi_{xy} : \boldsymbol{I} \to Y$ (one for each pair $x, y \in U$) that vary continuously with x, y and get small when x, y get closer together. We give an exact formulation of this intuitive statement in

(3.2) THEOREM. *Let Y be an ANR. Then there is a nbd U of the diagonal Δ in $Y \times Y$, and a continuous $\lambda : U \times \boldsymbol{I} \to Y$ such that $\lambda(a,b,0) = a$, $\lambda(a,b,1) = b$, and $\lambda(a,a,t) = a$ for every $(a,b) \in U$, $t \in \boldsymbol{I}$.*

PROOF. On the closed $L = (Y \times Y \times \{0\}) \cup (\Delta \times \boldsymbol{I}) \cup (Y \times Y \times \{1\}) \subset Y \times Y \times \boldsymbol{I}$, define $\psi : L \to Y$ by $(x,y,0) \mapsto x$, $(x,x,t) \mapsto x$, $(x,y,1) \mapsto y$. Since Y is

an ANR, this map extends over an open $W \supset L$ in $Y \times Y \times \boldsymbol{I}$; by the tube lemma there is an open $U \supset \Delta$ with $U \times \boldsymbol{I} \subset W$, and the restriction of ψ to $U \times \boldsymbol{I}$ is the required λ. □

The pair (λ, U) is called *equiconnecting data* for Y; for a pair $a, b \in Y$ with $(a, b) \in U$, $\lambda(a, b, t)$ is a path in Y running from a to b.

(3.3) COROLLARY. *Let (λ, U) be equiconnecting data for an ANR Y Then for each $a \in Y$ and each nbd W of a, there is a nbd V with $a \in V \subset W$ such that $\lambda(V, V, I) \subset W$ (i.e., the paths that vary continuously with the endpoints and that join points in V all lie in W).*

PROOF. Given (λ, U), we have $\lambda^{-1}(W)$ open in the open $U \times \boldsymbol{I}$, hence open in $Y \times Y \times \boldsymbol{I}$; moreover, $\{a\} \times \{a\} \times \boldsymbol{I} \subset \lambda^{-1}(W)$. By the tube lemma, there is an open $\widehat{V} \ni (a, a)$ with $\widehat{V} \times \boldsymbol{I} \subset \lambda^{-1}(W)$; and we can always choose $\widehat{V}$ of the form $V \times V$, where V is a nbd of a. □

The existence of such families of paths leads to the expectation that the ANR and AR have geometrical behavior analogous in some respects to that of locally convex sets in linear spaces. We give two such properties, which are important for our later purposes.

(3.4) THEOREM. *Every ANR is locally contractible (for each $a \in Y$ and a nbd W of a there exists a nbd V of a that is contractible in W).*

PROOF. Let (λ, U) be equiconnecting data for $Y \in$ ANR, and given W, find a nbd V of a such that $\lambda(V, V, I) \subset W$. Then $H : V \times \boldsymbol{I} \to W$ given by

$$(y, t) \mapsto \lambda(y, a, t)$$

is the required homotopy. □

Note, in particular, the strong kind of contractibility: one can, in fact, contract V to a point of W without ever moving any preassigned point of V

For the second application, we recall some terminology.

(3.5) DEFINITION. Let $f, g : X \to Y$ be two maps of a space X into a space Y, and let $\alpha = \{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover of Y. We say that f, g are *α-close* if for each $x \in X$ there is a $U_{\lambda(x)}$ containing both $f(x)$ and $g(x)$. We say that f, g are *α-homotopic* if there is a homotopy $h : f \simeq g$ such that $h(x, \boldsymbol{I})$ is contained in some $U_{\lambda(x)}$ for each $x \in X$. The maps f, g are *stationarily homotopic* if there is a homotopy never changing the image of any $x_0 \in \{x \mid f(x) = g(x)\}$.

We now have

(3.6) THEOREM. *Let Y be an ANR. Then each open cover $\alpha = \{U\}$ has an open refinement $\beta = \{V\}$ with the property: for every space X, any two β-close maps $f, g : X \to Y$ are stationarily α-homotopic.*

PROOF. Take some equiconnecting data for Y. Then (3.3) says that each $a \in Y$ has a nbd V_a contained in some U such that $\lambda(V_a, V_a, I)$ is also contained in that U. The open cover $\beta = \{V_a \mid a \in Y\}$ refines $\alpha = \{U\}$. If X is any space and $f, g : X \to Y$ are β-close, then $(f(x), g(x))$ belongs to a $V_a \times V_a$ for each x, and

$$h(x,t) = \lambda[f(x), g(x), t], \quad t \in \boldsymbol{I},$$

is a stationary α-homotopy of f to g. □

4. Pasting ANRs Together

The main burden of the development will be carried by the following general

(4.1) LEMMA (Kuratowski). *Let Y be a metrizable space such that $Y = Y_1 \cup Y_2$, where Y_1, Y_2, and $Y_1 \cap Y_2$ are ANRs. Let X be metrizable, $A \subset X$ closed, and let $f : A \to Y$. If X can be represented as the union $B_1 \cup B_2$ of two closed sets such that $f(A \cap B_1) \subset Y_1$, $f(A \cap B_2) \subset Y_2$, then f is extendable over a nbd of A in X.*

PROOF. We have $f : A \cap B_1 \cap B_2 \to Y_1 \cap Y_2$. Since $Y_1 \cap Y_2$ is an ANR and $A \cap B_1 \cap B_2$ is closed in X, f extends over a nbd in X, hence over the closure of a smaller nbd. We therefore have an H open in $B_1 \cap B_2$ with $A \cap B_1 \cap B_2 \subset H \subset B_1 \cap B_2$ and an extension $h : \overline{H} \to Y_1 \cap Y_2$ of $f|\overline{H} \cap A$.

I. Let $F = B_1 \cap B_2 - H$; then F is closed in $B_1 \cap B_2$, hence in X, and

$$F \cap A = \emptyset. \tag{1}$$

The two maps

$$f_i = \begin{cases} f & \text{on } A \cap B_i, \\ h & \text{on } \overline{H}, \end{cases} \qquad (i = 1, 2)$$

are consistently defined on $A \cap B_i \cap \overline{H} = A \cap B_1 \cap B_2$, so are continuous.

II. Each of the f_i sends the closed $(A \cap B_i) \cup \overline{H}$ into Y_i. Since Y_i is an ANR, there is an extension of f_i over a nbd U_i in X to $g_i : U_i \to Y_i$. Let $C_i = B_i - U_i$, $i = 1, 2$.

The C_i are closed in X and

$$A \cap C_i = \emptyset, \quad \overline{H} \cap C_i = \emptyset. \tag{2}$$

III. We now remove the possible inconsistencies. Define $U = X - (F \cup C_1 \cup C_2)$. Then U is open in X; moreover, $A \subset U$ by (1), (2). Set

$$f^* = \begin{cases} g_1 & \text{on } B_1 \cap U, \\ g_2 & \text{on } B_2 \cap U. \end{cases}$$

This is defined on the open U. Since each g_i is continuous on U and $U \cap F = \emptyset$, we have

$$B_1 \cap B_2 \cap U = B_1 \cap B_2 - F = H,$$

so g_1, g_2 are consistently defined on the intersection of two closed subsets of U, and therefore f^* is continuous on U. Thus, f extends over a nbd of A, and the proof is complete. □

In (4.1), Y_1, Y_2 are arbitrary ANRs. If they are assumed to be either both closed or both open, the required decomposition of X is always possible, and we thus obtain the following two important results:

(4.2) THEOREM (Aronszajn–Borsuk). *Let Y be a metrizable space and $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are closed.*
(a) *If Y_1, Y_2, and $Y_1 \cap Y_2$ are ANRs, then Y is an ANR.*
(b) *If Y_1, Y_2, and $Y_1 \cap Y_2$ are ARs, then Y is an AR.*

PROOF. (a) Let X be metrizable, $A \subset X$ closed, and let $f : A \to Y$ be a given map. The theorem will follow from (4.1) if we can show $X = B_1 \cup B_2$, where B_i are closed and $f(A \cap B_i) \subset Y_i$. The sets $f^{-1}(Y_1)$, $f^{-1}(Y_2)$, being closed in A, are closed in X.

Define

$$B_1 = \{x \mid \operatorname{dist}(x, f^{-1}(Y_1)) \leq \operatorname{dist}(x, f^{-1}(Y_2))\},$$
$$B_2 = \{x \mid \operatorname{dist}(x, f^{-1}(Y_2)) \leq \operatorname{dist}(x, f^{-1}(Y_1))\},$$

so that $X = B_1 \cup B_2$ and each B_i is closed in X. We show $f(A \cap B_i) \subset Y_i$ for $i = 1, 2$; by symmetry only $f^{-1}(Y_1) = A \cap B_1$ needs a proof: if $a \in f^{-1}(Y_1)$, then $a \in B_1$, since $\operatorname{dist}(a, f^{-1}(Y_1)) = 0$, so $a \in A \cap B_1$; if $a \notin f^{-1}(Y_1)$, then $a \in f^{-1}(Y_2)$ and $\operatorname{dist}(a, f^{-1}(Y_1)) > 0$, so $a \notin B_1$, i.e., $a \notin A \cap B_1$. Thus, $f^{-1}(Y_1) = A \cap B_1$, and the proof of (a) is complete.

The proof of (b) is similar and is left to the reader. □

(4.3) COROLLARY. *Let Y be an ANR and $B \subset Y$ a closed ANR. Then $Y \times \{0\} \cup B \times I$ is an ANR.*

PROOF. By (1.3), the product $B \times I$ is an ANR, and (4.2) applies. □

(4.4) COROLLARY. *Any finite union of closed metrizable convex sets in a locally convex space is an ANR.*

PROOF. We show by induction on n that $\bigcup_{i=1}^{n} C_i \in$ ANR for any closed convex metrizable C_i $(i = 1, \ldots, n)$. For $n = 1$, our statement is true because, by the Dugundji extension theorem (7.7.4), any convex metrizable set in a locally convex space is an AR. Assuming that it is true for all integers smaller than n, we have $Y = \bigcup_{i=1}^{n-1} C_i \in$ ANR, $C_n \in$ ANR, and hence $Y \cap C_n = \bigcup_{i=1}^{n-1}(C_i \cap C_n) \in$ ANR, because $C_i \cap C_n \in$ ANR. Thus $Y \cup C_n$ is the union of two closed ANRs whose intersection is an ANR, and because $Y \cup C_n$ is metrizable (in view of Theorem (4.4.19) in Engelking's book [1989], p. 286), our assertion follows from (4.2). □

(4.5) COROLLARY. *Any finite polyhedron is a compact ANR.* □

(4.6) COROLLARY. *Let Y be a space, $U \subset Y$ open, and $X \subset U$ compact. Assume furthermore that one of the following conditions is satisfied:*
(i) *Y is a Banach space.*
(ii) *Y is a metrizable compact convex subset of a locally convex linear topological space.*
Then there exists a compact set $C \in$ ANR such that $X \subset C \subset U$.

PROOF. In case (i), cover X by a finite number of closed balls $K_1, \dots, K_s \subset U$, and let $C_i = \mathrm{Conv}(K_i \cap X)$ for $i \in [s]$. Since Y is a Banach space, by Mazur's theorem (see Appendix), each $C_i \subset U$ is convex compact, and we infer that X is contained in the compact set $C = \bigcup C_i \subset U$. Now our assertion clearly follows from (4.4).

The proof of (ii) is analogous and is left to the reader. □

A consequence of (4.1) whenever $Y = Y_1 \cup Y_2$ with Y_i open is

(4.7) THEOREM. *Let Y be a metrizable space, and let $Y = U_1 \cup U_2$, where U_1, U_2 are open. If U_1, U_2 are ANRs, then so also is Y.*

PROOF. Note that we do not have to assume $U_1 \cap U_2 \in$ ANR, because it is open in $U_1 \in$ ANR and (1.2) applies.

Let X be metrizable, $A \subset X$ closed, and let $f : A \to Y$ be given. By (4.1) we need only show that $X = B_1 \cup B_2$, where B_i are closed and $f(A \cap B_i) \subset U_i$. To this end consider first

$$A_1 = A - f^{-1}(U_1), \quad A_2 = A - f^{-1}(U_2);$$

these are disjoint closed sets in X, so there is a continuous $\lambda : X \to \boldsymbol{I}$ with $\lambda(x) = 1$ for $x \in A_1$ and $\lambda(x) = 0$ for $x \in A_2$.

Let

$$B_1 = \{x \mid \lambda(x) \le \tfrac{1}{2}\}, \quad B_2 = \{x \mid \lambda(x) \ge \tfrac{1}{2}\},$$

so that B_1, B_2 are closed and $X = B_1 \cup B_2$. We have $f(A \cap B_1) \subset U_1$: for if $a \in A \cap B_1$, then $\lambda(a) \le \frac{1}{2}$, so $a \notin A_1$, i.e., $a \in A - (A - f^{-1}(U_1)) = f^{-1}(U_1)$. Similarly $f(A \cap B_2) \subset U_2$, and the proof is complete. □

(4.8) COROLLARY. *Let Y be a metrizable space that admits a finite open covering by ANRs. Then Y is an ANR.* □

(4.9) COROLLARY. *Any compact locally Euclidean space is an ANR.* □

5. Theorem of Hanner

The fact that (4.7) is valid with no additional assumption on $U_1 \cap U_2$ leads one to suspect that the ANR property may be local. In this section we shall establish the fundamental localization theorem.

(5.1) THEOREM. *Let Y be a metrizable space and $\{U_\lambda \mid \lambda \in \Lambda\}$ a family of pairwise disjoint open ANR subspaces. Then their union $\bigcup_\lambda U_\lambda$ is also an ANR.*

PROOF. Let X be metrizable, $A \subset X$ be closed, and let $f : A \to \bigcup_\lambda U_\lambda$ be continuous. Because of pairwise disjointness, the union of any family of the U_λ is closed in $\bigcup_\lambda U_\lambda$, so the $A_\lambda = f^{-1}(U_\lambda)$ are pairwise disjoint and the union of any family of the A_λ is closed in A, hence in X.

Putting $V_\lambda = \{x \in X \mid \operatorname{dist}(x, A_\lambda) < \operatorname{dist}(x, A - A_\lambda)\}$, we obtain a family $\{V_\lambda \mid \lambda \in \Lambda\}$ of open sets in X with $A_\lambda \subset V_\lambda$ for each λ; moreover, the V_λ are pairwise disjoint because $\operatorname{dist}(x, A - A_\lambda) \le \operatorname{dist}(x, A_{\lambda'})$ if $\lambda \neq \lambda'$. Since each U_λ is an ANR, we can extend each $f|A_\lambda$ over a nbd $W_\lambda \supset A_\lambda$ in X to an $F_\lambda : W_\lambda \to U_\lambda$; the map $F : \bigcup_\lambda (V_\lambda \cap W_\lambda) \to \bigcup_\lambda U_\lambda$ defined to be F_λ on $V_\lambda \cap W_\lambda$ is then consistently defined and represents a continuous extension of f over a nbd of A in X. □

(5.2) THEOREM. *Let Y be a metrizable space that is the union of a countable family $\{V_n \mid n = 1, 2, \ldots\}$ of open ANRs. Then Y is an ANR.*

PROOF. We may assume without loss of generality that $V_i \subset V_{i+1}$ for each $i = 1, 2, \ldots$ (otherwise, we would consider the family $\{V_n^* = \bigcup_{k=1}^n V_k \mid n = 1, 2, \ldots\}$), and that $Y \neq V_i$ for all $i = 1, 2, \ldots$. We now shrink $\{V_n\}$ to an open covering $\{W_n\}$ by putting

$$W_i = \{y \in Y \mid \operatorname{dist}(y, Y - V_i) > 1/2^i\} \quad (i = 1, 2, \ldots).$$

Clearly, W_i is open in V_i and hence an ANR, $\overline{W}_i \subset W_{i+1}$, and $\bigcup_{i=1}^\infty W_i = Y$.

We now form the "rings"

$$R_1 = W_1, \quad R_2 = W_2, \quad R_3 = W_3 - \overline{W}_1, \quad \ldots, \quad R_n = W_n - \overline{W}_{n-2}.$$

Because each W_n is an ANR, so is each R_n; moreover, $Y = \bigcup_{n=1}^\infty R_n$ because if $n = \min\{i \mid x \in W_i\}$, then $x \in W_n - W_{n-1} \subset R_n$.

Now $Y = \bigcup_{n=1}^\infty R_{2n} \cup \bigcup_{n=1}^\infty R_{2n-1}$; since $\{R_{2n} \mid n = 1, 2, \ldots\}$ is a disjoint family of open ANRs, it follows from (5.1) that $\bigcup_{n=1}^\infty R_{2n} \in$ ANR; similarly, $\bigcup_{n=1}^\infty R_{2n-1} \in$ ANR. Thus Y is the union of two open ANRs, so by (4.7) we conclude that $Y \in$ ANR. □

(5.3) THEOREM (Hanner). *Let Y be a metrizable space such that each $y \in Y$ admits a nbd U that is an ANR. Then $Y \in$ ANR.*

PROOF. By assumption, Y has an open cover $\{U_\lambda \mid \lambda \in \Lambda\}$ with each $U_\lambda \in$ ANR. According to (B.10) in the Appendix, this cover has a σ-discrete open refinement $\{U_{\lambda,n} \mid (\lambda, n) \in \Lambda \times Z^+\}$, where for each fixed integer n, the family $\{U_{\lambda,n}\}$ is pairwise disjoint and each $U_{\lambda,n}$ is contained in U_λ. It follows that each $U_{\lambda,n}$ is an ANR, and therefore, from (5.1), that $V_n = \bigcup_\lambda U_{\lambda,n}$ is

an ANR for each n; consequently, $Y = \bigcup_{n=1}^{\infty} V_n$ with each $V_n \in$ ANR. Now our assertion follows from (5.2). □

(5.4) COROLLARY. *Any metrizable Banach manifold is an ANR.* □

6. Homotopy Properties

We begin with the following useful

(6.1) LEMMA (Dowker). *Let X be a metric space, $A \subset X$ closed, and assume that $V \subset X \times \boldsymbol{I}$ is an open nbd of the partial cylinder*

$$M = X \times \{0\} \cup A \times \boldsymbol{I}.$$

Then there exists a map $\varphi : X \times \boldsymbol{I} \to V$ such that $\varphi(x,t) = (x,t)$ for all $(x,t) \in M$.

PROOF. By the tube lemma there exists an open $U \supset A$ in X such that $U \times \boldsymbol{I} \subset V$. Because the closed sets A and $X - U$ are disjoint, there exists a continuous $\lambda : X \to \boldsymbol{I}$ with $\lambda|A = 1$, $\lambda|X - U = 0$. Now letting

$$\varphi(x,t) = (x, \lambda(x)t) \qquad \text{for } (x,t) \in X \times \boldsymbol{I},$$

we observe that if $x \in U$, then $(x, \lambda(x)t) \in V$, and if $x \in X - U$, then $\varphi(x,t) = (x,0) \in M \subset V$. Because clearly $\varphi|M = \mathrm{id}_M$, we conclude that φ is the desired map. □

Let Y be an ANR, X metrizable, and $A \subset X$ closed. We know that every $f : A \to Y$ is extendable over an open $U \supset A$. We now address the problem of whether or not f is, in fact, extendable over the entire space X. The following theorem of Borsuk asserts that this problem reduces to one in the homotopy category: if any single $g : A \to Y$ homotopic to f is extendable, then so also will be f. In more pictorial terms: if we map X into an ANR and pull the image of A around, the image of X will always follow.

(6.2) THEOREM (Borsuk). *Let Y be an ANR, X a metrizable space, and $A \subset X$ closed. Let $f, g : A \to Y$ be homotopic. If f is extendable to an $F : X \to Y$, then g is also extendable to a $G : X \to Y$; moreover, G can be so chosen that the given homotopy of f to g extends to a homotopy of F to G.*

PROOF. Form the partial cylinder $M = X \times \{0\} \cup A \times \boldsymbol{I}$, and let $h : A \times \boldsymbol{I} \to Y$ be a homotopy joining f and g. Define $\Psi_0 : M \to Y$ by

$$\begin{aligned} \Psi_0(x,0) &= F(x) \qquad && \text{for } x \in X, \\ \Psi_0(a,t) &= h(a,t) \qquad && \text{for } (a,t) \in A \times \boldsymbol{I}. \end{aligned}$$

Since M is closed in $X \times \boldsymbol{I}$ and Y is an ANR, Ψ_0 extends over an open $V \supset M$ to $\Psi : V \to Y$. Using the Dowker lemma (6.1), we choose for this V

a map $\varphi : X \times \boldsymbol{I} \to V$ such that $\varphi|M = \mathrm{id}_M$, and define $H : X \times \boldsymbol{I} \to Z$ as the composite

$$X \times \boldsymbol{I} \xrightarrow{\varphi} V \xrightarrow{\Psi} Y.$$

Then, because for each $(x, t) \in M$,

$$H(x,t) = \Psi[\varphi(x,t)] = \Psi(x,t) = \Psi_0(x,t),$$

we infer that $x \mapsto G(x) = H(x, 1)$ is an extension of g over X, and H is the desired homotopy of F to G extending h over $X \times \boldsymbol{I}$. □

(6.3) COROLLARY. *Let Y be an ANR, X metrizable, and $A \subset X$ closed. If $f : A \to Y$ is nullhomotopic, then f is extendable to an $F : X \to Y$ that is also nullhomotopic.*

PROOF. The constant map of A into Y is evidently extendable over X, and Borsuk's theorem applies. □

The fact that an extension problem can be reduced to a homotopy question is very important. In fact, any space X (not necessarily metric) and any closed $A \subset X$ for which it is true that for every space Z (not necessarily an ANR) the extendability of maps of A into Z always reduces to a homotopy question, are called cofibered pairs. More precisely:

(6.4) DEFINITION. Let X be a space and $A \subset X$ closed. We say that (X, A) is a *cofibered pair* if for every space Z and two homotopic maps of A into Z, if one is extendable over X, so also is the other, and the extensions can be chosen homotopic by a homotopy extending the given one.

(6.5) THEOREM. *Let X be a metric space and $A \subset X$ closed. If the partial cylinder $M = X \times \{0\} \cup A \times \boldsymbol{I}$ is a nbd retract in $X \times \boldsymbol{I}$, then (X, A) is a cofibered pair.*

PROOF. Let Z be any space and $f, g : A \to Z$. Assume that $F : X \to Z$ is an extension of f and that $h : A \times \boldsymbol{I} \to Z$ is a homotopy joining f and g.

Define $\Psi_0 : M \to Z$ by

$$\begin{aligned} \Psi_0(x,0) &= F(x) \qquad &&\text{for } x \in X, \\ \Psi_0(a,t) &= h(a,t) \qquad &&\text{for } (a,t) \in A \times \boldsymbol{I}. \end{aligned}$$

By assumption there is an open U in $X \times \boldsymbol{I}$ and a retraction $r : U \to M$. Using the Dowker lemma, we choose for this U a map $\varphi : X \times \boldsymbol{I} \to U$ such that $\varphi|M = \mathrm{id}_M$, and define $H : X \times \boldsymbol{I} \to Y$ as the composite

$$X \times \boldsymbol{I} \xrightarrow{\varphi} U \xrightarrow{r} M \xrightarrow{\Psi_0} Z.$$

Then, because for each $(x, t) \in M$,

$$H(x,t) = \Psi_0[r\varphi(x,t)] = \Psi_0 r(x,t) = \Psi_0(x,t),$$

the map $y \mapsto H(y,1)$ is the desired extension G of g, and H is a homotopy of F to G extending h. □

(6.6) COROLLARY. *Let A be a closed ANR subspace of an ANR space X. Then (X, A) is a cofibered pair.*

PROOF. By (4.2) the partial cylinder $M = X \times \{0\} \cup A \times \boldsymbol{I}$ is an ANR, and hence, as a closed subset of $X \times \boldsymbol{I}$, it is a nbd retract in $X \times \boldsymbol{I}$. □

For the last property, we recall some terminology.

(6.7) DEFINITION. Let X be a space and $B \subset X$.

(a) By a *deformation* of B in X is meant a homotopy $D : B \times \boldsymbol{I} \to X$ with $D(b,0) = b$ for all $b \in B$; if moreover $D(B \times \{1\})$ is contained in a subspace $A \subset X$, then D is said to be a *deformation of B into A in X*.

(b) A closed set $A \subset B$ is called a *strong deformation retract* of B in X if there exists a deformation $D : B \times \boldsymbol{I} \to X$ of B into A in X such that $D(a,t) = a$ for all $(a,t) \in A \times \boldsymbol{I}$.

(c) The set A is a *deformation retract* of B in X if the last condition is valid only for $t = 1$.

(6.8) THEOREM. *Let A be a closed ANR subset of an ANR X. Then for some open $U \supset A$ the set A is a strong deformation retract of U in X.*

PROOF. Let $r : V \to A$ be a retraction of a closed nbd V of A in X. On the set

$$N = V \times \{0\} \cup A \times \boldsymbol{I} \cup V \times \{1\}$$

define a continuous $\Phi_0 : N \to X$ by

$$\begin{aligned} \Phi_0(x,0) &= x && \text{for } x \in V, \\ \Phi_0(a,t) &= a && \text{for } (a,t) \in A \times \boldsymbol{I}, \\ \Phi_0(x,1) &= r(x) && \text{for } x \in V. \end{aligned}$$

Since N is closed in $X \times \boldsymbol{I}$ and X is an ANR, this Φ_0 extends over an open $W \supset N$ to a map $\Phi : W \to X$. The tube lemma gives an open $U \supset A$, $U \subset V$ with $U \times \boldsymbol{I} \subset W$, and the restriction of Φ to $U \times \boldsymbol{I}$ gives the desired strong deformation retraction of U onto A in X. □

7. Generalized Leray–Schauder Principle in ANRs

In this section we extend to arbitrary ANRs some basic fixed point theorems for compact operators. The method is elementary and is based on some properties of compact maps into ANRs which use the notion of extendability in their formulation.

(7.1) DEFINITION. Let X, Y be two spaces, and let $\mathscr{K}(X,Y)$ be the set of all compact maps from X to Y.

(a) If $A \subset X$, then a map $f \in \mathscr{K}(A,Y)$ is said to be *compactly extendable* over X if there is a map $F \in \mathscr{K}(X,Y)$ with $F|A = f$.

(b) Two maps $f, g \in \mathscr{K}(X,Y)$ are called *compactly homotopic* if there exists a compact map $h : X \times \boldsymbol{I} \to Y$ with $h(x,0) = f(x)$, $h(x,1) = g(x)$ for each $x \in X$. The map h is a *compact homotopy* joining f and g, and we write $h : f \simeq g$ in $\mathscr{K}(X,Y)$; the one-parameter family of maps $\{h_t : X \to Y\}$, which is determined by h in a familiar manner, is also called a compact homotopy.

(c) A map $f \in \mathscr{K}(X,Y)$ is *compactly nullhomotopic* if it is compactly homotopic to a constant map from X to Y

An important property of ANRs is given in

(7.2) THEOREM. *Let Y be an ANR, X a metric space, and $A \subset X$ closed. Then any $f \in \mathscr{K}(A,Y)$ is compactly extendable over an open nbd U of A in X.*

PROOF. Let $K = \overline{f(X)}$; since K is compact, there is an embedding $h^{-1} : K \to \boldsymbol{I}^\infty$ onto a closed subset $\widehat{K} \subset \boldsymbol{I}^\infty$. The map $f : A \to Y$ can therefore be factored

$$A \xrightarrow{\varphi} \widehat{K} \xrightarrow{g} Y$$

as in the following diagram:

$$\begin{tikzcd}
X \arrow[r, dashed] & \boldsymbol{I}^\infty & Y \\
& \widehat{K} \arrow[u, hook] \arrow[ur, "g"] & \\
A \arrow[uu, hook] \arrow[ur, "\varphi"] \arrow[rr] & & K \arrow[ul, "h^{-1}"'] \arrow[uu, hook]
\end{tikzcd}$$

By the Tietze theorem, we extend φ to a map $\Phi : X \to \boldsymbol{I}^\infty$ Because Y is an ANR, g has an extension $G : W \to Y$ over some nbd W of $\widehat{K}$ in $\boldsymbol{I}^\infty$. Choose an open V with $\widehat{K} \subset V \subset \overline{V} \subset W$, and let $U = \Phi^{-1}(V)$. Then the map $F = G\Phi|U : U \to Y$ is an extension of f over the nbd $U \supset A$. Because $\overline{V}$ is compact, so also is $G(\overline{V})$ and from $F(U) = G\Phi[\Phi^{-1}(V)] \subset G(V) \subset G(\overline{V}) = \overline{G(V)}$ we conclude that the map F is compact. □

We now extend Borsuk's theorem to arbitrary compact maps.

(7.3) THEOREM. *Let Y be an ANR, X a metric space, $A \subset X$ closed, and let $f, g \in \mathscr{K}(A,Y)$ be two compactly homotopic maps. Assume that f is compactly extendable to an $F \in \mathscr{K}(X,Y)$. Then g is also compactly*

extendable to a $G \in \mathscr{K}(X,Y)$; *moreover,* G *can be so chosen that the given compact homotopy of* f *to* g *extends to a compact homotopy of* F *to* G.

PROOF. Form the partial cylinder $M = X\times\{0\}\cup A\times \boldsymbol{I}$ and let $h : A\times\boldsymbol{I} \to Y$ be a compact homotopy joining f and g. Define a compact map $\Psi_0 : M \to Y$ by

$$\begin{aligned} \Psi_0(x,0) &= F(x) && \text{for } x \in X, \\ \Psi_0(a,t) &= h(a,t) && \text{for } (a,t) \in A \times \boldsymbol{I}. \end{aligned}$$

Since M is closed in $X \times \boldsymbol{I}$ and Y is an ANR, this Ψ_0 extends, in view of (7.2), over an open $V \supset M$ to a compact $\Psi : V \to Y$. Using Dowker's lemma, choose for this V a map $\varphi : X\times\boldsymbol{I} \to V$ with $\varphi|M = \mathrm{id}_M$, and define a compact homotopy $H : X \times \boldsymbol{I} \to Y$ as the composite

$$X \times \boldsymbol{I} \xrightarrow{\varphi} V \xrightarrow{\Psi} Y.$$

Then, because for each $(x,t) \in M$,

$$H(x,t) = \Psi[\varphi(x,t)] = \Psi(x,t) = \Psi_0(x,t),$$

we infer that $x \mapsto G(x) = H(x,1)$ is a compact extension of g over X, and H is the desired compact homotopy of F to G extending h over $X \times \boldsymbol{I}$. □

As an immediate consequence we obtain

(7.4) THEOREM (Generalized Schauder theorem in ANRs). *Let* Y *be an ANR, and let* $f \in \mathscr{K}(Y,Y)$ *be a compactly nullhomotopic map. Then* f *has a fixed point.*

PROOF. Using the Arens–Eells theorem, embed Y as a closed subset in a normed linear space E. By assumption, $f \in \mathscr{K}(Y,Y)$ is compactly homotopic to a constant map $g : Y \to Y$. Because g is extendable over E, it follows from (7.3) that f extends over E to a compact $F \in \mathscr{K}(E,Y)$. For this F, by the Schauder fixed point theorem, $x_0 = F(x_0)$ for some $x_0 \in E$; but since the values of F lie in Y and $F|Y = f$, we get $x_0 = F(x_0) = f(x_0)$, completing the proof. □

To formulate the next result we introduce some terminology.

(7.5) DEFINITION. Let Y be a space, $U \subset Y$ open, and ∂U the boundary of U in Y. The set of all compact maps $f : \overline{U} \to Y$ such that the restriction $f|\partial U : \partial U \to Y$ is fixed point free is denoted by $\mathscr{K}_{\partial U}(\overline{U},Y)$. A compact homotopy $h_t : \overline{U} \to Y$ is said to be *admissible* if $h_t \in \mathscr{K}_{\partial U}(\overline{U},Y)$ for each $t \in \boldsymbol{I}$. Two maps $f,g \in \mathscr{K}_{\partial U}(\overline{U},Y)$ are called *admissibly homotopic*, written $f \simeq g$ in $\mathscr{K}_{\partial U}(\overline{U},Y)$, if there is an admissible compact homotopy $h_t : \overline{U} \to Y$ with $h_0 = f$, $h_1 = g$.

A compact map $f \in \mathscr{K}_{\partial U}(\overline{U}, Y)$ is said to be *compactly nullhomotopic* (notation $f \simeq 0$ in $\mathscr{K}_{\partial U}(\overline{U}, Y)$) if it is admissibly homotopic to a constant map g sending $\overline{U}$ into a single point $u_0 \in U$

With this terminology we now establish a general fixed point result formulated in purely topological terms.

(7.6) THEOREM. *Let Y be an ANR, $U \subset Y$ open, and let $f \in \mathscr{K}_{\partial U}(\overline{U}, Y)$ be a compactly nullhomotopic map. Then f has a fixed point.*

PROOF. Let $h : \overline{U} \times \boldsymbol{I} \to Y$ be an admissible compact homotopy joining the constant map g with $g(U) = u_0 \in U$ and the map $f \in \mathscr{K}_{\partial U}(\overline{U}, Y)$. Because g extends to a constant map $G : Y \to Y$, it follows from (7.3) that the homotopy h extends to a compact homotopy $H : Y \times \boldsymbol{I} \to Y$

Let $B = \{x \in Y - U \mid H(x,t) = x \text{ for some } t \in \boldsymbol{I}\}$. We may assume $B \neq \emptyset$ (otherwise, f would have a fixed point in U by (7.4)). Noting that B and $A = \overline{U}$ are closed and disjoint subsets of Y, choose $\lambda : Y \to \boldsymbol{I}$ with $\lambda | A = 1$, $\lambda | B = 0$. Define a compact map $F : Y \to Y$ by

$$F(y) = H(y, \lambda(y)) \quad \text{for } y \in Y$$

We claim that F is fixed point free on $Y - U$: for if

$$x_0 = F(x_0) = H(x_0, \lambda(x_0)) \quad \text{for } x_0 \in Y - U,$$

then $x_0 \in B$ and $x_0 = H(x_0, 0) = G(x_0) = u_0$, which contradicts the assumption that $u_0 \in U$. By (7.4), this implies that F must have a fixed point y_0 in U, and thus, because $F|U = f$, we have $y_0 = F(y_0) = f(y_0)$. □

Theorem (7.6) can be rephrased as

(7.7) THEOREM (Generalized Leray–Schauder principle in ANRs). *Let Y be an ANR, $U \subset Y$ open, and let g be a constant map sending $\overline{U}$ to a point $u_0 \in U$. Then any compact homotopy $h_t : \overline{U} \to Y$ joining $h_0 = g$ and $h_1 = f$ has one of the following two properties:*
(a) *f has a fixed point in U,*
(b) *there are $y_0 \in \partial U$ and $\lambda \in (0, 1]$ with $y_0 = h_\lambda(y_0)$.* □

We remark that (7.4) and the classical Leray–Schauder principle (6.5.1) are special cases of (7.7).

8. Miscellaneous Results and Examples

A. Identifications

Let X be a topological space, Y an arbitrary set, and $p : X \to Y$ a surjection. The *identification topology* $\mathbb{T}(p)$ induced by p on Y is

$$\mathbb{T}(p) = \{U \subset Y \mid p^{-1}(U) \text{ is open in } X\}.$$

If X and Y are two spaces, a continuous surjection $p : X \to Y$ is an *identification map* (or briefly an *identification*) if the topology in Y is exactly $\mathbb{T}(p)$.

(A.1) Show: (a) A continuous surjection $p : X \to Y$ is an identification if and only if for any $g : Y \to Z$, the continuity of gp implies that of g; (b) if a continuous surjection $p : X \to Y$ is an open (or closed) map, then p is an identification.

(A.2) Let $p : X \to Y$ be an identification, and $h : X \to Z$ be continuous such that hp^{-1} is single-valued [i.e., h is constant on each $p^{-1}(y)$]. Show:

(a) $hp^{-1} : Y \to Z$ is continuous and $h = (hp^{-1})p$.

(b) $hp^{-1} : Y \to Z$ is an open (respectively closed) map if and only if $h(U)$ is open (respectively closed) whenever U is a p-saturated open (respectively closed) set. ($A \subset X$ is *p-saturated* whenever $A = p^{-1}[p(A)]$.)

(A.3) Let $p : X \to Y$ be an identification, Z a set, and $g \cdot Y \to Z$ a surjection. Show: (a) $\mathbb{T}(gp) = \mathbb{T}(g)$; (b) if Z is a space, then gp is an identification if and only if g is.

(A.4) Let $p : X \to Y$ be an identification and K a locally compact space. Show: The map $p \times \mathrm{id} : X \times K \to Y \times K$ is an identification map.

B. Quotient spaces

Let R be an equivalence relation in a space X. The quotient set X/R with the identification topology determined by the projection $p : X \to X/R$ is called the *quotient space* of X by R. If $A \subset X$, then X/A is the quotient space obtained using the equivalence relation whose equivalence classes are A and the single points of $X - A$.

(B.1) Let X, Y be spaces with equivalence relations R, S, respectively, and let $f : X \to Y$ be a relation-preserving, continuous map. Show: (a) The induced map $\widehat{f} : X/R \to Y/S$ is continuous; (b) if f is an identification, then so is $\widehat{f}$.

(B.2) Let $f : X \to Y$ be continuous. Define the equivalence relation $K(f)$ in X by $x \sim x'$ if $fx = fx'$, and denote by $p : X \to X/K(f)$ the identification map; $X/K(f)$ is called the *decomposition space* of f. Show:

(a) f can always be factored as $X \xrightarrow{p} X/K(f) \xrightarrow{g} Y$ with p surjective and g injective.

(b) If the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ Z \arrow[r, "h"] & W \end{tikzcd}$$

commutes, then there is a continuous $\lambda : X/K(f) \to Z/K(h)$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "\varphi"'] & X/K(f) \arrow[r, "g"] \arrow[d, "\lambda"] & Y \arrow[d, "\psi"] \\ Z \arrow[r, "p'"] & Z/K(h) \arrow[r, "g'"] & W \end{tikzcd}$$

is commutative.

(B.3) Let X be a space and $J = [-1, +1]$. The *suspension* SX of X is the quotient space of $X \times J$ obtained by identifying $X \times \{+1\}$ and $X \times \{-1\}$ to points. The *cone* CX over X is the quotient space $CX = X \times [0,1]/X \times \{1\}$; the elements of SX and also of CX are denoted by $\langle x, t \rangle$. Show:

(a) $SX \approx CX/X \times \{0\}$.
(b) CX is homeomorphic to the subspace $\{(x,t) \in SX \mid t \geq 0\}$.
(c) A continuous $f : X \to Y$ induces a continuous map $Sf : SX \to SY$ and also $Cf : CX \to CY$ by $(x,t) \mapsto (f(x),t)$.
(d) The assignments $X \mapsto SX$, $f \mapsto Sf$ and $X \mapsto CX$, $f \mapsto Cf$ define functors $S, C : \mathbf{Top} \to \mathbf{Top}$.

(B.4) Let X, Y be two disjoint spaces, $A \subset X$ closed, and $f : A \to Y$ continuous. In the disjoint union $X + Y$, we generate an equivalence relation R by $a \sim f(a)$ for each $a \in A$. The quotient space $X \cup Y/R$ is the *space obtained by attaching X to Y by f*, and is written $Y \cup_f X$; let $p : X + Y \to Y \cup_f X$ be the projection. Show:
(a) $p|Y : Y \to Y \cup_f X$ is an embedding onto a closed subspace.
(b) $p|X - A : X - A \to Y \cup_f X$ is an embedding onto an open subspace.
(c) CX is obtained by attaching $X \times [0,1]$ to a point q_0 by $f(X \times \{1\}) = \{q_0\}$.
(d) SX is obtained by attaching $X \times J$ to a two-point set $\{q^+, q^-\}$ by $f(X \times \{1\}) = \{q^+\}$, $f(X \times \{-1\}) = \{q^-\}$.
(e) If X, Y, and A are compact, then so is the space $Y \cup_f X$.

(B.5) The *join* $X * Y$ of spaces X and Y is defined by attaching $X \times [0,1] \times Y$ to the disjoint union $X + Y$ by $(x,0,y) \mapsto x$, $(x,1,y) \mapsto y$. Show: $CX \approx X * q_0$ and $SX \approx X * \{q^+, q^-\}$.

(B.6) (*Relative bijections*) A map $f : (X,A) \to (Y,B)$ is called a *relative bijection* if $f|X - A$ is a bijective map of $X - A$ onto $Y - B$; clearly, $f^{-1}(B) = A$, and $f(A) \subset B$ (though possibly not equal to B). Let $f : X \to Y$ be an identification, and let $f : (X,A) \to (Y,B)$ be a relative bijection. Show:
(a) $X/A \approx Y/B$ (and this is natural, induced by f).
(b) If A is open or closed (or if f is an open or closed mapping), then $f : X - A \approx Y - B$.
[Consider the diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"'] & X/A \arrow[d, "f'"] \\ Y \arrow[r, "q"] & Y/B \end{tikzcd}$$

and show that $f' = (qf)p^{-1} : X/A \to Y/B$ is continuous and bijective.]

C. Deformation retracts

(C.1) Let $A \subset X$ be closed. Show: A is a deformation retract of X if and only if it has the following two properties:
(a) For every space Z, each continuous $f : A \to Z$ is extendable over X.
(b) For every space Z and any $f, g : X \to Z$, $f \simeq g$ whenever $f|A \simeq g|A$.

(C.2) Let $A \subset X$ be closed. We say that X is *deformable* into A if there exists a deformation $D : X \times I \to X$ such that $D(x,1) \in A$ for $x \in X$. Show: If X is deformable into A and A is a retract of X, then A is a deformation retract of X.
[If $r : X \to A$ is a retraction and $d : X \times I \to X$ a deformation of X into A, define

$$D(x,t) = \begin{cases} d(x,2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ r[d(x,2-2t)] & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

and prove that $D : X \times I \to X$ is the desired deformation retraction of X onto A.]

(C.3) Show: If A is a deformation retract of X, then the inclusion map $i \quad A \to X$ is a homotopy equivalence.

(C.4) Show: If $A \subset X$ is a deformation retract of X with the retraction $r : X \to A$, then any map $f : Y \to X$ is homotopic to the map $irf : Y \to X$.

(C.5) Let $D : X \times I \to X$ be a deformation of X into $A \subset X$ such that the points of A remain in A during the entire deformation. Show: X/A is contractible to $[A]$, keeping $[A]$ fixed.

[Consider the diagram

$$X/A \times I \xleftarrow{p\times 1} X \times I \xrightarrow{D} X \xrightarrow{p} X/A$$

in which $p \times 1$ is an identification by (A.4). Show that $(p \circ D)(p \times 1)^{-1}$ given by

$$(p \circ D)(p \times 1)^{-1}(\xi, t) = p \circ D[p^{-1}(\xi), t]$$

is single-valued and that it provides the required continuous deformation of X/A into $[A]$.]

(C.6) Let (X, A) be a compact pair and A be a strong deformation retract of X. Assume that $f \quad (X, A) \to (Y, B)$ is surjective and maps $X - A$ homeomorphically onto $Y - B$. Show: B is a strong deformation retract of Y (Spanier [1949]).

(C.7) Let $p_0 \in S^n$ Show: If $p \neq p_0$, then the set $(S^n \times \{p\}) \cup (\{p\} \times S^n)$ is a strong deformation retract of $S^n \times S^n - \{(p_0, p_0)\}$ (Borsuk [1937]).

[Let $f : (K^n, S^{n-1}) \to (S^n, p)$ be a map sending $K^n - S^{n-1}$ homeomorphically onto $S^n - \{p\}$, and let $x_0 = f^{-1}(p_0)$ be the center of K^n; define

$$\begin{aligned} g : (K^n \times K^n - \{(x_0, x_0)\}, (K^n \times S^{n-1}) \cup (S^{n-1} \times K^n)) \\ \to (S^n \times S^n - \{(p_0, p_0)\}, (S^n \times \{p\}) \cup (\{p\} \times S^n)) \end{aligned}$$

by $g(x, x') = (f(x), f(x'))$ and apply (C.6) to the map g.]

D. Deformation retracts and ANRs

(D.1) Let Y be an AR and $A \subset Y$ closed. Show: A is an AR if and only if A is a strong deformation retract of Y

(D.2) Let Y be an ANR and $A \subset Y$ closed. Show: If A is a deformation retract of Y, then it is also a strong deformation retract of Y (Samelson [1944]).

[Let $d : Y \times I \to Y$ be a deformation and $y \mapsto r(y) = d(y, 1)$ a retraction of Y onto A; on the set $N = Y \times \{0\} \cup A \times I \cup Y \times \{1\} \subset Y \times I$ define $h_t : N \to Y$, $t \in I$, by

$$h_t(y, s) = \begin{cases} y & \text{for } y \in X \text{ and } s = 0, \\ d(y, s(1-t)) & \text{for } y \in A \text{ and } s \in I, \\ d(r(y), 1-t) & \text{for } y \in X \text{ and } s = 1. \end{cases}$$

Then observe that h_0 extends to a map $H_0 : Y \times I \to Y$ by taking $H_0(y, s) = d(y, s)$ for $(y, s) \in Y \times I$, and apply the Borsuk theorem to get an extension H_1 of h_1 over $Y \times I$. Prove that $H_1 : Y \times I \to Y$ is the desired strong deformation retraction of Y onto A.]

(D.3) Let Y be an ANR, $A \subset Y$ closed, and $M = Y \times \{0\} \cup A \times I \subset Y \times I$. Prove: The following statements are equivalent: (i) M is a retract of $Y \times I$; (ii) A is an ANR; (iii) M is an ANR.

(D.4) Let X be compact metric and (Y, d) a metric space. Denote by (Y^X, ϱ) the metric space of all continuous maps $f : X \to Y$ with $\varrho(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}$. Show: The space Y^X is an ANR if and only if Y is an ANR.

(D.5) (*Borsuk fibration theorem*) A surjective map $p : E \to B$ is called a *fibration* provided for any space Z, any $g : Z \to E$, and any homotopy $h_t : Z \to B$ such that $pg = h_0$, there exists a homotopy $\hat{h}_t : Z \to Y$ such that $p\hat{h}_t = h_t$ for each t. Assume now that (X, A) is a pair of compact ANRs and Y is an arbitrary metric space. Let $E = Y^X$, $B = Y^A$ and $p : E \to B$ be the restriction map $f \mapsto f|A$. Prove: p is a fibration (Borsuk [1937]).

[Given $u : Z \times I \times A \to Y$, $v : Z \times X \to Y$ with $u(z, 0, a) = v(z, a)$ for $a \in A$ find $w : Z \times I \times X \to Y$ such that $w(z, t, a) = u(z, a)$ for $a \in A$, $w(z, 0, x) = v(z, x)$ for $x \in X$. Use the fact that $M = I \times A \cup \{0\} \times X$ is a retract of $I \times X$ and that $Z \times M$ is a retract of $Z \times I \times X$.]

E. The Lusternik Schnirelmann category and ANRs

Let X be a space. A closed subset $A \subset X$ is said to have *category* $\mathrm{Cat}_X(A) \leq n$ in X if $A = A_1 \cup \ldots \cup A_n$ and each A_i is closed and deformable to a point in X; when $A = X$ we write $\mathrm{Cat}\, X$ for $\mathrm{Cat}_X(X)$.

(E.1) Let A, B be closed in X. Show:
   (a) If $A \subset B$, then $\mathrm{Cat}_X(A) \leq \mathrm{Cat}_X(B)$.
   (b) $\mathrm{Cat}_X(A \cup B) \leq \mathrm{Cat}_X(A) + \mathrm{Cat}_X(B)$.
   (c) If $d : A \times I \to X$ is a deformation and $B = d(A, 1)$, then $\mathrm{Cat}_X(A) \leq \mathrm{Cat}_X(B)$.

(E.2) Let X be an ANR and $A \subset X$ be closed. Show: There exists a closed nbd W of A in X such that $\mathrm{Cat}_X(A) = \mathrm{Cat}_X(W)$.

(E.3) Let X be an ANR, $A \subset Y$ closed, and let $d : A \times I \to X$ be a deformation. Show: There exists a deformation $D : X \times I \to X$ such that $D|(A \times I) = d$.
   [Use Borsuk's theorem (6.2)].

(E.4) (*Essential ANRs*) A space M is called *essential* if there is no deformation $h : M \times I \to M$ such that $M \neq h(M, 1)$. Let M be an essential compact ANR such that $M = M_1 \times \ldots \times M_n$, where each M_i consists of more than one point. Show: $\mathrm{Cat}\, M > n$ (Eilenberg [1936]).

[Suppose to the contrary that $\mathrm{Cat}\, M \leq n$, i.e., there exists a decomposition $M = A_1 \cup \cdots \cup A_n$ such that for each $i \in [n]$, A_i is closed and there is a deformation $d_i : A_i \times I \to M$ with $d_i|A_i \times \{1\} = q_i \in M$. Use (E.3) to extend each d_i over $M \times I$ to a deformation D_i $M \times I \to M$. Letting $x = (x_1, \ldots, x_n) \in M$ with $x_i \in M_i$, define a deformation $D : M \times I \to M$ by

$$D(x, t) = (P_1 D_1(x, t), P_2 D_2(x, t), \ldots, P_n D_n(x, t)),$$

where $P_i : M \to M_i$ is the projection for each $i \in [n]$. To get a contradiction, choose a point $y = (y_1, \ldots, y_n)$ in M such that $y_i \neq P_i(q_i)$ for each $i \in [n]$, and show that $y \neq D(x, 1)$ for each $x \in M$.]

(E.5) (*Schnirelmann theorem*) Let $M_1, \ldots, M_n$ be compact manifolds with each M_i consisting of more than one point and $M = M_1 \times \ldots \times M_n$. Show: $\mathrm{Cat}\, M > n$.
   [Observe that a compact manifold is essential and use (E.4).]

(E.6) Let T^n be the n-dimensional torus (i.e., the product of n copies of S^1). Show: $\mathrm{Cat}\, T^n > n$.

F. Special ANRs

Throughout this subsection, E is a normed linear space, m is a fixed integer, and $\mathscr{C} = \{C_i\}_{i=1}^m$ stands for a family of m subsets of E; we denote by $M(\mathscr{C})$ the smallest lattice of sets containing $\mathscr{C}$, and by $N(\mathscr{C})$ the nerve of the family $\mathscr{C}$.

Given a multiindex $J = (j_1, \dots, j_k)$ with $j_1 < \dots < j_k$ and $|J| = k \le m$, we let $C_J = \bigcap\{C_i \mid i \in J\}$.

By an *m-special ANR* in E is meant a closed subset $C = \bigcup_{i=1}^m C_i$ of E that is equipped with a covering $\{C_i\}_{i=1}^m$ by m closed convex subsets of C.

We denote by $\mathbf{K}_m$ the category defined as follows:

(i) the objects of $\mathbf{K}_m$ are m-special ANRs,

(ii) the morphisms from C to D are continuous maps $f : C \to D$ such that $f(C_i) \subset D_i$ for $i \in [m]$.

By a pair in $\mathbf{K}_m$ is meant a pair (C, D) of m-special ANRs such that $i : D \hookrightarrow C$ is in $\mathbf{K}_m$ and $N(\{C_i\}) = N(\{D_i\})$.

(F.1) Let (C, D) be a pair in $\mathbf{K}_m$ and set $C^* = \bigcap_{i=1}^m C_i$. Show: If $C^* \neq \emptyset$, then there exists a retraction $s : D \cup C^* = Z \to D$ such that s sends $Z \cap C_i$ into C_i for each $i \in [m]$.

[Using (4.2)(b), prove by induction that $C^* \cap D$ is an ANR, and therefore there is a retraction $r : C^* \to C^* \cap D$; then $s = i \cup r$ is the desired retraction of Z onto D.]

(F.2) (*Regular coverings*) Let (C, D) be a pair in $\mathbf{K}_m$. Given any $J \subset [m]$ with $C_J \neq \emptyset$, we let U_J denote a closed nbd of $C_J \cap D$ in C_J. For each $p \in [m]$, consider the covering

$$\alpha_p = \{U_J \cup D \mid J \subset [m],\ |J| = p,\ (C_J = \emptyset) \Rightarrow (U_J = \emptyset)\}$$

of the set D, and let

$$Z_p = \bigcup\{U_J \cup D \mid |J| = p,\ (C_J = \emptyset) \Rightarrow (U_J = \emptyset)\}.$$

We say that α_p is a *regular covering* of D if there is a retraction $s_p : Z_p \to D$ such that $s_p(Z_p \cap C_i) \subset C_i$ for each $i \in [m]$. Show: For each $p \in [m]$ there exists a regular covering α_p of D.

[Proceed by induction on $|J|$, starting with $|J| = m$ and noting that in this case the assertion is true by (F.1). Suppose the existence of a regular α_p has been established for some $1 < p \le m$; thus, for each L with $|L| = p$, there is a closed nbd U_L of $C_L \cap D$ in C_L and a retraction $s_p : Z_p \to D$ with $s_p(Z_p \cap C_i) \subset C_i$ for each $i \in [m]$. To construct a regular covering α_{p-1} of D, proceed as follows: For each J with $|J| = p - 1$, extend $s_p : Z_p \cap C_J \to D \cap C_J$ over an open nbd U'_J of $Z_p \cap C_J$ in C_J to a map $s_J : U'_J \to D \cap C_J$ and then show successively:

(i) The set $W_J = [C_J - \bigcup\{C_L \mid |L| = p\}] \cup \bigcup\{U_L \cap C_J \mid |L| = p\}$ is a nbd of $C_J \cap D$ in C_J.

(ii) If U_J is a closed nbd of $C_J \cap D$ in C_J with $U_J \subset W_J \cap U'_J$, and $\alpha_{p-1} = \{U_J \cup D \mid |J| = p - 1\}$, then the map $s_{p-1} : Z_{p-1} \to D$ given by

$$s_{p-1} = s_p \cup \bigcup\{s_J | U_J \mid |J| = p - 1\}$$

is well defined. (If $|J| = |J'| = p - 1$, $J \neq J'$ and $x \in U_J \cap U_{J'} \subset C_{J \cup J'}$, then $s_J(x) = s_{J'}(x)$; use the fact that $|J \cup J'| \ge p$ implies that $x \in U_K$ for some K with $|K| = p$ and also that $s_J | U_K = s_{J'} | U_K$.)

(iii) If $x \in U_J \cap C_i$, $|J| = p - 1$, then $s_{p-1}(x) \in C_i$.]

(F.3) Let (C, D) be a pair in $\mathbf{K}_m$. Show: There exists an open nbd V of D in C and a retraction $s : V \to D$ such that $s(V \cap C_i) \subset C_i$ for each $i \in [m]$.

(F.4)* Let $C \in \mathbf{K}_m$, and let $\{O_J \mid J \subset \{1,\dots,m\},\ C_J \neq \emptyset\}$ be an open covering of C such that each O_J is an open nbd of C_J. Show: There exists an open covering $\{U_J \mid J \subset \{1,\dots,m\},\ C_J \neq \emptyset\}$ of C such that:

(i) $U_J \subset O_J$,

(ii) if $|K| \geq |J|$ and $J \not\subset K$, then $U_K \cap U_J = \emptyset$,

(iii) $\overline{U}_K \cap C_i = \emptyset$ for each $i \notin K$.

[Proceed inductively starting from $|K| = m$; then consider $|K| = m-1$ and so on. For $|K| = m$ define $U_K = O_K$ and note that $\{U_K \mid |K| \geq m\}$ satisfies (i)–(iii). Suppose the existence of open sets U_L in C has been established for all L with $|L| \geq r$, for some r with $1 < r \leq m$, such that $\{U_L \mid |L| \geq r\}$ satisfies (i)–(iii) and $\bigcup\{C_L \mid |L| \geq r\} \subset \bigcup\{U_L \mid |L| \geq r\}$.

To construct $\{U_K \mid |K| \geq r-1\}$, proceed as follows:

(i) Fix K with $|K| = r-1$ and let $A_K = C_K - \bigcup\{U_L \mid |L| \geq r\}$; then observing that $A_K \cap C_j = \emptyset$ whenever $j \notin K$, choose an open nbd $V_K \supset A_K$ with $V_K \subset O_K$ and $\overline{V}_K \cap C_j = \emptyset$ for $j \notin K$.

(ii) Consider any $L \subset \{1,\dots,m\}$ with $|L| > |K| = r-1$ and $L \not\supset K$, and choose $j \in K - L$. By the inductive hypothesis applied to (L, j), we have $\overline{U}_L \cap C_j = \emptyset$, so $U_L \cap W_{(L,j)} = \emptyset$ for some open nbd $W_{(L,j)}$ of C_j; observe that $W_{(L,j)}$ is, in fact, an open nbd of A_K, since $C_j \supset C_K \supset A_K$. Next define $W_K = \bigcap\{W_{(L,j)} \mid |L| > |K|,\ |L| \not\supset K,\ j \in K - L\}$.

(iii) For any K with $|K| = r-1$ find an open nbd Z_K of A_K such that if K and K' are any two distinct subsets of $[m]$ with $|K| = |K'| = r-1$, then $Z_K \cap Z_{K'} = \emptyset$.

(iv) Lastly, let $U_K = V_K \cap W_K \cap Z_K$, and show that $\{U_K \mid |K| \geq r-1\}$ satisfies the inductive hypothesis.]

(F.5) Let (C, D) be a pair in $\mathbf{K}_m$. Show:

(i) There exists a retraction $r : C \to D$ such that $r(C_i) \subset C_i \cap D$ for each $i \in [m]$.

(ii) D is a deformation retract of C.

(iii) Given $\varepsilon > 0$, there exist open nbds $\tilde{O}_K$ in E of D_K for $K \subset \{1,\dots,m\}$ with $D_K \neq \emptyset$ such that if $C_K \subset \tilde{O}_K$ for all K, then the retraction r as in (i) can be so chosen that $\|r(x) - x\| \leq \varepsilon$ for all $x \in C$.

[For each K with $D_K \neq \emptyset$ let $r_K : E \to D_K$ be a retraction and $\tilde{O}_K = \{x \in E \mid \|r_K(x) - x\| < \varepsilon\}$. Define an open covering $\{O_K\}$ of C by

$$O_K = \begin{cases} \tilde{O}_K \cap C & \text{if } C_K \subset \tilde{O}_K, \\ \text{any open nbd of } C_K \text{ in } C & \text{otherwise.} \end{cases}$$

Apply (F.4) to get an open cover $\{U_K\}$ of C satisfying (i)–(iii) of (F.4); then for each $L \subset \{1,\dots,m\}$, choose an open nbd W_L of $\overline{U}_L$ such that $W_L \cap C_j = \emptyset$ for $j \notin L$. Using (F.3), take a retraction $s : V \to D$ (where V is an open nbd of D) with $s(V \cap C_i) \subset C_i$ for all $i \in [m]$. Define an open nbd W of D by the condition $x \in W \Leftrightarrow \|s(x) - x\| < \varepsilon$ and $s(x) \in W_L$ if $x \in \overline{U}_L$ for some L. Choose a closed nbd $W^* \subset W$ of D, and take a partition of unity $\{\lambda, \lambda_K\}$ subordinate to the open covering $\{W, (C - W^*) \cap U_K\}_K$ of C. Define $R : C \to D$ by $R(x) = \lambda(x)s(x) + \sum_K \lambda_K(x) r_K(x)$ (the sum is over K with $D_K \neq \emptyset$) and show that R is the desired retraction of C onto D.]

(F.6) Let C be an m-special ANR with $C = \bigcup_{i=1}^m C_i$. Show: There exists in $\mathbf{K}_m$ a compact finite-dimensional map $R : C \to C$ (i.e., $R(C_i) \subset C_i$ for each $i \in [m]$).

[For each $J \subset \{1,\dots,m\}$ with $C_J \neq \emptyset$, select $x_J \in C_J$ and then proceed as in (F.5).]

(The above results are due to Nussbaum [1971].)

G. Leray-Schauder spaces

In this subsection, unless stated otherwise, Y stands for a completely regular space. If $U \subset Y$ is open, we let as usual $\mathscr{K}(\overline{U}, Y)$ be the set of compact maps from $\overline{U}$ to Y, and $\mathscr{K}_{\partial U}(\overline{U}, Y) = \{f \in \mathscr{K}(\overline{U}, Y) \mid \mathrm{Fix}(f|\partial U) = \emptyset\}$. The relations of compact (respectively admissible) homotopy in $\mathscr{K}(\overline{U}, Y)$ (respectively in $\mathscr{K}_{\partial U}(\overline{U}, Y)$) (see (7.1), (7.5)) divide the above sets into disjoint homotopy classes; the homotopy class of a map f is denoted by $[f]$.

(G.1) (*Homotopically nontrivial maps*) A map f in $\mathscr{K}(\overline{U}, Y)$ (respectively in $\mathscr{K}_{\partial U}(\overline{U}, Y)$) is called *homotopically nontrivial* (written $[f] \neq 0$) provided all maps g in the homotopy class $[f]$ in $\mathscr{K}(\overline{U}, Y)$ (respectively in $\mathscr{K}_{\partial U}(\overline{U}, Y)$) have $\mathrm{Fix}(g) \neq \emptyset$. Show:

(a) If $f \in \mathscr{K}_{\partial U}(\overline{U}, Y)$ is essential (see (0.3.1)), then $[f] \neq 0$ in $\mathscr{K}_{\partial U}(\overline{U}, Y)$.

(b) If Y is a convex set in a linear topological space, then $f \in \mathscr{K}_{\partial U}(\overline{U}, Y)$ is essential $\Leftrightarrow$ $[f] \neq 0$ in $\mathscr{K}_{\partial U}(\overline{U}, Y)$.

[For (a), see (0.3.3).]

(G.2) Given $X \subset Y$ and a compact homotopy $h : X \times I \to Y$, we let

$$\mathrm{FIX}(h) = \{(x, t) \in X \times I \mid h(x, t) = x\}.$$

Let $U \subset Y$ be open, $g : Y \to Y$ the constant map sending Y to $u_0 \in U$, and $h : \overline{U} \times I \to Y$ an admissible compact homotopy with $h(x, 0) = g(x)$ for all $x \in \overline{U}$. Show: There exists a compact homotopy $\widehat{h} : Y \times I \to Y$ such that:

(i) $\mathrm{FIX}(h) = \mathrm{FIX}(\widehat{h})$,

(ii) $\widehat{h}(y, 0) = u_0$ for all $y \in Y$

[Observe that $C = \{x \in \overline{U} \mid h(x, t) = x$ for some $t \in I\}$ is nonempty and compact; then using complete regularity of Y and $C \cap \partial U = \emptyset$, choose $\lambda . \overline{U} \to I$ such that $\lambda|C = 1$, $\lambda|\partial U = 0$, and verify that $\widehat{h} : Y \times I \to Y$ given by

$$\widehat{h}(y, t) = \begin{cases} h(y, \lambda(y)t) & \text{for } (y, t) \in \overline{U} \times I, \\ u_0 & \text{for } y \in Y - \overline{U},\ t \in I, \end{cases}$$

is the required compact homotopy.]

(G.3) Let $g : Y \to Y$ be the constant map sending Y to $u_0 \in Y$ Show: The following conditions are equivalent:

(i) g is homotopically nontrivial,

(ii) for each open $U \subset Y$ containing u_0, the restriction $g|\overline{U}$ is homotopically nontrivial in $\mathscr{K}_{\partial U}(\overline{U}, Y)$

(G.4) (*The fixed point class* $\mathscr{F}_0$) A space Y is called a *fixed point space for compactly nullhomotopic maps* (written $Y \in \mathscr{F}_0$) if every $f \in \mathscr{K}(Y, Y)$ such that $f \simeq 0$ in $\mathscr{K}(Y, Y)$ has a fixed point. Show:

(a) $Y \in \mathscr{F}_0 \Leftrightarrow$ [every constant map $g \in \mathscr{K}(Y, Y)$ is homotopically nontrivial in $\mathscr{K}(Y, Y)$].

(b) If Y is in $\mathscr{F}_0$, then so also is every open subset of Y

(c) If Y is in $\mathscr{F}_0$, then so also is every retract of Y

(G.5) (*Leray–Schauder spaces*) A space Y is called a *Leray–Schauder space* if for every open $U \subset Y$ and any compact homotopy $h_t : \overline{U} \to Y$ such that h_0 is the constant map sending $\overline{U}$ to $u_0 \in U$, one of the following properties holds: either $\mathrm{Fix}(h_1|U) \neq \emptyset$, or $y_0 = h_\lambda(y_0)$ for some $y_0 \in \partial U$ and $\lambda \in (0, 1)$.

(a) Prove: For a space Y, the following assertions are equivalent:
   (i) Y is a Leray Schauder space.
   (ii) If $U \subset Y$ is open and $f \in \mathscr{K}_{\partial U}(\overline{U}, Y)$ is compactly nullhomotopic, then $\mathrm{Fix}(f) \neq \emptyset$.
   (iii) If $U \subset Y$ is open, then any constant map sending $\overline{U}$ to $u_0 \in U$ is homotopically nontrivial in $\mathscr{K}_{\partial U}(\overline{U}, Y)$.
(b) Show: Y is a Leray Schauder space $\Leftrightarrow Y \in \mathscr{F}_0$.
(The results (G.2)–(G.5) are due to Horvath–Lassonde [2001].)

H. Homotopy extension property for compact maps

In this subsection only metrizable spaces are considered, and by a pair of spaces we understand a pair (X, A) with $A \subset X$ closed. A space Y is said to have the *homotopy extension property for compact maps* if given any pair (X, A) and a compact map $f \in \mathscr{K}(X, Y)$, every partial compact homotopy $h_t : A \to Y$ of f admits a compact extension $\widehat{h}_t : X \to Y$ such that $\widehat{h}_0 = f$.

(H.1) Assume that Y has the homotopy extension property for compact maps. Show:
(a) If $f \in \mathscr{K}(Y, Y)$ is nullhomotopic, then $\mathrm{Fix}(f) \neq \emptyset$.
(b) If $U \subset Y$ is open and $g \in \mathscr{K}_{\partial U}(\overline{U}, Y)$ is nullhomotopic, then $\mathrm{Fix}(g) \neq \emptyset$.

(H.2) (*Leray–Schauder principle*) Let Y be a space having the homotopy extension property for compact maps, $U \subset Y$ open, and let g be a constant map sending $\overline{U}$ to a point $u_0 \in U$. Show: Any compact homotopy $h_t : \overline{U} \to Y$ joining $h_0 = g$ and $h_1 = f$ has one of the following two properties:
(i) f has a fixed point in U;
(ii) there are $y_0 \in \partial U$ and $\lambda \in (0, 1]$ such that $y_0 = h_\lambda(y_0)$.

(H.3) Prove: Given a space Y, the following two conditions are equivalent:
(a) Y is NES(compact metric).
(b) For any pair (X, A), each $f \in \mathscr{K}(A, Y)$ admits an extension $F \in \mathscr{K}(U, Y)$ over a nbd $U \supset A$ in X.

(H.4) Show: Every Y that is NES(compact metric) has the homotopy extension property for compact maps.

9. Notes and Comments

Absolute neighborhood results

Theorem (3.6) asserting that any two sufficiently close maps into an ANR are homotopic was proved by Dugundji [1965]. Aronszajn–Borsuk [1932] proved Theorem (4.2) for compact ARs, and Borsuk [1932] extended the result to the case of compact ANRs. The fundamental localization theorem (5.3) was proved by Hanner [1951]. The homotopy extension theorem (6.2) was established by Borsuk [1937]; this result in more special cases (for example, in the case of maps with compact domain into S^n) was, in fact, known and used in the early thirties (cf. Borsuk [1931] and Eilenberg [1936]). The proof of the homotopy extension theorem given in the text, which uses Lemma

(6.1), is essentially due to Dowker. The homotopy extension theorem for compact maps (7.3) and its applications to fixed point results for compact maps in ANRs (Theorems (7.4) and (7.7)) are due to Granas and appear here for the first time; another direct proof (without using the fixed point index theory) of (7.4), (7.7) was given earlier by Horvath–Lassonde [2001].

The following more special results are of importance in fixed point theory:

(i) (*Hanner–Dugundji domination theorem*) Let Y be an ANR and α an open cover of Y. Then there exists a polytope P with the weak topology and maps $f : Y \to P$ and $g : P \to Y$ such that $g \circ f$ and 1_Y are α-homotopic.

(ii) (*Bothe embedding theorem*) For each $n = 1, 2, \ldots$ there is a compact $(n+1)$-dimensional absolute retract $B^{(n)}$ such that any separable metric space of dimension $\leq n$ is homeomorphic to a subset of $B^{(n)}$.

LC^n *spaces*

This class was introduced in the context of compact metric spaces in Lefschetz's book [1930]. A metric space X is *n-locally connected* if for every point $x \in X$ and every nbd U of x there is a nbd V of x such that any $f : S^k \to V$, $k \leq n$, is homotopic to a constant map into U

The class of n-locally connected spaces is denoted by LC^n, and we let $\mathrm{LC}^\infty = \bigcap_{n=0}^{\infty} \mathrm{LC}^n$; clearly, the class LC of locally contractible spaces is contained in LC$^\infty$.

The following result (proved by Kuratowski [1935] for separable metric spaces, and by Dugundji [1958] for the general case) provides a useful characterization of the LC^n spaces: *For a metric space Y the following conditions are equivalent*:

(i) $Y \in \mathrm{LC}^n$ *for some* $n > 0$,

(ii) *if A is closed in a metric space X and* $\dim(X - A) \leq n+1$, *then for each $f_0 : A \to Y$ there is an $f : U \to Y$ that extends f_0 over a nbd U of A in X.*

ENRs and finite-dimensional ANRs

A space Y is a *Euclidean neighborhood retract* if it is homeomorphic to a nbd retract $X \subset \mathbf{R}^n$. The class of all such spaces (denoted by ENR) is smaller than the class of finite-dimensional ANRs, and is characterized as follows: *A separable metric space X is an ENR if and only if it is a locally compact ANR with* $\dim X < \infty$. For details concerning ENRs, the reader is referred to Dold's book [1980].

Borsuk [1932] gave the following characterization of compact ENRs: *If $X \subset \mathbf{R}^n$ is compact, then* $X \in \mathrm{ENR} \Leftrightarrow X \in \mathrm{LC}$. Borsuk [1935] introduced

the operation of attaching spaces by continuous maps and (using the above characterization of compact ENRs) proved the following theorem: *If* $f : A \to Y$ *is continuous and* $A \subset X$, X *and* Y *are compact ENRs, then so also is the adjunction space* $X \cup_f Y$. J.H.C. Whitehead [1948] generalized Borsuk's theorem to arbitrary compact ANRs and used the extended result as an effective tool in his theory of CW-complexes.

For many years, the infinite-dimensional case remained an open problem until Borsuk [1948] showed that the solution is negative. He constructed a closed subset of the Hilbert cube that is locally contractible and that admits retractions onto subsets homeomorphic to $\boldsymbol{S}^n$ for any $n > 0$. Taking the join of this set with a single point yields an example of a compact contractible and locally contractible space that is not an absolute retract.

Borsuk's example is constructed as follows: Regard the Hilbert cube $\boldsymbol{I}^\infty$ as the cartesian product $\prod_{i=1}^\infty I_i$ of a countable family of unit intervals I_i, and for each $k = 1, 2, \ldots$ let C_k be the k-cube

$$C_k = \{x \in \boldsymbol{I}^\infty \mid 1/(k+1) \le [x]_1 \le 1/k,\ 0 \le [x]_i \le 1 \text{ for } 2 \le i \le k \\ \text{and } [x]_i = 0 \text{ for } i > k\},$$

where we write $[x]_i$ for the ith coordinate of x. Let $B_k \approx \boldsymbol{S}^{k-1}$ be the boundary of C_k, and let $B_0 = \{x \in \boldsymbol{I}^\infty \mid [x]_1 = 0\}$. Borsuk's locally contractible non-ANR is the compact subspace $B = \bigcup_{i=0}^\infty B_i \subset \boldsymbol{I}^\infty$. For each integer $N > 0$ there is a retraction $\varrho_N : B \to B_N$ given by

$$[\varrho_N(x)]_1 = \begin{cases} 1/(N+1) & \text{if } [x]_1 \le 1/(N+1), \\ [x]_1 & \text{if } 1/(N+1) \le [x]_1 \le 1/N, \\ 1/N & \text{if } 1/N \le [x]_1, \end{cases}$$

$$[\varrho_N(x)]_i = \begin{cases} [x]_i & \text{if } 2 \le i \le N, \\ 0 & \text{if } i > N. \end{cases}$$

We remark that if a metrizable and locally contractible space Y has the homotopy extension property with respect to the class of metric spaces, then Y is an ANR (Hanner [1951]).

Generalizing Borsuk's result of 1932, Kuratowski [1935] proved the following theorem: *If* Y *is separable metric and the covering dimension* $\dim Y$ *is finite, then the following properties are equivalent:* (i) Y *is* LC^n *for some* $n \ge \dim Y$; (ii) Y *is locally contractible*; (iii) Y *is an ANR*. Dugundji [1958] extended this theorem to arbitrary metric spaces.

Equiconnected and locally equiconnected spaces

A metric space Y is called *locally equiconnected* if there is a map $\lambda : U \times \boldsymbol{I} \to Y$, where U is a nbd of the diagonal in $Y \times Y$, such that $\lambda(a, b, 0) = a$, $\lambda(a, b, 1) = b$, and $\lambda(a, a, t) = a$; if $U = Y \times Y$, then Y is called *equiconnected*.

From (3.2) it follows that every ANR is locally equiconnected (and every AR is equiconnected). Dugundji [1965] found conditions implying that a locally equiconnected space is an ANR. The longstanding problem of whether an arbitrary locally equiconnected space is an ANR was resolved in the negative by Cauty [1994], who constructed a metric linear space (and thus an equiconnected space) that is not an absolute retract; Cauty's example also shows that in the statement of the Dugundji extension theorem the assumption of local convexity of the target space cannot be weakened.

Cauty [2002] established recently that any compact equiconnected space has the fixed point property (and more generally that any compact locally equiconnected space is a Lefschetz space (cf. §15)).

Neighborhood extensor spaces

Let Q be any of the following classes of spaces: $\mathscr{N}$ = normal, $\mathscr{P}$ = paracompact, $\mathscr{K}$ = compact, $\mathscr{M}$ = metric, $\mathscr{KM}$ = compact metric.

A space Y is called a *neighborhood extensor space for* Q if for each $X \in Q$, every closed $A \subset X$, and $f_0 : A \to Y$, there exists a map $f : U \to Y$ that extends f over some nbd U of A in X; if in the above definition f_0 extends over X, we call Y an *extensor space for* Q.

The corresponding classes of spaces are denoted by NES(Q) and ES(Q), respectively, and their elementary and frequently used properties are:

(i) ES(Q) $\subset$ NES(Q).
(ii) A retract of a NES(Q) (respectively ES(Q)) is a NES(Q) (respectively ES(Q)).
(iii) An open subset of a NES(Q) is a NES(Q).
(iv) If $Q' \subset Q$ then NES(Q) $\subset$ NES(Q').

Thus various classes of NES(Q) spaces satisfy the inclusions

$$\mathrm{NES}(\mathscr{N}) \subset \mathrm{NES}(\mathscr{P}) \subset \{\mathrm{NES}(\mathscr{K}), \mathrm{NES}(\mathscr{M})\} \subset \mathrm{NES}(\mathscr{KM}).$$

We list some examples of ES(Q) spaces:

- ES($\mathscr{N}$) spaces: (i) arbitrary products of lines; (ii) Tychonoff cubes; (iii) separable Banach spaces.
- ES($\mathscr{P}$) spaces: Banach spaces (Arens theorem).
- ES($\mathscr{M}$) spaces: (i) convex sets in locally convex linear topological spaces (Dugundji extension theorem (7.7.4)); (ii) polytopes with the weak topology.
- ES($\mathscr{K}$) spaces: (i) ARs, (ii) normed linear spaces; (iii) complete metric linear spaces that are admissible in the sense of Klee [1960].

Using this list and the above elementary properties (i)–(iii), we obtain numerous further examples of NES(Q) spaces.

Hanner [1951] established a number of important properties of NES(Q) spaces, which are analogous to the properties of ANRs given in the text:

(i) The product of any collection of ES(Q)'s is an ES(Q).

(ii) The product of any finite collection of NES(Q)'s is a NES(Q).

(iii) Any local NES($\mathscr{P}$) is a NES($\mathscr{P}$).

A space Y is called an *absolute neighborhood retract* (respectively an *absolute retract*) *for the class* Q provided Y is in Q and whenever it is embedded in another space of Q as a closed subset, it is a nbd retract (respectively a retract) of the ambient space; the corresponding classes of spaces are denoted by ANR(Q) and AR(Q).

We note that AR(Q) $\subset$ ANR(Q), and ANR($\mathscr{M}$) coincides with the class of ANRs; it can be shown that any ANR is a NES(compact). As for relations between NES(Q) and ANR(Q) spaces, Hanner [1951] proved that ANR(Q) $= Q \cap$ NES(Q).

Historical note

The basic notions of the theory (retracts, deformation retracts, ARs and ANRs, local contractibility, adjunction of ANRs) were introduced and studied by Borsuk [1931]; the term "retraction" was suggested by Mazurkiewicz and that of an "absolute retract" by Aronszajn. In the thirties the theory was systematically developed by Borsuk (in the context of compact metric spaces), by Kuratowski (for separable metric spaces), and by Lefschetz (theory of LC^n spaces, fixed points for maps of compact ANRs).

After Stone's fundamental discovery that every metric space is paracompact, the general theory of ANRs was extended to arbitrary metric spaces by Dowker [1948] and Dugundji [1951]; at the same time generalizations of the theory to more general classes of spaces (NES(Q) spaces and ANR(Q) spaces) were made by Hanner [1951], [1952], Michael [1953], and others. For further details, the reader may consult Hu's book [1965], concerned for the most part with the general theory of NES spaces.

The related notions of approximate ANRs were introduced by Noguchi [1953], Clapp [1971], and Gauthier [1983]. The concepts of approximate NES(compact) spaces and of related classes of spaces appear in the lecture notes of Granas [1980].

For more details and further study of the theory of retracts the reader is referred to the monographs of Borsuk, Lefschetz, and van Mill. The reader interested in more recent research on ANRs and related topics may consult the survey article by Mardešić [1999].

§12. Fixed Point Index in ANRs

This paragraph is devoted to the concept of the Leray–Schauder fixed point index. With the aid of the topological degree in $\boldsymbol{R}^n$ and some of the geometric methods developed in Chapter II, our objective is to establish the theory in the setting of arbitrary ANRs. The homological part of the theory will be treated separately in §16, in which the Hopf-Lefschetz normalization property of the index will be established, and thus the entire theory will be rounded out.

Throughout this paragraph, the following terminology and notation are used. Let X be a space, $U \subset X$ open, and $f : U \to X$ continuous. We say that f is *compactly fixed* if the fixed point set $\operatorname{Fix}(f)$ is compact. The set of all compactly fixed maps $f : U \to X$ is denoted by $\mathscr{F}(U, X)$. A homotopy $h_t : U \to X$ is said to be *compactly fixed* if $\bigcup_{t\in I} \operatorname{Fix}(h_t)$ is compact.

1. Fixed Point Index in $\boldsymbol{R}^n$

Let $U \subset \boldsymbol{R}^n$ be open and $f : U \to \boldsymbol{R}^n$ be a continuous map. Recall that $Z(f)$ denotes the zero set of f. Observing that

$$\operatorname{Fix}(f) = Z(\mathrm{id} - f)$$

we will use Theorem (10.8.5) to develop an index that indicates the minimal number of fixed points that f must have.

(1.1) DEFINITION. Let $U \subset \boldsymbol{R}^n$ be open, and let $\mathscr{F}(U, \boldsymbol{R}^n)$ be the set of all compactly fixed maps $f : U \to \boldsymbol{R}^n$. The *fixed point index function* $I : \mathscr{F}(U, \boldsymbol{R}^n) \to \boldsymbol{Z}$ is defined for $f \in \mathscr{F}(U, \boldsymbol{R}^n)$ by

$$I(f) = I(f, U) = D(\mathrm{id} - f, U),$$

and the integer $I(f, U)$ is called the *fixed point index* of f.

For compactly fixed maps on an open $U \subset \boldsymbol{R}^n$ the properties of the degree for compactly rooted maps given in (10.8.5) immediately translate to properties of the index:

(I) (Normalization) *If $f : U \to \boldsymbol{R}^n$ is the constant map $u \mapsto u_0$ with $u_0 \in U$, then $I(f, U) = 1$.*

(II) (Additivity) *For every pair of disjoint open $V_1, V_2 \subset U$, if $\operatorname{Fix}(f) \subset V_1 \cup V_2$, then $I(f, U) = I(f, V_1) + I(f, V_2)$.*

(III) (Homotopy) *Let $H_t : U \to \boldsymbol{R}^n$ be a compactly fixed homotopy. Then $I(H_0, U) = I(H_1, U)$.*

(IV) (Existence) *If $I(f, U) \neq 0$, then f has a fixed point in U.*

PROOF. $D(\mathrm{id}-f, U) \neq 0$, so $\mathrm{id}-f$ has a zero on U. $\square$

(V) (Excision) *If* $V \subset U$ *is open and* $\mathrm{Fix}(f) \subset V$, *then* $I(f,V) = I(f,U)$.

PROOF. This is simply the excision property of degree. $\square$

(VI) (Multiplicativity) *Let* $U \subset \boldsymbol{R}^n$, $V \subset \boldsymbol{R}^m$, *and let* $f : U \to \boldsymbol{R}^n$ *and* $g : V \to \boldsymbol{R}^m$ *be compactly fixed maps. Then*

$$I(f \times g, U \times V) = I(f,U) \cdot I(g,V).$$

PROOF. It is easily seen that $\mathrm{Fix}(f \times g) = \mathrm{Fix}(f) \times \mathrm{Fix}(g)$, so that the product map $f \times g : U \times V \to \boldsymbol{R}^n \times \boldsymbol{R}^m$ is compactly fixed. Consider the map

$$\begin{aligned}(x,y) \mapsto (x,y) - (f(x), g(y)) &= (\mathrm{id}_{U\times V} - (f \times g))(x,y)\\ &= [x - f(x)] \times [y - g(y)].\end{aligned}$$

By multiplicativity of degree for compactly rooted maps, we get

$$D\{[x - f(x)] \times [y - g(y)], U \times V\} = D[x - f(x), U] \cdot D[y - g(y), V]. \quad \square$$

The crucial property of the index, which is important in extending the concept to more general spaces, is that of commutativity:

(VII) (Commutativity) *Let* $U \subset \boldsymbol{R}^n$, $W \subset \boldsymbol{R}^m$ *be open, and let* $f : U \to \boldsymbol{R}^m$, $g : W \to \boldsymbol{R}^n$. *Consider the maps*

$$gf : V = f^{-1}(W) \to \boldsymbol{R}^n \quad \textit{and} \quad fg : V' = g^{-1}(U) \to \boldsymbol{R}^m.$$

If gf *is compactly fixed, then so is* fg, *and*

$$I(gf, V) = I(fg, V').$$

PROOF. Consider the fixed point sets

$$\mathrm{Fix}(gf) = \{x \in f^{-1}(W) \mid x = gf(x)\}, \ \mathrm{Fix}(fg) = \{y \in g^{-1}(U) \mid y = fg(y)\}$$

and observe that $f : \mathrm{Fix}(gf) \to \mathrm{Fix}(fg)$ and $g : \mathrm{Fix}(fg) \to \mathrm{Fix}(gf)$ are inverse to each other, and hence the sets $\mathrm{Fix}(gf)$ and $\mathrm{Fix}(fg)$ are homeomorphic. Thus, if one of them is compact, then so is the other.

Assuming that both of them are compact, we now start the proof:

(a) $I(gf, V) = I((gfx, 0), V \times \boldsymbol{R}^m)$.

For $(x,y) \mapsto (gfx, 0)$ is the product of the two maps $gf : V \to \boldsymbol{R}^n$ and $0 : \boldsymbol{R}^m \to \boldsymbol{R}^m$, and so by multiplicativity, and because $I(0, \boldsymbol{R}^m) = 1$, we get

$$I((gfx, 0), V \times \boldsymbol{R}^m) = I(gf, V) \cdot I(0, \boldsymbol{R}^m) = I(gf, V).$$

(b) $I((gfx, 0), V \times \boldsymbol{R}^m) = I((gfx, fx), V \times \boldsymbol{R}^m)$.

For (gfx, fx) and $(gfx, 0)$ can be joined by the compactly fixed homotopy $\alpha_t : V \times \boldsymbol{R}^m \to \boldsymbol{R}^n \times \boldsymbol{R}^m$ given by

$$\alpha_t(x, y) = (gfx, (1-t)fx), \qquad t \in [0, 1].$$

Note: $\bigcup_{t \in I} \operatorname{Fix}(\alpha_t) \subset V \times \boldsymbol{R}^m$ is compact because it coincides with the image of the compact set $\operatorname{Fix}(gf) \times I$ under the map $(x, t) \mapsto (x, (1-t)fx)$.

(c) $I[(gfx, fx), V \times \boldsymbol{R}^m] = I[(gfx, fx), V \times V']$.

For if $gfx = x$ and $y = fx$ with $x \in V = f^{-1}(W)$, then $gy = x$ shows $y \in g^{-1}(U) = V'$; thus all the fixed points of $(x, y) \mapsto (gfx, fx)$ in $V \times \boldsymbol{R}^m$ are in a smaller set $V \times V'$, so we can cut down the domain of consideration (by excision).

(d) $I[(gfx, fx), V \times V'] = I[(gy, fx), V \times V']$.

For $(x, y) \mapsto (gfx, fx)$ and $(x, y) \mapsto (gy, fx)$ can be joined by the compactly fixed homotopy $H_t : V \times V' \to \boldsymbol{R}^n \times \boldsymbol{R}^m$ given by

$$H_t(x, y) = [(1-t)gfx + tgy, fx].$$

Note: $\operatorname{Fix}(H_t) = \{(x, y) \mid (1-t)gfx + tgy = x,\ y = fx\} = \{(x, y) \mid gfx = x,\ y = fx\}$ is compact.

Now we go down the line in the opposite direction:

(e) $I[(gy, fx), V \times V'] = I[(gy, fgy), V \times V']$.

We use a compactly fixed homotopy $\alpha_t : V \times V' \to \boldsymbol{R}^n \times \boldsymbol{R}^m$,

$$\alpha_t(x, y) = (gy, (1-t)fx + tfgy).$$

(f) $I[(gy, fgy), V \times V'] = I[(gy, fgy), \boldsymbol{R}^n \times V']$.

(g) $I[(gy, fgy), \boldsymbol{R}^n \times V'] = I[(0, fgy), \boldsymbol{R}^n \times V'] = I(fg, V')$.

Thus the proof is complete. □

We can now formulate a basic theorem:

(1.2) THEOREM (Fixed point index in $\boldsymbol{R}^n$). *Let U be any open set in $\boldsymbol{R}^n$ and $\mathscr{F}(U, \boldsymbol{R}^n)$ the set of all compactly fixed maps $f : U \to \boldsymbol{R}^n$. Then:*
(A) *There exists a fixed point index function $f \mapsto I(f) = I(f, U)$ for $f \in \mathscr{F}(U, \boldsymbol{R}^n)$ with properties* (I)–(VII).
(B) *The function $I : \mathscr{F}(U, \boldsymbol{R}^n) \to \boldsymbol{Z}$ is uniquely determined by the normalization, additivity, and homotopy properties* (I)–(III).

PROOF. The existence of the function I has already been established. Its uniqueness reduces to that of the degree, since each fixed point index function $\hat{I}$ (i.e., a function satisfying (I)–(VII)) determines a degree $D_{\hat{I}}$ by the

formula $D_{\hat{I}}(f, U) = \hat{I}(\mathrm{id} - f, U)$; specifically, it follows from (10.8.5) that the normalization, additivity, and homotopy properties uniquely determine the fixed point index function in $\boldsymbol{R}^n$. $\square$

2. Axioms for the Index

(2.1) DEFINITION. Let C be a category of topological spaces and continuous maps, and let

$$\mathscr{F} = \mathscr{F}(C) \subset \{f \in \mathscr{F}(U, X) \mid X \in C,\ U \text{ open in } X\}$$

be a distinguished class of compactly fixed maps. A *fixed point index* on $\mathscr{F}$ is an integer-valued function $f \mapsto I(f, U)$ for $f : U \to X$ in $\mathscr{F}$ which satisfies the following conditions (for $V \subset U$ open, we write simply $I(f, V)$ for $I(f|V, V)$):

(I) (*Normalization*) If $f : U \to X$ in $\mathscr{F}$ is the constant map $u \mapsto u_0$ with $u_0 \in U$, then $I(f, U) = 1$.

(II) (*Additivity*) For $f \in \mathscr{F}$ and every pair of disjoint open $V_1, V_2 \subset U$, if $\mathrm{Fix}(f) \subset V_1 \cup V_2$, then $I(f, U) = I(f, V_1) + I(f, V_2)$.

(III) (*Homotopy*) If $h_t : U \to X$ is a compactly fixed homotopy and $h_t \in \mathscr{F}$ for all $t \in \boldsymbol{I}$, then $I(h_0, U) = I(h_1, U)$.

(IV) (*Existence*) If $f \in \mathscr{F}$ and $I(f, U) \neq 0$, then f has a fixed point in U.

(V) (*Excision*) If $f \in \mathscr{F}$ and $\mathrm{Fix}(f) \subset V$ for some open $V \subset U$, then $f|V \in \mathscr{F}$ and $I(f, U) = I(f, V)$.

(VI) (*Multiplicativity*) If $f_1 : U_1 \to X_1$ and $f_2 : U_2 \to X_2$ are in $\mathscr{F}$, then so is their product $f_1 \times f_2 : U_1 \times U_2 \to X_1 \times X_2$ and $I(f_1 \times f_2, U_1 \times U_2) = I(f_1, U_1) I(f_2, U_2)$.

(VII) (*Commutativity*) Let $U \subset X$, $U' \subset X'$ be open and $f : U \to X'$, $g : U' \to X$ be maps in the category C. Consider the maps

$$gf : V = f^{-1}(U') \to X, \qquad fg : V' = g^{-1}(U) \to X'.$$

If both gf and fg are in $\mathscr{F}$, then $I(gf, V) = I(fg, V')$.

REMARK. Note that the set of properties in (2.1) is not an independent set of axioms for the index. For example, excision follows from additivity: indeed, if we take $U = U_1 = U_2 = \emptyset$, then additivity implies formally that $I(f, \emptyset) = 0$; then for $f \in \mathscr{F}(U, X)$, if $\mathrm{Fix}(f) \subset V \subset U$, then because the fixed points of f all lie in the disjoint sets V and $\emptyset$, additivity gives $I(f, V) = I(f, U)$.

On the other hand, the excision axiom implies (IV): for if $f \in \mathscr{F}(U, X)$ and $\mathrm{Fix}(f) = \emptyset$, then $I(f, U) = I(f, \emptyset) = 0$.

The interdependencies of the axioms will be discussed in §16.

We now extend the index to any n-dimensional normed linear space E^n. If $U \subset E^n$ is open and $f : U \to E^n$ is compactly fixed, choose any linear isomorphism $h : \boldsymbol{R}^n \to E^n$ and define

$$(*) \qquad I(f,U) = I(h^{-1}fh, h^{-1}(U)).$$

The commutativity of the index in $\boldsymbol{R}^n$ implies that $I(f,U)$ is independent of the particular isomorphism used in its definition. Properties (I)–(VII) for this index follow from those of the index in $\boldsymbol{R}^n$.

Thus we may state the results of Section 1 in the following slightly more general form.

(2.2) THEOREM (Fixed point index in E^n). *Let C be the category of finite-dimensional normed linear spaces, and $\mathscr{F} = \mathscr{F}(C)$ the corresponding class of all compactly fixed maps. Then the integer-valued function $f \mapsto I(f,U)$ defined for $f : U \to E^n$ in $(*)$ has properties* (I)–(VII).

We note that commutativity implies the following property of the index:

(VIII) (Contraction) *Let $U \subset E^n$ be open and $f : U \to E^n$ a compactly fixed map such that $f(U) \subset L$, where L is some linear subspace of E^n. Denote by $f_{U\cap L} : U \cap L \to L$ the contraction of f. Then*

$$I(f,U) = I(f_{U\cap L}, U \cap L).$$

PROOF. Let $i : L \hookrightarrow E^n$ be the inclusion; then commutativity gives

$$I(f,U) = I(if,U) = I(fi, i^{-1}(U)) = I(f_{U\cap L}, U \cap L). \qquad \square$$

3. The Leray–Schauder Index in Normed Linear Spaces

This section develops the Leray–Schauder fixed point index for compact and compactly fixed maps in normed linear spaces. In the subsequent discussion, given a compact map f and $\varepsilon > 0$, any finite-dimensional ε-approximation of f will be simply referred to as an *ε-approximation* of f.

We begin with a strengthened version of the Schauder approximation theorem (6.2.3) which, though more general than now required, is stated explicitly for future use:

(3.1) THEOREM. *Let X be a space, U an open subset of a normed linear space E, and $f : X \to U$ a compact map. Then for each sufficiently small $\varepsilon > 0$ there exists a polytope $K_\varepsilon \subset U$ and a finite-dimensional map $f_\varepsilon : X \to U$ such that:*
 (i) *$\|f(x) - f_\varepsilon(x)\| < \varepsilon$ for all $x \in X$,*
 (ii) *$f_\varepsilon(X) \subset K_\varepsilon$,*
 (iii) *$f_\varepsilon \simeq f$.*

PROOF. Let $0 < \varepsilon < \operatorname{dist}(\overline{f(X)}, \partial U)$, and let $f_\varepsilon : X \to E$ be an ε-approximation to f with $f_\varepsilon(X) \subset E^n$ for some n-dimensional linear subspace $E^n \subset E$ (see (6.2.3)); by the choice of ε, it is clear that $\overline{f_\varepsilon(X)} \subset U$ and that (i) and (iii) are satisfied. Observe now that that if $W \subset \boldsymbol{R}^n$ is open and $K \subset W$ is compact, then there always exists a polytope P_ε with $K \subset P_\varepsilon \subset W$: taking a cubical grating of $\boldsymbol{R}^n$ with mesh $< \alpha/2$, where $\alpha < \operatorname{dist}(K, \partial W)$, and keeping only the n-cubes that meet K provides such a polytope. Using this observation, and the fact that E^n is linearly homeomorphic to $\boldsymbol{R}^n$, we find a polytope K_ε such that $f_\varepsilon(X) \subset K_\varepsilon \subset U \cap E^n \subset U$; thus, property (ii) is also established. □

We also state a simple general result that will be frequently used:

(3.2) LEMMA. *Let V be open in a normed linear space E and assume that $f : \overline{V} \to E$ is a compact map with no fixed points on ∂V Then:*
(a) *the number*
$$\eta = \inf\{\|x - f(x)\| \mid x \in \partial V\}$$
is positive,
(b) *if $0 < \varepsilon < \eta$, then any ε-approximation f_ε of f is fixed point free on ∂V,*
(c) *if $\varepsilon < \eta/2$, then given any two ε-approximations f'_ε, f''_ε of f, the formula $h_t(x) = (1-t)f'_\varepsilon(x) + tf''_\varepsilon(x)$ defines a finite-dimensional η-homotopy joining f'_ε and f''_ε without fixed points on ∂V*

PROOF. The proof, analogous to that of (10.4.1), is left to the reader. □

Let U be an open subset of a normed linear space E, and let $f \in \mathscr{F}(U, E)$ be a compact map. Given $\varepsilon > 0$, we let $f_\varepsilon : U \to E$ be an ε-approximation of f. Take an open set $V \subset \overline{V} \subset U$ such that $\operatorname{Fix}(f) \subset V$ and note that by (3.2) the number $\eta = \inf\{\|x - f(x)\| \mid x \in \partial V\}$ is positive. From (3.2) and the definition of η it follows that:
(i) if $0 < \varepsilon < \eta$, then the map $f_\varepsilon|V = g_\varepsilon : V \to E$ is an ε-approximation of $g = f|V$ and g_ε is compactly fixed,
(ii) if $0 < \varepsilon < \eta/2$ and $f'_\varepsilon, f''_\varepsilon : U \to E$ are any two ε-approximations of f, and $g'_\varepsilon = f'_\varepsilon|V$, $g''_\varepsilon = f''_\varepsilon|V$, then $h_t(x) = (1-t)g'_\varepsilon(x) + tg''_\varepsilon(x)$, $(x, t) \in V \times \boldsymbol{I}$, defines a compactly fixed finite-dimensional homotopy $h_t : V \to E$ joining g'_ε and g''_ε such that each h_t is η-close to f.

Let $f \in \mathscr{F}(U, E)$ be a compact map and $g_\varepsilon : V \to E$ be an ε-approximation of $g = f|V$ as above. Denote by E^n a finite-dimensional subspace of E that contains $g_\varepsilon(V)$ and by $g'_\varepsilon : E^n \cap V \to E^n$ the evident contraction of g_ε.

Set $I(f, V) = I(g'_\varepsilon, E^n \cap V)$. It follows from (i) and (ii) and the homotopy and contraction properties of the index in E^n that $I(f, V)$ is independent of the choice of E^n and also of the finite-dimensional approximation chosen.

Moreover, given V_1, V_2 with the same properties as V, we have

$$(*) \qquad I(f, V_1) = I(f, V_2).$$

Indeed, if $V_1 \subset V_2$, our assertion follows from the excision property of the index in E^n, and the general case reduces evidently to this one.

(3.3) DEFINITION. Let $U \subset E$ be open and $f \in \mathscr{F}(U, E)$ compact. We define the *Leray Schauder fixed point index* $I(f) = I(f, U)$ of f by

$$I(f) = I(f, V) = I(g'_\varepsilon, E^n \cap V).$$

It follows from $(*)$ that this definition is independent of the choice of V, and thus $I(f)$ is well defined.

We may now state the first main result of this paragraph.

(3.4) THEOREM. *Let C be the category of normed linear spaces, and let $\mathscr{F}$ be the class of all compact maps $f \in \mathscr{F}(U, E)$ for all $E \in C$ and all $U \subset E$ open. Then the Leray-Schauder index $I : \mathscr{F} \to Z$ has all the properties* (I)–(VII) *of* (2.1) *provided that in* (III) *all homotopies are assumed to be compact, and in* (VII) *it is required that either f and gf be compact, or g and fg be compact.*

PROOF. With the aid of the Schauder approximation theorem (3.1) and Lemma (3.2), the properties of the index in E^n yield in a straightforward manner the corresponding properties (I)–(VI) of the Leray–Schauder index.

As an example, we give the proof of property (III). Let $h : U \times \boldsymbol{I} \to E$ be a compact and compactly fixed homotopy, and let $h_t : U \to E$ be the corresponding family of compactly fixed compact maps; we want to show that $I(h_0, U) = I(h_1, U)$. Take an open set $V \subset \overline{V} \subset U$ such that $K = \bigcup_{t \in \boldsymbol{I}} \operatorname{Fix}(h_t) \subset V$, and let $\varepsilon > 0$ be smaller than $\eta = \operatorname{dist}(K, \partial V)$. Consider the restriction $H = h|\overline{V} \times \boldsymbol{I}$ and choose an ε-approximation $H^{(\varepsilon)} : \overline{V} \times \boldsymbol{I} \to E$ of the compact map $H : \overline{V} \times \boldsymbol{I} \to E$ with $H^{(\varepsilon)}(\overline{V} \times \boldsymbol{I}) \subset E^n$, where E^n is a finite-dimensional subspace of E. We now define $g_t : V \cap E^n \to E^n$ by $g_t = H_t^{(\varepsilon)}|V \cap E^n$ and observe that because $\{H_t^{(\varepsilon)}\}$ has no fixed points on ∂V, the homotopy $\{g_t\}$ is compactly fixed. Hence, by property (III) of the index in E^n, we have $I(g_0, V \cap E^n) = I(g_1, V \cap E^n)$. On the other hand, by the definition (3.3) of the Leray–Schauder index,

$$I(h_0, U) = I(g_0, V \cap E^n), \qquad I(h_1, U) = I(g_1, V \cap E^n),$$

and thus the proof of the homotopy property of the Leray–Schauder index is complete.

The proof of properties (I), (II) and (IV)–(VI) is similar and is left to the reader.

The proof of property (VII), which is somewhat more sophisticated and requires some attention to detail, is given separately in the next section. □

4. Commutativity of the Index

In this proof we shall use the fact that the Leray–Schauder index has properties (I), (III), (V), and (VI).

Let E and E' be normed linear spaces, $U \subset E$ and $U' \subset E'$ be open, and let $f : U' \to E$ and $g : U \to E'$ be continuous. Consider the composites

$$gf : V' = f^{-1}(U) \to E', \qquad fg : V = g^{-1}(U') \to E,$$

and their fixed point sets

$$\mathrm{Fix}(gf) = \{x \in f^{-1}(U) \mid x = gf(x)\},\ \mathrm{Fix}(fg) = \{y \in g^{-1}(U') \mid y = fg(y)\}.$$

We note that the maps $f : \mathrm{Fix}(gf) \to \mathrm{Fix}(fg)$ and $g : \mathrm{Fix}(fg) \to \mathrm{Fix}(gf)$ are inverse to each other, and hence the fixed point sets $\mathrm{Fix}(gf)$ and $\mathrm{Fix}(fg)$ are homeomorphic; thus, if one of them is compact, then so is the other. Assuming that gf (and hence also fg) is compactly fixed, we are going to show that

$$I(gf, f^{-1}(U)) = I(fg, g^{-1}(U')).$$

We proceed in two steps.

Step 1 (Special case: both f and g are compact). The proof in this case is essentially a repetition of that in $\boldsymbol{R}^n$ (Section 1); we shall therefore only indicate the main points, leaving the details to the reader.

We define compact homotopies

$$\begin{aligned} h_t, \widehat{h}_t &: V' \times V \to E' \times E, \\ H_t &: V' \times E \to E' \times E, \\ \widehat{H}_t &: E' \times V \to E' \times E, \end{aligned}$$

by setting

$$(*) \qquad \begin{aligned} h_t(x,y) &= (tgf(x) + (1-t)g(y), f(x)), && (x,y,t) \in V' \times V \times \boldsymbol{I}, \\ \widehat{h}_t(x,y) &= (g(y), tfg(y) + (1-t)f(x)), && (x,y,t) \in V' \times V \times \boldsymbol{I}, \\ H_t(x,y) &= (gf(x), (1-t)f(x)), && (x,y,t) \in V' \times E \times \boldsymbol{I}, \\ \widehat{H}_t(x,y) &= ((1-t)g(y), fg(y)), && (x,y,t) \in E' \times V \times \boldsymbol{I}. \end{aligned}$$

Observe that $\mathrm{Fix}(h_0)$ and $\mathrm{Fix}(gf)$ are homeomorphic under $(x,y) \mapsto x$, $x \mapsto (x, f(x))$; a simple calculation gives $\mathrm{Fix}(h_0) = \mathrm{Fix}(h_t) = \mathrm{Fix}(\widehat{h}_t)$, thus showing that the homotopies h_t and $\widehat{h}_t$ are compactly fixed. In a similar way, one verifies that H_t and $\widehat{H}_t$ are also compactly fixed.

Further, since in view of $(*)$ we have

$$h_0 = \widehat{h}_0, \quad h_1 = H_0 | V' \times V, \quad \widehat{h}_1 = \widehat{H}_0 | V' \times V,$$

the homotopy and excision properties imply that

$$\begin{aligned} I(H_1, V' \times E) &= I(H_0, V' \times V) = I(h_1) = I(h_0) = I(\widehat{h}_0) = I(\widehat{h}_1) \\ &= I(\widehat{H}_0, V' \times V) = I(\widehat{H}_1, E' \times V), \end{aligned}$$

and hence $I(H_1) = I(\widehat{H}_1)$.

Finally, since $H_1 = (gf) \times 0$ and $\widehat{H}_1 = 0 \times (fg)$, where 0 is the constant map $y \mapsto 0$ for H_1 and $x \mapsto 0$ for $\widehat{H}_1$, in view of the multiplicativity property we have $I(gf) \cdot I(0) = I(0) \cdot I(fg)$, and because $I(0) = 1$ by normalization, the assertion is proved.

STEP 2 (General case: f and gf are compact). In this case, because f is compact, so also is fg. Assuming as before that fg and gf are compactly fixed, we must show that

$$I(gf, f^{-1}(U)) = I(fg, g^{-1}(U')),$$

and we begin by reducing the problem.

First, by excision, we may assume (by taking smaller open sets if necessary) that f and g are in fact defined on $\overline{U'}$ and $\overline{U}$, respectively. Take a smaller open set $O \subset \overline{O} \subset U$ with $\operatorname{Fix}(fg) \subset O$ and set $O' = f^{-1}(O)$. Clearly $\operatorname{Fix}(gf) \subset O'$, so by excision, our assertion will be established if we show that

$$I(gf, f^{-1}(O)) = I(fg, g^{-1}(O')).$$

We now start the proof. Set

$$\begin{aligned} \eta_1 &= \inf\{\|x - gf(x)\| \mid x \in \partial f^{-1}(O)\}, \\ \eta_2 &= \inf\{\|y - fg(y)\| \mid y \in \partial g^{-1}(O')\}, \\ \eta &= \min(\eta_1, \eta_2). \end{aligned}$$

We note that $\eta > 0$ by (3.2) (because $(gf)|\overline{f^{-1}(O)}$ and $(fg)|\overline{g^{-1}(O')}$ are fixed point free on $\partial f^{-1}(O)$ and $\partial g^{-1}(O')$, respectively).

Let K^* be a compact set satisfying $f(U') \subset K^*$ and observe that on $\overline{f^{-1}(O)} \subset f^{-1}(\overline{O})$ the map f has values in the compact set $K = K^* \cap \overline{O} \subset U$.

We claim that

(**) there exists a positive $\delta < \delta^* = \operatorname{dist}(K, \partial U)$ such that for all y, y' in U,

$$[y' \in B(y, \delta) \text{ and } y \in K] \Rightarrow [g(y') \in B(g(y), \eta)].$$

Indeed, supposing the contrary, we find sequences $\{y_n\}$, $\{y_n'\}$ with $y_n \in K$, $y_n' \in U$ such that for every $n \in N$,

$$\text{(i) } \|y_n - y_n'\| < \delta^*/n. \qquad \text{(ii) } \|g(y_n) - g(y_n')\| \geq \eta.$$

Since K is compact, we may assume $y_n \to y_0 \in K$, and hence $y'_n \to y_0$ by (i). From this and the continuity of g, we obtain $\|g(y_n) - g(y'_n)\| \to 0$, which contradicts (ii); thus our assertion (**) is proved.

We set $V_\delta = \bigcup_{y \in K} B(y, \delta)$ and $\varepsilon = \min\{\delta/2, \eta/2\}$. Let $f_\varepsilon : U' \to E$ be an ε-approximation of f and

$$h_t(x) = tf(x) + (1-t)f_\varepsilon(x) \qquad \text{for } (x,t) \in U' \times \boldsymbol{I}.$$

We make the following observations:

1° Since $\varepsilon < \delta/2$, we see that for any $x \in \overline{f^{-1}(O)}$ the values of $h_t(x)$, $t \in \boldsymbol{I}$, are in the convex open ball $B(f(x), \delta/2) \subset V_{\delta/2}$, and therefore $h_t : \overline{f^{-1}(O)} \to V_\delta$ is a compact δ-homotopy joining $f, f_\varepsilon : \overline{f^{-1}(O)} \to V_\delta$.

2° From 1° and (**), it follows that $gh_{t'}(x), gh_{t''}(x) \in B(gf(x), \eta)$ for all $x \in \overline{f^{-1}(O)}$ and $t', t'' \in \boldsymbol{I}$; this implies that the composition

$$\overline{f^{-1}(O)} \xrightarrow{h_t} V_\delta \xrightarrow{g} E'$$

is a compact 2η-homotopy with no fixed points on $\partial f^{-1}(O)$, by (3.2) and the choice of η.

3° In view of 2° and (3.2), we see that on $f^{-1}(O)$ the homotopy $\{gh_t\}$ joining $gf, gf_\varepsilon : f^{-1}(O) \to E'$ is compactly fixed, and therefore, by the homotopy invariance of the index,

$$I(gf, f^{-1}(O)) = I(gf_\varepsilon, f^{-1}(O)).$$

We now reduce the problem further. Since $f_\varepsilon(U') \subset E^n$ for some $E^n \subset E$, we have $f_\varepsilon(f^{-1}(O)) \subset E^n \cap V_\delta$, and we note that on $f^{-1}(O)$, gf_ε can be expressed as the composition

$$f^{-1}(O) \xrightarrow{\tilde{f}_\varepsilon} E^n \cap V_\delta \xrightarrow{\tilde{g}} E',$$

where $\tilde{f}_\varepsilon$ is defined by f_ε and $\tilde{g} = g|E^n \cap V_\delta$; therefore,

$$I(gf_\varepsilon, f^{-1}(O)) = I(\tilde{g}\tilde{f}_\varepsilon, f^{-1}(O)).$$

Moreover, since $E^n \cap V_\delta$ is bounded, the map $\tilde{g}$ is compact, and thus (because $\tilde{f}_\varepsilon$ and $\tilde{g}$ now satisfy the requirements of the special case of commutativity established in Step 1) we get

$$I(\tilde{g}\tilde{f}_\varepsilon, f^{-1}(O)) = I(\tilde{f}_\varepsilon\tilde{g}, \tilde{g}^{-1}(f^{-1}(O))).$$

Comparing this with the preceding formulas and since $O' = f^{-1}(O)$, we obtain

(***) $$I(gf, f^{-1}(O)) = I(\tilde{f}_\varepsilon\tilde{g}, \tilde{g}^{-1}(O')).$$

Next, on $\overline{g^{-1}(O')} \subset g^{-1}(\overline{O'})$ consider the composition

$$\overline{g^{-1}(O')} \xrightarrow{g} \overline{f^{-1}(O)} \xrightarrow{h_t} V_\delta.$$

Clearly, because $\varepsilon < \eta$, $\{h_t g\}$ is a compact η-homotopy, and therefore, by (3.2), it is fixed point free on $\partial g^{-1}(O')$. This implies that $h_t g : g^{-1}(O') \to E$ is compactly fixed, and thus, by homotopy invariance,

$$I(fg, g^{-1}(O')) = I(f_\varepsilon g, g^{-1}(O')).$$

To conclude, we observe that since the values of $f_\varepsilon g$ are in $E^n \cap V_\delta$, by the definition of the Leray–Schauder index applied to $f_\varepsilon g$ we obtain

$$I(f_\varepsilon g, g^{-1}(O')) = I(f_\varepsilon g, g^{-1}(O') \cap V_\delta \cap E^n) = I(\widetilde{f}_\varepsilon \widetilde{g}, \widetilde{g}^{-1}(O')).$$

Comparing this with the preceding formula and then with (***) we get $I(gf, f^{-1}(O)) = I(fg, g^{-1}(O'))$, and thus the proof of commutativity is complete. □

5. Fixed Point Index for Compact Maps in ANRs

In this section the Leray–Schauder fixed point index is extended to compact maps $f \in \mathscr{F}(U, X)$, where X is an ANR. The basic idea of the extension is to use commutativity in conjunction with the fact that every ANR is r-dominated by an open subset of some normed linear space.

(5.1) DEFINITION. Let X be an ANR, $U \subset X$ open, and $f \in \mathscr{F}(U, X)$ a compact map. Take an open set V in a normed linear space E that r-dominates X. Let $s : X \to V$ and $r : V \to X$ be such that $rs = 1_X$. Since the composite $r^{-1}(U) \xrightarrow{r} U \xrightarrow{f} X \xrightarrow{s} V$ is compact and compactly fixed (because $\mathrm{Fix}(f) \approx \mathrm{Fix}(sfr)$), its Leray–Schauder index is defined and we let

$$(*) \qquad I(f, U) = I(f) = I(sfr, r^{-1}(U)).$$

This is independent of the choice of V, r, and s. Indeed, let $V' \subset E'$ be another open set in a normed linear space E' that r-dominates X, with $s' : X \to V'$, $r' : V' \to X$, $r's' = 1_X$. Then, because the second of the maps $sr' : V' \to V$, $s'fr : r^{-1}(U) \to V'$ is compact, and so is the composite $(sr')(s'fr)$, the commutativity property of the Leray–Schauder index gives

$$I((s'fr) \circ (sr'), (sr')^{-1}(r^{-1}(U))) = I((sr') \circ (s'fr), r^{-1}(U)).$$

Since $(s'fr) \circ (sr') = s'fr'$, $(sr') \circ (s'fr) = sfr$ and $(sr')^{-1}(r^{-1}(U)) = r'^{-1}(U)$, we get $I(s'fr', r'^{-1}(U)) = I(sfr, r^{-1}(U))$, which proves that our definition is independent of the choices involved.

We are now ready to state the main result of this chapter.

(5.2) THEOREM. *Let C be the category of ANRs, and let $\mathscr{F}$ be the class of all compact maps $f \in \mathscr{F}(U, X)$ for all $X \in C$ and $U \subset X$ open. Then the fixed point index $I : \mathscr{F} \to \mathbb{Z}$ defined by formula $(*)$ has all the*

properties (I)- (VII) *of* (2.1) *provided that in* (III) *it is assumed that all homotopies are compact, and in* (VII) *it is required that either* f *and* gf *be compact, or* g *and* fg *be compact.*

PROOF. The proof consists of evident reductions to the corresponding properties of the Leray -Schauder index. As an example, we prove property (VII).

Let X and X' be ANRs, let $f : U' \to X$, $g : U \to X'$ be two maps, and assume that f and gf are compact. Let V (respectively V') be an open set in a normed linear space E (respectively E') that r-dominates X (respectively X'); let $X \xrightarrow{s} V \xrightarrow{r} X$ and $X' \xrightarrow{s'} V' \xrightarrow{r'} X'$ satisfy $rs = 1_X$ and $r's' = 1_{X'}$.

Consider the maps

$$sfr' : r'^{-1}(U') \to V, \quad s'gr : r^{-1} : r^{-1}(U) \to V'$$

and note that the first of them is compact, and so is the composition $(s'gr)(sfr') = s'gfr'$. By commutativity of the Leray-Schauder index,

$$I((sfr')(s'gr), (s'gr)^{-1}r'^{-1}(U')) = I((s'gr)(sfr'), (sfr')^{-1}r^{-1}(U)),$$

and hence, because

$$(s'gr)^{-1}r'^{-1}(U') = r^{-1}g^{-1}(U'), \quad (sfr')^{-1}r^{-1}(U) = r'^{-1}f^{-1}(U),$$

we have $I(sfgr, r^{-1}g^{-1}(U')) = I(s'gfr', r'^{-1}f^{-1}(U))$. In view of Definition (5.1), this gives

$$I(fg, g^{-1}(U')) = I(gf, f^{-1}(U)).$$

The proofs of other properties are similar and are left to the reader. □

We remark that commutativity implies the following property of the index:

(VIII) (Contraction) *Let* (X, A) *be a pair of ANRs with* A *closed in* X. *Assume that* $U \subset X$ *is open and* $f \in \mathscr{F}(U, X)$ *is compact with* $f(U) \subset A$, *and denote by* $f_{U \cap A} : U \cap A \to A$ *the contraction of* f. *Then* $I(f, U) = I(f_{U\cap A}, U \cap A)$.

PROOF. Note that if $i : A \hookrightarrow X$ is the inclusion, then both if and f are compact; hence commutativity gives

$$I(f, U) = I(if, U) = I(fi, i^{-1}(U)) = I(f_{U\cap A}, U \cap A). \qquad \square$$

As a special case of Theorem (5.2) we obtain

(5.3) THEOREM. *Let* C *be the category of compact ANRs, and let* $\mathscr{F}$ *be the class of all continuous maps* $f \in \mathscr{F}(U, X)$ *for all* $X \in C$ *and all* U *open in* X. *Then there exists on* $\mathscr{F}$ *a fixed point index function* $I : \mathscr{F} \to Z$ *with properties* (I)–(VII). □

6. The Leray–Schauder Continuation Principle in ANRs

In this section, using the notation and terminology of (11.7.4), we first state the properties of the Leray–Schauder index in a slightly different but equivalent setting for maps in $\mathscr{K}_{\partial U}(\overline{U}, X)$, where X is an ANR. Next, we establish a general version of homotopy invariance of the index, and with its aid we prove the Leray–Schauder continuation principle in ANRs.

Let X be an ANR and U open in X. Recall that by $\mathscr{K}(\overline{U}, X)$ we denote the set of all compact maps from $\overline{U}$ to X, and by $\mathscr{K}_{\partial U}(\overline{U}, X)$ the set of all maps $f \in \mathscr{K}(\overline{U}, X)$ that have no fixed points on ∂U. Observing that for $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ the map $f|U$ is in $\mathscr{F}(U, X)$, we can use Theorem (5.2) to develop an index for maps in $\mathscr{K}_{\partial U}(\overline{U}, X)$.

(6.1) Definition. Let X be an ANR, U open in X, and $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$. The *fixed point index* $i(f, U)$ of f is given by

$$i(f, U) = I(f|U, U).$$

The properties of the index for compact maps in $\mathscr{F}(U, X)$ immediately translate to properties of the index for maps in $\mathscr{K}_{\partial U}(\overline{U}, X)$, and we obtain the following basic theorem:

(6.2) Theorem. *Let X be an ANR, $U \subset X$ an arbitrary open subset, and $\mathscr{K}_{\partial U}(\overline{U}, X)$ the set of all compact admissible maps from $\overline{U}$ to X. Then there exists an integer-valued fixed point index function $f \mapsto i(f)$ for $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ with the following properties*:

(I) (Normalization) *If $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ is a constant map $u \mapsto u_0$, then $i(f, U) = 1$ or 0 depending on whether or not $u_0 \in U$.*

(II) (Additivity) *If $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ and* $\mathrm{Fix}(f) \subset U_1 \cup U_2 \subset U$ *with U_1, U_2 open and disjoint, then*

$$i(f, U) = i(f, U_1) + i(f, U_2).$$

(III) (Homotopy) *If $h_t : \overline{U} \to X$ is an admissible compact homotopy in $\mathscr{K}_{\partial U}(\overline{U}, X)$, then $i(h_0, U) = i(h_1, U)$.*

(IV) (Existence) *If $i(f, U) \neq 0$, then* $\mathrm{Fix}(f) \neq \emptyset$.

(V) (Excision) *If V is an open subset of U and if $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ has no fixed points in $U - V$, then $i(f, U) = i(f, V)$.*

(VI) (Multiplicativity) *If $f_1 \in \mathscr{K}_{\partial U_1}(\overline{U}_1, X_1)$ and $f_2 \in \mathscr{K}_{\partial U_2}(\overline{U}_2, X_2)$, then $f_1 \times f_2 \in \mathscr{K}_{\partial(U_1 \times U_2)}(\overline{U}_1 \times \overline{U}_2, X_1 \times X_2)$ and*

$$i(f_1 \times f_2, U_1 \times U_2) = i(f_1, U_1) \cdot i(f_2, U_2).$$

(VII) (Commutativity) *Let X, X' be ANRs, let $U \subset X$, $U' \subset X'$ be open, and let $f : \overline{U} \to X'$, $g : \overline{U'} \to X$ be continuous maps, at least one of them being compact. Define $V = U \cap f^{-1}(U')$ and $V' = U' \cap g^{-1}(U)$. Then:*

(i) *the maps*

$$gf : \overline{V} \to X, \quad fg : \overline{V'} \to X'$$

are compact,

(ii) *if* $\mathrm{Fix}(gf) \subset V$ *and* $\mathrm{Fix}(fg) \subset V'$, *then*

$$i(gf, V) = i(fg, V').$$

We list further frequently used properties of the index:

(VIII) (Contraction) *Let (X, A) be a pair of ANRs with A closed in X, $U \subset X$ open, and $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ with $f(\overline{U}) \subset A$. Let $\widehat{f} = f_{\overline{U \cap A}} : \overline{U \cap A} \to A$ be the contraction of f. Then $\widehat{f} \in \mathscr{K}_{\partial(U \cap A)}(\overline{U \cap A}, A)$ and $i(f, U) = i(\widehat{f}, U \cap A)$.*

(IX) (Localization) *Let $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ and* $\mathrm{Fix}(f) \subset U_1 \cup U_2 \subset U$ *with U_1, U_2 open and disjoint. Suppose $i(f, U) \neq 0$ and $i(f, U_1) = 0$. Then* $\mathrm{Fix}(f|U_2) \neq \emptyset$.

(X) (Multiplicity) *Let $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ and* $\mathrm{Fix}(f) \subset U_1 \cup U_2 \subset U$ *with U_1, U_2 open and disjoint. Suppose $i(f, U) = 0$ and $i(f, U_1) \neq 0$. Then* $\mathrm{Fix}(f|U_1) \neq \emptyset$ *and* $\mathrm{Fix}(f|U_2) \neq \emptyset$.

We remark that the contraction property follows from the commutativity of the index; the localization and multiplicity properties are obvious consequences of additivity and normalization.

We now establish, for the index in ANRs, the general homotopy invariance property, which has numerous applications in nonlinear analysis.

Let X be an ANR and $J = [a, b] \subset \boldsymbol{R}$; if M is a subset of $X \times J$, then for each $t \in J$, the *t-slice* of M, written M_t, is given by

$$M_t = \{x \in X \mid (x, t) \in M\}.$$

Let $U \subset X \times J$ be open and assume that U_a and U_b are both nonempty. We call $\partial U - [(U \cap (X \times \{a\})) \cup (U \cap (X \times \{b\}))]$ the *vertical boundary* of U, and denote it by $\hat{\partial} U$ If $f : \overline{U} \to X$ is a map, we let

$$S = \{(x, t) \in \overline{U} \mid f(x, t) = x\}.$$

For $t \in J$, $f_t : \overline{U}_t \to X$ is given by $f_t(x) = f(x, t)$; as usual we use the fact that f determines the parametrized family $\{f_t : \overline{U}_t \to X\}_{t \in J}$ and vice versa.

With this notation, the basic property of the fixed point index is:

(6.3) THEOREM (General homotopy invariance). *Let X be an ANR, $J = [a, b] \subset \boldsymbol{R}$, and U open in $X \times J$. Let $f : \overline{U} \to X$ be a compact map such that $S \cap \hat{\partial} U = \emptyset$. Then $s \mapsto i(f_s, U_s)$ is a constant function on J; in particular, $i(f_a, U_a) = i(f_b, U_b)$.*

PROOF. We first show that $s \mapsto i(f_s, U_s)$ is a locally constant function on J. Fix $t_0 \in J$. Select an open nbd $J_1 \subset J$ of t_0 and an open set V_0 with $\mathrm{Fix}(f_{t_0}) \subset V_0 \subset \overline{V}_0 \subset U_{t_0}$ such that $V_0 \times J_1 \subset U$. We now claim that there exists an open nbd $J_0 \subset J_1$ of t_0 such that:

(a) $V_0 \times J_0 \subset U$,

(b) $\mathrm{Fix}(f_s) \subset V_0$ for each $s \in J_0$.

Indeed, supposing our assertion to be false, we find a sequence $(x_n, t_n) \in (X - V_0) \times [t_0 - 1/n, t_0 + 1/n]$ such that $f(x_n, t_n) = x_n$ for any $n \in N$. Because f is compact, we may suppose that $x_n \to x_0 \in X - V_0$, and since $t_n \to t_0$, we obtain $f(x_n, t_n) \to f(x_0, t_0) = x_0 \in X - V_0$; but this contradicts $\mathrm{Fix}(f_{t_0}) \subset V_0$, and thus our assertion is proved.

We remark that (a) and (b) imply $\mathrm{Fix}(f_s|\partial V_0) = \emptyset$ for each $s \in J_0$.

Now define a compact homotopy $h : V_0 \times J_0 \to X$ by

$$h(x,t) = f(x,t) \quad \text{for } (x,t) \in V_0 \times J_0.$$

By the homotopy property, $i(h_s, V_0)$ is locally constant in a nbd $J^* \subset J_0$ of t_0, i.e.,

$$i(f_s, V_0) = i(f_{t_0}, V_0) \quad \text{for all } s \in J^*.$$

From this, and because by (b) and excision,

$$i(f_s, V_0) = i(f_s, U_s), \quad i(f_{t_0}, V_0) = i(f_{t_0}, U_{t_0}),$$

we obtain $i(f_s, U_s) = i(f_{t_0}, U_{t_0})$ for all $s \in J^*$. Since $t_0 \in J$ was fixed arbitrarily, we see that $s \mapsto i(f_s, U_s)$ is locally constant on J, and thus because J is connected, it is constant on the entire interval. □

To prepare for applications of (6.3), we first establish a result from general topology.

Let X be a space, and let A and B be two subsets of X. We say that X is *disconnected between A and B* if there is a closed-open set K_A such that $A \subset K_A$ and $K_A \cap B = \emptyset$; otherwise, X is said to be *connected between A and B*. We remark that X is connected between two points $a, b \in X$ if a and b belong to the same quasi-component of X (the *quasi-component* of a is the intersection of all closed-open subsets of X containing a).

(6.4) THEOREM (Kuratowski–Mazurkiewicz separation theorem). *Let A and B be two disjoint closed subsets of a compact space X. Then one of the following properties holds*:

(i) *there exist two closed-open subsets K_A and K_B of X such that $X = K_A \cup X_B$, $A \subset K_A$, $B \subset K_B$, and $K_A \cap K_B = \emptyset$,*

(ii) *there is a continuum $C \subset X$ such that $C \cap A \neq \emptyset \neq C \cap B$.*

PROOF. Assume first that X is disconnected between A and B; in this case if K_A is closed-open in X with $A \subset K_A$ and $K_A \cap B = \emptyset$, then $K_B = X - K_A$

is closed-open in X with $B \subset K_B$ and $K_A \cap K_B = \emptyset$, and thus property (i) is true.

Assume next that X is connected between A and B; in this case we are going to show that property (ii) holds. For the proof we first establish that X is in fact connected between a certain pair of points $a \in A$ and $b \in B$. Supposing the contrary, given any $(a,b) \in A \times B$, we find a closed-open N_{ab} such that $a \in N_{ab}$ and $b \notin N_{ab}$. For any $b \in B$, the open sets $\{N_{ab} \mid a \in A\}$ form a cover for A, and hence compactness of A shows $A \subset \bigcup\{N_{ab} \mid a \in A'\}$ with a finite $A' \subset A$. Letting $N_b = \bigcup\{N_{ab} \mid a \in A'\}$ for $b \in B$, we see that N_b is a closed-open subset of X with $A \subset N_b$ and $b \notin N_b$. This implies that the open sets $\{X - N_b \mid b \in B\}$ form a cover for B, and therefore, by compactness, $B \subset \bigcup\{X - N_b \mid b \in B'\}$ where $B' \subset B$ is finite. It follows readily that $K = \bigcap\{N_b \mid b \in B'\}$ is a closed-open subset of X such that $A \subset K$ and $K \cap B = \emptyset$, which contradicts our assumption.

Thus for a certain pair $(a,b) \in A \times B$, our assertion is established, i.e., the points a and b belong to the same quasi-component of X. Now because X is compact, and in a compact space components coincide with quasi-components (see Engelking's book [1989]), it follows that the component C of $a \in A$ must also contain $b \in B$. Thus $C \cap A \neq \emptyset \neq C \cap B$, and the proof is complete. □

We are now ready to prove a basic theorem:

(6.5) THEOREM (Leray–Schauder continuation principle). *Let X be an ANR, U an open subset of $X \times [a,b]$, and let $f : \overline{U} \to X$ be a compact map such that:*

(i) *the sets $\widehat{\partial} U$ and $S = \{(x,t) \in \overline{U} \mid f(x,t) = x\}$ are disjoint,*

(ii) $i(f_a, U_a) \neq 0$.

Then there exists a continuum $C \subset S$ joining the sets

$$A = S \cap (X \times \{a\}) \quad \textit{and} \quad B = S \cap (X \times \{b\}).$$

PROOF. We first observe that because of (ii), (6.3) implies that A and B are nonempty. Suppose to the contrary that there is no continuum $C \subset S$ such that $A \cap C \neq \emptyset \neq B \cap C$. Then, since S, A, and B are compact and nonempty, by the Kuratowski–Mazurkiewicz theorem (6.4) we can find disjoint compact sets $S_A, S_B \subset S$ such that $S = S_A \cup S_B$, $A \subset S_A$, and $B \subset S_B$. Let $\varepsilon < \frac{1}{2}\operatorname{dist}(S_A, S_B)$ and denote by W an ε-nbd of S_B in U. Letting now $V = U - \overline{W}$, we see that V is open in U and has the following properties:

(a) S and the vertical boundary $\widehat{\partial} V$ of V are disjoint,

(b) $\operatorname{Fix}(f_a) \subset V_a$, $\operatorname{Fix}(f_b) \cap V_b = \emptyset$.

By excision, $i(f_a, V_a) = i(f_a, U_a) \neq 0$, and because $S \cap \widehat{\partial} V = \emptyset$, the general homotopy invariance (6.3) gives $i(f_a, V_a) = i(f_b, V_b) \neq 0$. On the other hand,

$\mathrm{Fix}(f_b) \cap V_b = \emptyset$ shows that $i(f_b, V_b) = 0$. This contradiction completes the proof. □

We conclude this section by deriving, in the framework of index theory, a strengthened version of Borsuk's theorem.

(6.6) THEOREM (Antipodal theorem). *Let E be a normed linear space and U a bounded centrally symmetric neighborhood of the origin in E. Let $f \in \mathscr{K}_{\partial U}(\overline{U}, E)$ be such that $f(x) = -f(-x)$ for each $x \in \partial U$. Then $i(f, U)$ is odd.*

PROOF. Let $0 < \varepsilon < \inf\{\|x - f(x)\| \mid x \in \partial U\}$. Using the notation of (6.2.2), choose a finite subset $N \subset E$ symmetric with respect to the origin, such that $f(\overline{U}) \subset (N, \varepsilon)$. It follows that the finite-dimensional approximation $f_\varepsilon = p_\varepsilon \circ f$ is still antipode-preserving on ∂U. Let E^k be a finite-dimensional subspace of E with $f_\varepsilon(\overline{U}) \subset E^k$, and set $\widehat{f}_\varepsilon = f_\varepsilon | \overline{U} \cap E^k$ Now $U^k = U \cap E^k$ is an open, bounded, centrally symmetric neighborhood of 0 in E^k, and $\widehat{f}_\varepsilon \in \mathscr{K}_{\partial U^k}(\overline{U^k}, E^k)$ is antipode-preserving on ∂U^k, so by Theorem (10.9.2), because $i(f, U) = i(\widehat{f}_\varepsilon, U^k) = d(I - \widehat{f}_\varepsilon, U^k)$, the conclusion follows. □

7. Simple Consequences and Index Calculations

In many problems arising naturally in analysis, ANR spaces appear as convex sets in normed linear spaces. Because of their richer structure, the properties of the index lead to a number of further useful results.

Let E be a normed linear space, C a convex (not necessarily closed) subset of E, and $U \subset C$ open with $0 \in U$. As before, we denote by $\mathscr{K}_{\partial U}(\overline{U}, C)$ the set of compact maps $f : \overline{U} \to C$ that are fixed point free on ∂U.

We begin with two simple consequences of the homotopy property of the index:

(7.1) PROPOSITION. *Let $f, g \in \mathscr{K}_{\partial U}(\overline{U}, C)$ and assume that one of the following conditions holds:*

(i) $f|\partial U = g|\partial U$,
(ii) $(1 - t)f(x) + tg(x) \neq x$ *for each* $(x, t) \in \partial U \times I$,
(iii) $\sup_{x \in \partial U} \|f(x) - g(x)\| \leq \inf_{x \in \partial U} \|x - f(x)\|$.

Then $i(f, U) = i(g, U)$.

PROOF. By (6.4.2), any of (i)–(iii) implies $f \simeq g$ in $\mathscr{K}_{\partial U}(\overline{U}, C)$, so the conclusion follows from the homotopy property of the index. □

We now show that $i(f, U)$ does not change under small modifications of f on ∂U.

(7.2) PROPOSITION. *Let* $f \in \mathscr{K}_{\partial U}(\overline{U}, C)$ *and* $0 < \varepsilon < \inf_{x \in \partial U} \|x - f(x)\|$. *Assume also that* $g : \overline{U} \to C$ *is a compact map with* $\|f(x) - g(x)\| < \varepsilon$ *for all* $x \in \partial U$. *Then:*
(a) $g \in \mathscr{K}_{\partial U}(\overline{U}, C)$,
(b) $i(f, U) = i(g, U)$.

PROOF. (a) is obvious by definition of ε; (b) follows from (7.1). □

We now show that for some simple maps the index can be easily calculated.

(7.3) THEOREM. *Let* $f \in \mathscr{K}_{\partial U}(\overline{U}, C)$ *be such that*

$$x \neq \lambda f(x) \quad \text{for all } (x, \lambda) \in \partial U \times (0, 1).$$

Then $i(f, U) = 1$.

PROOF. Let $g : \overline{U} \to C$ be the constant map $x \mapsto 0$ and consider the compact homotopy $h_t : \overline{U} \to C$ given by $h_t(x) = (1 - t)f(x)$ joining f to g. By assumption, the homotopy $\{h_t\}$ is fixed point free on ∂U, so by the homotopy and normalization properties, the conclusion follows. □

(7.4) COROLLARY. *Let* $|\ |$ *be any norm in* E, *let* $f \in \mathscr{K}_{\partial U}(\overline{U}, C)$, *and assume that one of the following conditions holds for all* $x \in \partial U$:
(i) $|f(x)| \leq |x|$,
(ii) $|f(x)| \leq |x - f(x)|$,
(iii) $|f(x)|^2 \leq |x|^2 + |x - f(x)|^2$,
(iv) $\langle x, f(x) \rangle \leq \langle x, x \rangle$, *where* $\langle\ ,\ \rangle$ *is a scalar product in* E.
Then $i(f, U) = 1$. □

Invariant directions for compact maps

(7.5) DEFINITION. We say that a map $f \in \mathscr{K}_{\partial U}(\overline{U}, C)$ has an *invariant direction* if there are $x \in \partial U$ and $\mu > 0$ such that $x = \mu f(x)$; the element x is called an *eigenvector* of f, and the number μ is a *characteristic value* of f.

The following immediate consequence of (7.3) provides useful information about the existence of invariant directions:

(7.6) THEOREM. *Let* $U \subset C$ *be open with* $0 \in U$, *and let* $f \in \mathscr{K}_{\partial U}(\overline{U}, C)$ *satisfy* $i(f, U) \neq 1$. *Then* f *has at least one eigenvector on* ∂U *with characteristic value* $\mu \in (0, 1)$. □

Before giving examples of compact maps that have invariant directions, we recall some terminology.

(7.7) DEFINITION. A convex subset W of a normed linear space $(E, \|\ \|)$ is said to be a *wedge* if $\lambda W \subset W$ for all $\lambda \geq 0$; a wedge W is *proper*

if it is not a linear subspace of E. A wedge W is called a *cone* if $W \cap (-W) = \{0\}$.

If $W \subset E$ is a wedge and $\varrho > 0$, we let

$$B_\varrho = \{x \in W \mid \|x\| < \varrho\}, \quad S_\varrho = \{x \in W \mid \|x\| = \varrho\}, \quad K_\varrho = S_\varrho \cup B_\varrho.$$

(7.8) THEOREM. *Let E be a normed linear space, $W \subset E$ a proper wedge, and let $f \in \mathscr{K}_{S_\varrho}(K_\varrho, W)$ be such that $\inf\{\|f(x)\| \mid x \in S_\varrho\} > 0$ and $\lambda f(x) \neq x$ for all $(x, \lambda) \in S_\varrho \times (1, \infty)$. Then:*
(a) $i(f, B_\varrho) = 0$,
(b) *f has an eigenvector on S_ϱ with characteristic value $\mu \in (0,1)$.*

PROOF. Clearly, (a)⇒(b) by (7.6). To show that $i(f, B_\varrho) = 0$, we proceed in a few steps.

Since $\beta = \inf\{\|f(x)\| \mid x \in S_\varrho\} > 0$, we can choose a ball $B_d \subset B_\varrho$ such that $B_d \cap f(S_\varrho) = \emptyset$. Let

$$\alpha_t(x) = (1 + td^{-1}\varrho)f(x), \qquad (x,t) \in K_\varrho \times \boldsymbol{I},$$

and observe that $\{\alpha_t\}$ is a compact homotopy from $f = \alpha_0$ to α_1 in $\mathscr{K}_{S_\varrho}(K_\varrho, W)$, where $\alpha_1(x) = (1 + d^{-1}\varrho)f(x)$.

Next select $v \in W$ with $\|v\| = 1$ such that $-v \notin W$ (v exists because W is not a linear subspace) and define

$$\beta_t(x) = \begin{cases} \|\alpha_1(x)\| \dfrac{(1-t)\alpha_1(x) + tv}{\|(1-t)\alpha_1(x) + tv\|} & \text{if } \alpha_1(x) \neq 0, \\ 0 & \text{if } \alpha_1(x) = 0. \end{cases}$$

From the estimate

$$\|\alpha_1(x)\| = (1 + d^{-1}\varrho)\|f(x)\| \geq \beta \qquad \text{on } S_\varrho,$$

we see that $\{\beta_t\}$ is a compact homotopy joining $\beta_0 = \alpha_1$ to β_1 in $\mathscr{K}_{S_\varrho}(K_\varrho, W)$. Lastly, define

$$\gamma_t(x) = \beta_1(x) + 2t\varrho v \qquad \text{for } (x,t) \in K_\varrho \times \boldsymbol{I}$$

and note that because $\beta_1(S_\varrho) \subset \{\lambda v \mid \lambda > 0\}$, $\{\gamma_t\}$ is a compact homotopy joining β_1 to γ_1 in $\mathscr{K}_{S_\varrho}(K_\varrho, W)$; moreover, from $\gamma_1(K_\varrho) \subset W - K_\varrho$, it follows that $\mathrm{Fix}(\gamma_1) = \emptyset$, and hence, since $f = \alpha_0 \simeq \gamma_1$ in $\mathscr{K}_{S_\varrho}(K_\varrho, W)$, we conclude, by the homotopy and existence axioms, that $i(f, B_\varrho) = i(\gamma_1, B_\varrho) = 0$. □

(7.9) COROLLARY. *Let $f \in \mathscr{K}_{S_\varrho}(K_\varrho, W)$ satisfy $\|f(x)\| \geq \|x\|$ for all $x \in S_\varrho$. Then:*
(a) $i(f, B_\varrho) = 0$,
(b) *f has an eigenvector on S_ϱ with characteristic value $\mu \in (0,1)$.*

PROOF. (a) Suppose $i(f, B_\varrho) \neq 0$; then, by (7.8), $x = \lambda f(x)$ for some $x \in S_\varrho$ and $\lambda > 1$, and hence $\|f(x)\| \geq \lambda\|f(x)\|$, giving $\lambda \leq 1$, a contradiction; (a) implies (b) by (7.6). □

(7.10) COROLLARY. *Let $f \in \mathscr{K}_{S_\varrho}(K_\varrho, W)$ satisfy one of the following conditions:*

(i) $\|f(x)\| \geq \alpha$ *for some* $\alpha > 0$ *and all* $x \in S_\varrho$,

(ii) $\|f(x)\| \geq \|x - f(x)\|$ *for all* $x \in S_\varrho$.

Then f has an eigenvector on S_ϱ with characteristic value $\mu > 0$. □

As our last example, we have

(7.11) THEOREM. *Let E be an infinite-dimensional normed linear space, and U an open bounded subset of E with $0 \in U$. Let $f \in \mathscr{K}_{\partial U}(\overline{U}, E)$ satisfy one of the following conditions:*

(i) $f(\partial U) \cap \operatorname{conv} U = \emptyset$,

(ii) *there is a point $x_0 \in E - \{0\}$ such that $x \neq f(x) + \lambda x_0$ for all $x \in \partial U$ and $\lambda \geq 0$,*

(iii) $\inf\{\|f(x)\| \mid x \in \partial U\} > 0$ *and $f(x) \neq \lambda x$ for all $x \in \partial U$ and $\lambda \in (0,1)$.*

Then:

(a) $i(f, U) = 0$,

(b) *f has at least one eigenvector on ∂U with characteristic value $\mu \in (0,1)$.*

PROOF. Clearly, by (7.6) it suffices to show that $i(f, U) = 0$. As an illustration, we prove the assertion under hypotheses (i) and (ii).

(i) Choose a small ball $B(0, m) = B \subset \overline{B} \subset U$ and (using the fact that $S = \partial B$ is an AR) take a retraction $r : E \to E - B$. Observe that $g(x) = rf(x)$, $x \in \overline{U}$, defines a compact map $g : \overline{U} \to E$ with $g|\partial U = f|\partial U$ and $g(x) \neq 0$ for $x \in \overline{U}$. Let

$$\varrho = \sup\{\|y\| \mid y \in \partial U\} \quad \text{and} \quad h(x) = (2\varrho/m)g(x)$$

for $x \in \overline{U}$. An easy verification shows that $h \in \mathscr{K}_{\partial U}(\overline{U}, E)$, $h(\overline{U}) \cap \overline{U} = \emptyset$ and $h \simeq g$ in $\mathscr{K}_{\partial U}(\overline{U}, E)$. Thus, $0 = i(h, U) = i(g, U) = i(f, U)$.

(ii) Arguing indirectly, assume that $i(f, U) \neq 0$. For each $n \in N$, define $f_n \in \mathscr{K}_{\partial U}(\overline{U}, E)$ by $f_n(x) = f(x) + nx_0$ for $x \in \overline{U}$; observe that by assumption, the formula $h_t(x) = f(x) + tnx_0$ for $(x, t) \in \overline{U} \times I$ defines a compact homotopy joining f to f_n in $\mathscr{K}_{\partial U}(\overline{U}, E)$, and thus, by homotopy, $i(f_n, U) \neq 0$. Hence there exists a sequence $\{y_n\} \subset U$ with $y_n = f(y_n) + nx_0$ for all n, implying that $n\|x_0\| = \|y_n - f(y_n)\| \to \infty$, which is impossible, because the sequence $\{y_n - f(y_n)\}$ is bounded. □

Localization and multiplicity results

We conclude this section with a few simple applications of the index to problems of localization and existence of multiple fixed points.

(7.12) THEOREM (Krasnosel'skiĭ). *Let E be a normed linear space, $W \subset E$ a proper wedge, and assume that $f : W \to W$ is a completely continuous map such that for some numbers r and R with $0 < r < R$, one of the following conditions is satisfied:*
(a) $\|f(x)\| \le \|x\|$ *for* $x \in S_r$, *and* $\|f(x)\| \ge \|x\|$ *for* $x \in S_R$.
(b) $\|f(x)\| \ge \|x\|$ *for* $x \in S_r$, *and* $\|f(x)\| \le \|x\|$ *for* $x \in S_R$.
Then f has a fixed point x with $r \le \|x\| \le R$.

PROOF. We may assume that f has no fixed points on S_r and S_R. We make the following observations:
(i) if (a) holds, then $i(f, B_R) = 0$ by (7.9) and $i(f, B_r) = 1$ by (7.4),
(ii) if (b) holds, then $i(f, B_R) = 1$ by (7.4) and $i(f, B_r) = 0$ by (7.9).
In each case, the conclusion follows from the multiplicity or localization property of the index. □

(7.13) THEOREM (Nussbaum). *Let C be a convex subset of a normed linear space, $U \subset C$ open, and $f \in \mathscr{K}_{\partial U}(\overline{U}, C)$. Assume that:*
(a) *there is an open set $V \subset \overline{V} \subset U$ and a convex set $C_0 \subset V$ such that $f(\partial V) \subset C_0$,*
(b) *there is a point $x_0 \in C - \overline{U}$ such that $(1-t)f(x) + tx_0 \ne x$ for all $(x, t) \in \partial U \times \boldsymbol{I}$.*
Then f has at least two fixed points, $x_1 \in V$ and $x_2 \in U - \overline{V}$.

PROOF. We first calculate $i(f, U)$ and $i(f, V)$. Let $g \in \mathscr{K}_{\partial U}(\overline{U}, C)$ be the constant map $x \mapsto x_0 \in C - \overline{U}$. Clearly, $f \simeq g$ in $\mathscr{K}_{\partial U}(\overline{U}, C)$ by (b), and hence, by normalization and homotopy, $i(f, U) = 0$. Similarly, if $\widehat{g} \in \mathscr{K}_{\partial V}(\overline{V}, C)$ is the constant map $x \mapsto y_0 \in C_0 \subset V$, then because $\widehat{f} = f|V \in \mathscr{K}_{\partial V}(\overline{V}, C)$ and $(1-t)\widehat{f}(x) + ty_0 \ne x$ for $(x, t) \in \partial V \times \boldsymbol{I}$, by (a) we see that $\widehat{f} \simeq \widehat{g}$ in $\mathscr{K}_{\partial V}(\overline{V}, C)$; consequently, by normalization and homotopy, $i(f, V) = i(\widehat{g}, V) = 1$. Now, since $\operatorname{Fix}(f) \subset V \cup (U - \overline{V})$, our assertion follows from the multiplicity property of the index. □

(7.14) THEOREM. *Let E be an infinite-dimensional normed linear space, V an open bounded subset of E with $0 \in V$, and $U \subset \overline{U} \subset V$ a bounded centrally symmetric neighborhood of the origin in E. Let $f \in \mathscr{K}_{\partial V \cup \partial U}(\overline{V}, E)$ be such that:*
(i) $f(x) = -f(-x)$ *for* $x \in \partial U$,
(ii) *f satisfies on ∂V one of the conditions* (i)–(iii) *of* (7.11).
Then f has at least two fixed points, $x_1 \in U$ and $x_2 \in V - \overline{U}$

PROOF. By (6.6) and (7.11), the assertion follows from the multiplicity property of the index. □

8. Local Index of an Isolated Fixed Point

In this section we define the concept of the local index, which gives information about the behavior of isolated fixed points. Our aim is to establish the basic index formula for calculating the local index for compact differentiable maps in a Banach space.

(8.1) DEFINITION. Let X be an ANR, $U \subset X$ open, and $x_0 \in U$ an isolated fixed point of $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$. The *local index* of x_0 for f is defined by

$$J(f, x_0) = i(f, B_\delta),$$

where $B_\delta = B(x_0, \delta) \subset U$ is an open ball such that $\operatorname{Fix}(f|B_\delta) = \{x_0\}$.

By the excision property, $J(f, x_0)$ does not depend on the choice of δ.

As an immediate consequence of additivity we obtain

(8.2) THEOREM. *If $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ has only finitely many fixed points $x_1, \dots, x_k \in U$, then*

$$i(f, U) = \sum_{i=1}^{k} J(f, x_i). \qquad \square$$

We now calculate the local index for some simple maps.

EXAMPLE 1. If $X = \boldsymbol{R}$, $f : \boldsymbol{R} \to \boldsymbol{R}$ is continuous and $U = (a, b) \subset \boldsymbol{R}$, then

$$i(f, (a, b)) = \frac{1}{2}\operatorname{sgn}[b - f(b)] - \frac{1}{2}\operatorname{sgn}[a - f(a)],$$

provided that $f(a) \neq a$ and $f(b) \neq b$. In particular, if f is differentiable at $y_0 = f(y_0)$ and $f'(y_0) \neq 1$, then

$$J(f, y_0) = \operatorname{sgn} \frac{d}{dt}[t - f(t)]\bigg|_{t=y_0}.$$

EXAMPLE 2. If $T : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a linear map such that 1 is not an eigenvalue of T, then

$$J(T, 0) = \operatorname{sgn}\det(I - T) = (-1)^\beta,$$

where $\beta = \operatorname{card}\{i \mid \lambda_i > 1\}$ and $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of T. Indeed, since $I - T$ is invertible, this follows at once by applying (10.7.3) to $I - T$.

EXAMPLE 3. Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be differentiable at 0 with $f(0) = 0$ and assume that 1 is not an eigenvalue of $T = f'(0) \in \mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^n)$. We show

that 0 is an isolated fixed point for f and

$$J(f,0) = J(T,0).$$

Indeed, since $I-T$ is invertible, we have $\|x-Tx\| \geq 2\alpha\|x\|$ for all $x \in \boldsymbol{R}^n$ and a suitable $\alpha > 0$. Choose $B(0,\varepsilon) = B_\varepsilon$ so small that $\|f(x)-Tx\| \leq \alpha\|x\|$ on $\overline{B}_\varepsilon$. Then on $\overline{B}_\varepsilon$ we have $\|x-f(x)\| \geq \|x-Tx\| - \|f(x)-Tx\| \geq \alpha\|x\|$, showing that 0 is an isolated fixed point for f Moreover,

$$\|f(x)-Tx\| \leq \|x-Tx\| \quad \text{on } \partial B_\varepsilon,$$

so Rouché's theorem (10.8.3) gives

$$J(f,0) = i(f,B_\varepsilon) = i(T,B_\varepsilon) = J(T,0).$$

In the remaining part of this section our aim is to extend the index formula of Example 3 to infinite-dimensional Banach spaces and calculate the local index for compact differentiable maps.

Let E be a Banach space, $T \in \mathscr{K}(E,E)$ a completely continuous linear operator, and $r(T) = \{\nu \in \boldsymbol{R} \mid \operatorname{Ker}(I-\nu T) \neq 0\}$ the set of characteristic values of T By the Riesz–Schauder theory (cf. Appendix), the following facts are known:

(i) Every closed interval $[a,b] \subset \boldsymbol{R}$ contains only a finite number of characteristic values of T.

(ii) For any $\nu \in r(T)$ with $\nu \neq 1$, there exists an $s \in \boldsymbol{N}$ such that

$$\mathcal{N}^\nu = \operatorname{Ker}(I-\nu T)^s = \bigcup_{k=1}^{\infty} \operatorname{Ker}(I-\nu T)^k$$

(called the *generalized kernel* of $I-\nu T$) is of finite dimension; the integer $m_\nu = \dim \mathcal{N}^\nu$ is called the *algebraic multiplicity* of ν.

(iii) We have the splitting

$$E = \mathcal{N}^\nu \oplus \mathcal{R}^\nu,$$

where $\mathcal{R}^\nu = \operatorname{Im}(I-\nu T)^s = \bigcap_{k=1}^{\infty} \operatorname{Im}(I-\nu T)^k$ is a closed subspace of E.

(iv) We have the inclusions

$$T(\mathcal{N}^\nu) \subset \mathcal{N}^\nu \quad \text{with} \quad r(T|\mathcal{N}^\nu) = \{\nu\},$$
$$T(\mathcal{R}^\nu) \subset \mathcal{R}^\nu \quad \text{with} \quad \nu \notin r(T|\mathcal{R}^\nu).$$

(v) If $\mu,\nu \in r(T)$ and $\mu \neq \nu$, then $\mathcal{N}^\nu \subset \mathcal{R}^\mu$

(8.3) THEOREM. *Let E be a Banach space, and $T \in \mathscr{K}(E,E)$ a linear completely continuous map such that $1 \notin r(T)$. Then*

$$J(T,0) = (-1)^\beta,$$

where β is the sum of the algebraic multiplicities of the characteristic values of T lying in $(0,1)$.

PROOF. We proceed in a few steps.

(a) For the characteristic values $\nu_1, \ldots, \nu_k \in r(T) \cap (0,1)$, take the corresponding finite-dimensional spaces $\mathcal{N}^{\nu_i}$, and let

$$\mathcal{N} = \mathcal{N}^{\nu_1} \oplus \cdots \oplus \mathcal{N}^{\nu_k}$$

with $\dim \mathcal{N} = \sum_{j=1}^{k} m_j = \beta$, where $m_j = \dim \mathcal{N}^{\nu_j}$ is the algebraic multiplicity of ν_j.

Now, because for each $j \in [k]$, $T(\mathcal{N}^{\nu_j}) \subset \mathcal{N}^{\nu_j}$ and $r(T|\mathcal{N}^{\nu_j}) = \{\nu_j\}$, it follows that $T(\mathcal{N}) \subset \mathcal{N}$ and $r(T|\mathcal{N}) = \{\nu_1, \ldots, \nu_k\}$. Since $\{1/\nu_j\}_{j=1}^{k}$ are the eigenvalues of $T|\mathcal{N}$ with $1/\nu_j > 1$, the finite-dimensional formula of Example 2 applied to the nonsingular linear map $(I-T)|\mathcal{N}$ gives

$$(*) \qquad J(T|\mathcal{N}, 0) = d((I-T)|\mathcal{N}, B \cap \mathcal{N}) = \operatorname{sgn}\det((I-T)|\mathcal{N}) = (-1)^{\beta}.$$

where $\beta = \sum_{j=1}^{k} m_j$ and B is the open unit ball in E.

(b) For any $j \in [k]$, take the linear projection $Q_j : \mathcal{N}^{\nu_j} \oplus \mathcal{R}^{\nu_j} \to \mathcal{N}^{\nu_j}$ of E onto $\mathcal{N}^{\nu_j}$ and the linear projection $Q : E \to \mathcal{N}$ defined by $Q = \sum_{j=1}^{k} Q_j$; because each Q_j commutes with T, it follows that so also does Q. Now from (v) it follows that the linear map $P = I - Q$ maps E onto the closed subspace $\mathcal{R} = \bigcap_{j=1}^{k} \mathcal{R}^{\nu_j}$ of E, which is invariant under T and satisfies

$$E = \mathcal{N} \oplus \mathcal{R} \quad \text{and} \quad [0,1] \cap r(T|\mathcal{R}) = \emptyset.$$

(c) Consider the compact homotopy $H_t : \overline{B \cap \mathcal{R}} \to \mathcal{R}$ given by $H_t(x) = (1-t)Tx$ for $x \in \overline{B \cap \mathcal{R}}$ and joining $T = T|\overline{B \cap \mathcal{R}}$ to the constant map $x \mapsto 0$; because $1 \notin r(H_t|\mathcal{R})$, $\{H_t\}$ is fixed point free on $\partial(B \cap \mathcal{R})$, and therefore, by homotopy and normalization,

$$(**) \qquad i(T, B \cap \mathcal{R}) = 1.$$

Now, by multiplicativity,

$$i(T, B) = i(T|\mathcal{N}, B \cap \mathcal{N}) \cdot i(T|\mathcal{R}, B \cap \mathcal{R}),$$

and thus in view of $(*)$ and $(**)$ the desired conclusion follows. □

(8.4) COROLLARY. *Let $T \in \mathscr{K}(E, E)$ be a linear completely continuous map, and ν a characteristic value of T with algebraic multiplicity m_ν. Then*

$$J((\nu - \delta)T, 0) = (-1)^{m_\nu} J((\nu + \delta)T, 0)$$

for all sufficiently small $\delta > 0$. □

We now establish the main result of this section.

(8.5) THEOREM (Leray–Schauder formula). *Let E be a Banach space, $U \subset E$ open, and $F : \overline{U} \to E$ a compact map with $F(x_0) = x_0 \in U$.*

Assume that F is differentiable at x_0 and that 1 is not an eigenvalue of $F'(x_0) \in \mathscr{K}(E,E)$. Then:
(a) *x_0 is an isolated fixed point for F and $J(F,x_0) = J(F'(x_0),0)$,*
(b) *$J(F,x_0) = (-1)^\beta$, where β is the sum of the algebraic multiplicities of the characteristic values of $F'(x_0)$ lying in $(0,1)$.*

PROOF. The proof of (a) is strictly analogous to that of the index formula in Example 3 and is left to the reader; (b) follows from (a) and (8.2). □

9. Miscellaneous Results and Examples

A. Essential fixed points

Let (X,d) be a metric space and $f : X \to X$ a compact map. A fixed point x_0 of f is called *essential* (or *stable*) if for each ball $B(x_0,\delta)$ in X there exists an $\varepsilon > 0$ such that any compact map $g : X \to X$ with $d(f(x),g(x)) < \varepsilon$ for $x \in X$ has a fixed point in $B(x_0,\delta)$.

(A.1) Let (X,d) be an ANR and $f : X \to X$ a compact map. Prove: If x_0 is an isolated fixed point for f and $J(f,x_0) \neq 0$, then x_0 is an essential fixed point.

(A.2) Let (X,d) be an AR, and $(\mathscr{K}(X,X),\varrho)$ the space of compact maps $f : X \to X$ with the sup metric. Let $\Gamma : \mathscr{K}(X,X) \to 2^X$ be given by $\Gamma(f) = \operatorname{Fix}(f)$, where 2^X is the space of nonempty compacta in X with the Hausdorff metric. Prove: Γ is upper semicontinuous.

[Given $U \subset X$ open and $f_0 \in \mathscr{K}(X,X)$ with $\Gamma(f_0) \subset U$, observe that $d(x,f_0(x)) \geq \alpha > 0$ for $x \notin U$. Then, given any g with $\varrho(g,f_0) < \alpha/2$, we have $\Gamma(g) \subset U$: for if $d(x,g(x)) = 0$ for some $x \notin U$, then $d(x,f_0(x)) \leq d(x,g(x)) + d(g(x),f_0(x)) \leq \alpha/2$, which is a contradiction.]

(A.3) Let (X,d), $(\mathscr{K}(X,X),\varrho)$ and Γ be as in (A.2), and let $f \in \mathscr{K}(X,X)$. Prove:
(a) If every fixed point of f is essential, then Γ is lower semicontinuous at f.
(b) If Γ is lower semicontinuous at f, then every fixed point of f is essential.
(c) Every fixed point of f is essential if and only if $\Gamma : \mathscr{K}(X,X) \to 2^X$ is continuous at f.

[For (a), to show that Γ is l.s.c at f, let $U \subset X$ be open with $\Gamma(f) \cap U \neq \emptyset$; choose $x \in \Gamma(f) \cap U$ and find an open nbd $V(f)$ of f such that $g \in V(f) \Rightarrow \operatorname{Fix}(g) \cap U \neq \emptyset$. For (c), show that given $\varepsilon > 0$, there is a nbd $V(f)$ of f such that $\operatorname{Fix}(g) \cap B(x,2\varepsilon) \neq \emptyset$ for all $g \in V(f)$ and $x \in \operatorname{Fix}(f)$: cover $\Gamma(f)$ by finitely many ε-balls $B(x_i,\varepsilon)$: then for each $x_i \in \operatorname{Fix}(f)$ find $V_i(f)$ from l.s.c., and take $V(f) = \bigcap V_i(f)$. Use (a), (b), and (A.2).]

(A.4) Let (X,d) and $\mathscr{K}(X,X)$ be as in (A.2), and let $\mathcal{E}_X = \{f \in \mathscr{K}(X,X) \mid \text{every } x \in \operatorname{Fix}(f) \text{ is essential}\}$. Prove: If the space X is complete, then $\mathcal{E}_X$ is dense in $(\mathscr{K}(X,X),\varrho)$.

[Show that $(\mathscr{K}(X,X),\varrho)$ is complete and then use the following theorem of Kuratowski–Fort: *If X is complete and $\Gamma : \mathscr{K}(X,X) \to 2^X$ is semicontinuous, then the points of discontinuity of Γ form a set of the first category* (cf. Fort [1951], where a reference to an earlier result of Kuratowski (for separable spaces) is given).]

(A.5) Let X be an ANR and $f : X \to X$ a compact map with $\operatorname{Fix}(f) \neq \emptyset$. A component C of $\operatorname{Fix}(f)$ is called *essential* if for each nbd U of C there exists a nbd $V(f)$ of f (in $(\mathscr{K}(X,X),\varrho)$) such that if $g \in V(f)$, then g has a fixed point in U. Prove: If the fixed

point set of a compact map $f : X \to X$ has finitely many components and $i(f, X) \neq 0$, then f has at least one essential fixed point component.

[Letting $\{C_j\}$ be the components of Fix(f), find finitely many nonintersecting open sets U_j with $C_j \subset U_j$ for each j; using additivity, find a j with $i(f, U_j) \neq 0$ and show that the component C_j is essential.]

B. Fixed points in wedges and cones

Throughout this subsection, E is a normed linear space and W is a wedge in E; for $\varrho > 0$, we let $B_\varrho = \{x \in W \mid \|x\| < \varrho\}$, $S_\varrho = \partial B_\varrho$ and $K_\varrho = S_\varrho \cup B_\varrho$.

(B.1) Let $f \in \mathscr{K}_{S_\varrho}(K_\varrho, W)$, and let $p : E \to \boldsymbol{R}^+$ be any (not necessarily continuous) function such that $p^{-1}(0) = \{0\}$ and $p(\lambda x) = \lambda p(x)$ for all $\lambda > 0$ and $x \in E$. Assume that one of the following conditions holds:

(i) $p(f(x)) \leq \max\{p(x), p(f(x) - x)\}$ for $x \in S_\varrho$.

(ii) $p(f(x)) \leq \{(p(x))^k + (p(f(x) - x))^k\}^{1/k}$ for $x \in S_\varrho$, where $k > 1$.

Show: $i(f, B_\varrho) = 1$.

(B.2) (*Krasnosel'skii theorem*) Let C be a cone in E, and let $f : C \to C$ be completely continuous. Assume that for some numbers r and R with $0 < r < R$, one of the following conditions is satisfied:

(a) $x - f(x) \notin C$ for $x \in S_r$ and $f(x) - x \notin C$ for $x \in S_R$.

(b) $f(x) - x \notin C$ for $x \in S_r$ and $x - f(x) \notin C$ for $x \in S_R$.

Prove: f has a fixed point x with $r \leq \|x\| \leq R$ (Krasnosel'skii) [1960].

(B.3) Let $W \subset E$ be a proper wedge, and let $f : W \to W$ be completely continuous. Assume that for some numbers r and R with $0 < r < R$, one of the following conditions is satisfied:

(a) $\|f(x)\| \leq \|f(x) - x\|$ for $x \in S_r$ and $\|f(x)\| \geq \|x\|$ for $x \in S_R$.

(b) $\|f(x)\| \geq \|x\|$ for $x \in S_r$ and $\|f(x)\| \leq (\|x\|^2 + \|x - f(x)\|^2)^{1/2}$ for $x \in S_R$.

Prove: f has a fixed point x with $r \leq \|x\| \leq R$.

(B.4) Let $X \subset E$ be a closed ANR, $U \subset X \times [a, b]$ open with vertical boundary $\widehat{\partial} U$, and $f : \overline{U} \to X$ a compact map. Let $S = \{(x, \lambda) \in \overline{U} \mid f(x, \lambda) = x\}$. Prove: If $S_a \cap (\widehat{\partial} U)_a = \emptyset$ and $i(f_a, U_a) \neq 0$, then there exists a continuum $C \subset S$ joining $S_a \times \{a\}$ and $\widehat{\partial} U \cup S_b \times \{b\}$.

(B.5) Let $W \subset E$ be a closed wedge, $U \subset W \times [a, b]$ open, and $f : \overline{U} \to W$ completely continuous. Let $S = \{(x, \lambda) \in \overline{U} \mid f(x, \lambda) = x\}$ and assume that S_a is bounded and $S_a \cap (\widehat{\partial} U)_a = \emptyset$. Prove: If $i(f_a, U_a) \neq 0$, then there exists a component $C \subset S$ with $C_a \neq \emptyset$ such that either C is unbounded or $C \cap (\widehat{\partial} U \cup S_b \times \{b\}) \neq \emptyset$.

(B.6) Let $W \subset E$ be a closed wedge, and let $h : W \times \boldsymbol{R}^+ \to W$ be a completely continuous map such that $h(x, 0) = 0$ for all $x \in W$. Let $C_+(W)$ be the component of $S = \{(x, \lambda) \in W \times \boldsymbol{R}^+ \mid h(x, \lambda) = x\}$ containing $(0, 0)$. Show: $C_+(W)$ is unbounded.

C. Geometric approach to the index for ENRs

Throughout this subsection, we deal with ENRs without isolated points. We recall that ENRs are (up to homeomorphism) neighborhood retracts of Euclidean spaces.

Consider compactly fixed maps $f : U \to X$, where U is open in X and X is an ENR. Let $\mathscr{F}$ be the set of (homeomorphism classes) of such maps. In $\mathscr{F}$ we introduce the following "homotopy" and "excision" relations:

(HTP) Let $f_0, f_1 \in \mathscr{F}$; then

$$f_0 \overset{\text{HTP}}{\sim} f_1 \Leftrightarrow \begin{cases} \text{there exists a compactly fixed homotopy} \\ h_t : U \to X \text{ such that } h_0 = f_0,\ h_1 = f_1. \end{cases}$$

(EXC) Let $f : U \to X$ and $f_0 : U_0 \to X_0$ be in $\mathscr{F}$; then

$$f \overset{\text{EXC}}{\sim} f_0 \text{ (or } f_0 \overset{\text{EXC}}{\sim} f) \Leftrightarrow \begin{cases} X_0 \subset X \text{ and } U_0 \text{ is open in } U, \\ \operatorname{Fix}(f_0) = \operatorname{Fix}(f), \\ f_0(x) = f(x) \text{ for } x \in U_0. \end{cases}$$

These two relations generate an equivalence relation $\sim$ in $\mathscr{F}$; the equivalence class of f is denoted by $[f]$, and the set of equivalence classes by $\mathrm{FIX} = \mathscr{F}/\sim$.

(C.1) Let $g : W \to Y$ be in $\mathscr{F}$, and let $p \in \operatorname{Fix}(g)$. Show:
(a) If $\varphi : I \to Y$ is a path joining $\varphi(0) = p$ to $\varphi(1) = q$, then the map $h : W \times \{0\} \cup \{p\} \times I \to Y$ given by

$$h(x,t) = \begin{cases} g(x), & (x,t) \in W \times \{0\}, \\ \varphi(t), & (x,t) \in \{p\} \times I, \end{cases}$$

extends to a map $\widehat{h} : W \times I \to Y$.
(b) If $V \subset W$ is an open nbd of $\operatorname{Fix}(g)$, then there exists $g_p : W \to Y$ in $\mathscr{F}$ satisfying:
(i) $g \overset{\text{HTP}}{\sim} g_p$,
(ii) $g_p|W - V = g|W - V$,
(iii) $g_p(p) \neq p$.

[For (a), use the homotopy extension property. For (b), take $\widehat{h}$ as in (a); then, choosing $\lambda : W \to I$ with $\lambda(p) = 1$, $\lambda|W - V = 0$, define $H : W \times I \to Y$ by $H(x,t) = \widehat{h}(x, t\lambda(x))$, and let $g_p(x) = H(x,1)$.]

(C.2) Let $g : W \to Y$ be in $\mathscr{F}$, and let V be an open nbd of $\operatorname{Fix}(g)$. Show:
(a) There exist $g_0, g_1, \dots, g_s : W \to Y$ in $\mathscr{F}$ such that for each $i = 0, 1, \dots, s$,
(i) $g_i \overset{\text{HTP}}{\sim} g$,
(ii) $g_i|W - V = g|W - V$,
(iii) $\bigcap_{i=0}^{s} \operatorname{Fix}(g_i) = \emptyset$.
(b) There exist Urysohn functions $\varrho_0, \varrho_1, \dots, \varrho_s : Y \to I$ such that $\varrho_i|\operatorname{Fix}(g_i) = 1$ and $\varrho_i|Y - W = 0$ for each $i = 0, 1, \dots, s$, and $\min_{0 \le i \le s} \varrho_i = 0$.

[(a): Proceeding for each $p \in \operatorname{Fix}(g)$ as in (C.1), obtain $[\bigcap_{p \in \operatorname{Fix}(g)} \operatorname{Fix}(g_p)] \cap \operatorname{Fix}(g) = \emptyset$; from this, by compactness of $\operatorname{Fix}(g)$, get $[\bigcap_{i=1}^{s} \operatorname{Fix}(g_{p_i})] \cap \operatorname{Fix}(g) = \emptyset$ for some $p_1, \dots, p_s$; letting $g_i = g_{p_i}$ for $i > 0$, take $g_0 = g$ to complete the proof.

(b): For $k = 0, 1, \dots, s$, choose $\lambda_k : Y \to I$ with $\operatorname{Fix}(g_k) = \lambda_k^{-1}(0)$, $\lambda_k|Y - W = 1$, and define $\varrho_k(y) = 1 - \lambda_k(y)/\max_{0 \le i \le s} \lambda_i(y)$.]

(C.3)* Let $g : W \to Y$ be in $\mathscr{F}$, and X a retract of Y; form the partial cylinder $M = Y \times \{0\} \cup X \times I$ with $r : M \to Y$, $j : Y \to M$ given by $r(y,t) = y$, $j(y) = (y,0)$, so $rj = \mathrm{id}_Y$, and consider a map in $\mathscr{F}$ given by $h = jgr : r^{-1}(W) \to M$. Show: h and g are equivalent in $\mathscr{F}$

[Let $M = M_0 \supset M_1 \supset \cdots \supset M_s \supset M_{s+1} = Y \times \{0\}$ be $s+2$ subsets of M given for $k = 1, \dots, s+1$ by

$$M_k = \{(y,t) \in M \mid t \le \min\{\varrho_0(y), \dots, \varrho_{k-1}(y)\}\}.$$

where $\varrho_0, \varrho_1, \ldots, \varrho_s : Y \to I$ are from (C.2); observe that $(y,t) \mapsto (y, \min\{t, \varrho_0(y), \ldots \ldots, \varrho_{k-1}(y)\})$ determines a retraction of M onto M_k, and this implies (because M is an ENR) that each M_k is an ENR. For each $k = 0, 1, \ldots, s+1$, let

$$h_k = jgr : M_k \cap r^{-1}(W) \to M_k, \qquad h_k(y,0) = (g(y),0),$$

and observe that $h_{s+1} = g$. Prove inductively that $h \sim h_k$ for each k:

(i) Observe first that $h_0 = h$; then assuming $h \sim h_k$, take the maps $\{g_k\}_{k=0}^{s}$ from (C.2), and note that $h_k = jgr \overset{\text{HTP}}{\sim} jg_kr$ (since $g \overset{\text{HTP}}{\sim} g_k$ by (C.2)).

(ii) Consider the restriction $jg_kr|M_{k+1} : M_{k+1} \cap r^{-1}(W) \to M_{k+1}$ and observe that $\text{Fix}(jg_kr) \subset \{(y,t) \in M_k \mid t = 0\}$; then use excision to remove all $(y,t) \in M_k$ with $t > \varrho_k(y)$, giving $jg_kr \overset{\text{EXC}}{\sim} jg_kr|M_{k+1}$.

(iii) From (ii), since $jg_kr|M_{k+1} \overset{\text{HTP}}{\sim} jgr|M_{k+1} = h_{k+1}$, deduce $h \sim h_{k+1}$, completing the induction and giving $h \sim h_{s+1} = g$.]

(C.4)* Let Y be an ENR, and X a retract of Y with $X \xrightarrow{i} Y \xrightarrow{\varrho} X$, $\varrho i = \text{id}_X$. Show: If $f : V \to X$ is in $\mathscr{F}$, then so is $g = if\varrho : \varrho^{-1}(V) \to Y$, and f and g are equivalent in $\mathscr{F}$

[(i) As in (C.3), form the partial cylinder $M = X \times I \cup Y \times \{0\}$ and consider the map $h = jgr = j(if\varrho)r : r^{-1}\varrho^{-1}(V) \to M$; note that $h \sim g$ as in (C.3).

(ii) Consider the homotopy $k_t f\varrho r : r^{-1}\varrho^{-1}(V) \to M$, where $\{k_t : X \to M\}_{t\in I}$ is given by $x \mapsto (x,t)$; observe that $h = j(if\varrho)r = k_0 f\varrho r \overset{\text{HTP}}{\sim} k_1 f\varrho r$.

(iii) Since $\text{Fix}(k_1 f\varrho r) \subset X \times \{1\}$, excise all (y,t) with $t < \frac{1}{2}$, so that $X \times [\frac{1}{2}, 1]$ remains, and note that, with $q : V \times [\frac{1}{2}, 1] \to V$ being the projection, the maps $k_1 f\varrho r, k_1 fq : V \times [\frac{1}{2}, 1] \to X \times [\frac{1}{2}, 1]$ are equivalent in $\mathscr{F}$ and $k_1 fq(v,t) = (f(v), 1)$.

(iv) To conclude, use (i)–(iii) to get $g \sim h = k_0 f\varrho r \sim k_1 f\varrho r \sim k_1 fq \sim f$.]

(C.5) Let $\widehat{\mathscr{L}}(\boldsymbol{R}^n) = \{B \in \mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^n) \mid \text{Fix}(B) = \{0\}\}$ and assume A is in $\widehat{\mathscr{L}}(\boldsymbol{R}^n)$. Show:

(a) There is a path in $\widehat{\mathscr{L}}$ joining A to the linear map determined by one of the $n \times n$ matrices $\begin{bmatrix} 1\pm 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) A is equivalent in $\mathscr{F}$ to one of the self-maps of $\boldsymbol{R}$: $x \mapsto 0$, $x \mapsto 2x$.

[For (a): because $I - A \in \text{GL}(\boldsymbol{R}^n)$, there exists a path in $\text{GL}(\boldsymbol{R}^n)$ joining $I - A$ to the linear map determined by one of the matrices

$$\begin{bmatrix} \pm 1 & & 0 \\ & 1 & \\ & & \\ 0 & & 1 \end{bmatrix}$$

For (b): use (C.4) and (a).]

(C.6) Let $f : V \to X$ be any map in $\mathscr{F}$. Show: There exists an open subset U of $\boldsymbol{R}^n$ for some n and a map $g : U \to \boldsymbol{R}^n$ in $\mathscr{F}$ such that f and g are equivalent in $\mathscr{F}$.

[Use (C.4) and the fact that X is (up to homeomorphism) a nbd retract of some $\boldsymbol{R}^n$.]

(C.7) Given $f_0 : U_0 \to X_0$ and $f_1 : U_1 \to X_1$ in $\mathscr{F}$, consider their topological sum $f_0 \oplus f_1 : U_1 \oplus U_2 \to X_0 \oplus X_1$. Show:

(a) The addition $\oplus$ in $\mathscr{F}$ is compatible with $\sim$, and so determines an addition $+$ in FIX: $[f_0] + [f_1] = [f_0 \oplus f_1]$.

(b) The pair (FIX, $+$) is a commutative and associative monoid with zero $[\emptyset]$, where $\emptyset$ is the "empty map"

(C.8)* Let $f \cdot W \to \boldsymbol{R}^n$ be compactly fixed. Prove: f is equivalent in FIX to a finite sum $\bigoplus f_i$, where each $f_i : V_i \to \boldsymbol{R}^n$ is an affine map with $\mathrm{Fix}(f_i) = \{x_i\} \subset V_i$ and such that $x - f_i(x) = L_i(x) - x_i$ for $x \in V_i$ and some $L_i \in \mathrm{GL}(\boldsymbol{R}^n)$

[Assuming without loss of generality that f is defined on $\overline{W}$ and that W is a polyhedral domain, modify slightly the map $x \mapsto x - f(x)$ on $\overline{W}$ to get a PL-generic map $g_\sharp$ from $(W, \partial W)$ to $(\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$; then $\mathrm{id} - g_\sharp$ has isolated fixed points, and $\mathrm{id} - g_\sharp \sim f$. To conclude, apply excision to the map $\mathrm{id} - g_\sharp$.]

(C.9) (*Dold theorem*) Prove:

(a) Each compactly fixed $f : W \to X$ is equivalent to a sum of maps $\boldsymbol{R} \to \boldsymbol{R}$ of the form $x \mapsto 0$ or $x \mapsto 2x$, and these maps add up to 0 in FIX.

(b) The monoid FIX is an infinite cyclic group generated by $e = [\boldsymbol{R} \to \boldsymbol{R}, x \mapsto 0] = -[\boldsymbol{R} \to \boldsymbol{R}, x \mapsto 2x]$.

(c) The projection $f \mapsto [f] \in \mathrm{FIX}$ gives an integer that coincides with the fixed point index $I(f, W)$ of f.

(The above results are due to Dold [1974].)

10. Notes and Comments

Fixed point index for polyhedra and compact ANRs

Fixed point index theory for polyhedra, due to Hopf, is classic and can be found in the treatise of Alexandroff–Hopf. The index for compactly fixed maps of ENRs, due to Dold [1965], is presented in full detail in his book [1972]; more general theory for compact ANRs is presented in R.F. Brown's book [1971]. The construction of the index in the above books is based on tools of algebraic topology.

Fixed point index for compact maps of ANRs

In this monograph, the fixed point index theory for compact maps of ANRs is presented in two parts: the first "geometric" part makes no use of algebraic topology, and the second "algebraic" part described in §16 combines the Lefschetz–Hopf theory with the geometric part of the index.

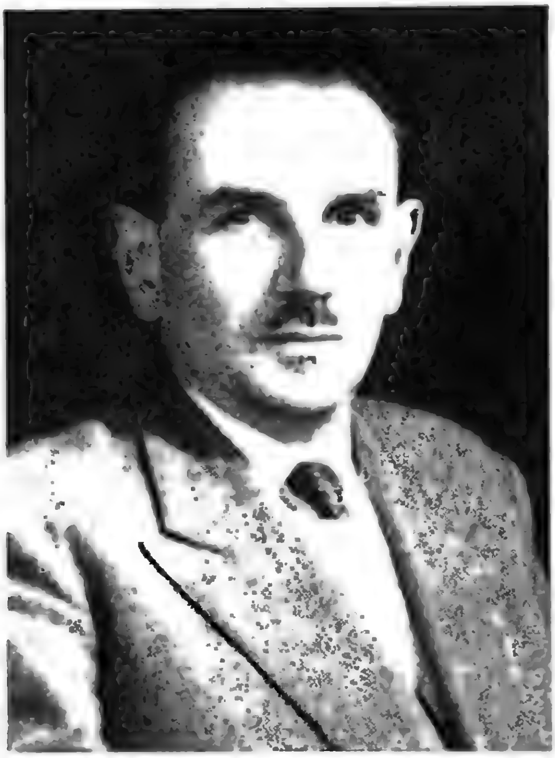
The index theory in a setting similar to that presented in the text was established in Granas [1972], on the basis of Dold's [1965] index for compactly fixed maps in $\boldsymbol{R}^n$ Closely related partial results were obtained earlier by Browder [1969] and Granas [1969].

§12 develops only the "geometric" part of the theory. Later on, in §16, the concept of the index will be discussed from the viewpoint of algebraic topology; specifically (after establishing the Hopf index theorem relating the Lefschetz number to the index), we will show that the "geometric" part of the index can be embodied in a suitably and axiomatically defined "algebraic" part of the theory. We remark that the index theory presented in §12 decomposes into two closely related parts, namely, the "I-index" for compactly fixed maps and the "i-index" for maps that are fixed point free on

the boundary. These parts are so closely analogous that there is no need to treat them in full detail separately; the results of one almost automatically apply to the other.

The presentation of the "I-index" is carried out in the following steps:

1. Construction of the index for compactly fixed maps in $\boldsymbol{R}^n$.
2. Extension to compact and compactly fixed maps in normed linear spaces (the Leray Schauder index).
3. Extension to compact and compactly fixed maps of ANRs, based on the commutativity of the Leray–Schauder index.



Jean Leray, 1953

Steps 2 and 3 closely follow Granas [1972]. Step 2 relies on the degree for compactly rooted maps in $\boldsymbol{R}^n$ (§10). The proof of the commutativity of the index in $\boldsymbol{R}^n$ (Section 1) is due to Dold [1965]; also, the argument in the first part of the proof of the commutativity of the Leray–Schauder index is analogous to the above proof. A special feature of the entire development is that the index for compact ANRs and the index for compact maps of ANRs are both derived as simple corollaries of the Leray–Schauder theory.

We remark that the index theory for compact maps of ANRs appeared first (see the lecture notes of Granas [1969 1970]) in the following somewhat less general setting:

Let $\mathscr{A}$ denote the set of all triples (X, f, U), where X is an ANR, $U \subset X$ open, and $f : X \to X$ a compact map with $\mathrm{Fix}(f|\partial U) = \emptyset$. Then there exists on $\mathscr{A}$ an integer-valued function $(X, f, U) \mapsto i_X(f, U)$ satisfying:

(I) (*Strong normalization*) If $U = X$, then $i_X(f, X) = \Lambda(f)$, where $\Lambda(f)$ is the generalized Lefschetz number of f.

(II) (*Additivity*) If $(X, f, U) \in \mathscr{A}$ and $U_1, U_2 \subset U$ are disjoint, and f has no fixed points in $U - \overline{U_1 \cup U_2}$, then $i_X(f, U) = i_X(f, U_1) + i_X(f, U_2)$.

(III) (*Homotopy*) If $h_t \; X \to X$ is a compact homotopy and $\mathrm{Fix}(h_t|\partial U) = \emptyset$ for $t \in I$, then $i_X(h_0, U) = i_X(h_1, U)$.

(IV) (*Commutativity*) If X, Y are ANRs and $f : X \to Y$ $g : Y \to X$ are maps such that either f or g is compact, then $i_X(gf, U) = i(fg, g^{-1}(U))$.

The proof of this theorem relies on the index theory for compact ANRs and on the fact that ANRs are dominated by polytopes with weak topology.

Kuratowski–Mazurkiewicz separation theorem

Theorem (6.4) is of importance in the fixed point index theory and its applications. It was initially attributed to G.T. Whyburn (*Topological Analysis*, Princeton Univ. Press, 1958; see, for example, Rabinowitz [1971]). Later on, J.C. Alexander [1981] pointed out that the result can be traced back to the early days of point-set topology and is contained in Kuratowski's monograph [1968]. In Chapter V of that book, Kuratowski introduces for a space X the following property:

$$(\mathrm{M}) \quad \begin{cases} \text{If } A, B \subset X \text{ are closed and } X \text{ is connected between } A \text{ and } B, \text{ then} \\ X \text{ is also connected between a pair of points } a \in A \text{ and } b \in B. \end{cases}$$

He attributes the notion to Mazurkiewicz (CRAS Paris 151 (1910), p. 296), and proves that:

(i) If X is compact, then X has property (M).

(ii) If X has property (M), then its components coincide with its quasi-components.

The argument to establish (i) (which is also given in the proof of (6.4)) is strictly similar to that used in the above mentioned note of Mazurkiewicz.

Index for k-set contractions and for condensing maps

The fixed point index has been extended to wider classes of maps. We outline two such extensions: to the classes of k-set contractions and condensing maps.

Let C be a closed convex subset of a Banach space E. Recall that the *Kuratowski measure of noncompactness* of $X \subset C$ is defined by

$$\alpha(X) = \inf\{\delta > 0 \mid X \text{ has a finite cover by sets of diameter} > 0\}.$$

The main properties of α are:

(I) If $\{A_n\}$ is a descending sequence of nonempty closed subsets of C such that $\alpha(A_n) \to 0$, then $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty and compact and each nbd U of A_∞ contains all the A_n with sufficiently large n.

(II) $\alpha(A) = 0$ if and only if A is compact.

(III) $\alpha(\mathrm{Conv}(A)) = \alpha(A)$ for all bounded $A \subset C$.

(IV) $\alpha(A \cup B) \leq \max\{\alpha(A), \alpha(B)\}$ for all bounded $A, B \subset C$.

DEFINITION. Let $X, Y \subset C$, $f : X \to Y$ be continuous and $k \in (0, 1)$.

(a) f is a *k-set contraction* if $\alpha(f(A)) \leq k\alpha(A)$ for all bounded $A \subset X$.

(b) f is *condensing* if $\alpha(f(A)) < \alpha(A)$ for all $A \subset X$ with $\alpha(A) \neq 0$.

We now describe the definition of the fixed point index for k-set contractions. The key result here is the following:

LEMMA. *Let U be a bounded open subset of C and $\Phi : \overline{U} \to C$ be a k-set contraction. Then:*

(i) *there exists a compact convex $C_\infty \subset C$ such that $\Phi(\overline{U} \cap C_\infty) \subset C_\infty$.*

(ii) *there exists a compact map $F : \overline{U} \to C_\infty$ extending the restriction $\Phi|\overline{U} : \overline{U} \cap C_\infty \to C_\infty$ and satisfying* $\mathrm{Fix}(F) = \mathrm{Fix}(\Phi)$; *any two such extensions are homotopic via a compact homotopy $H_t : \overline{U} \to C_\infty$ such that* $\mathrm{Fix}(H_t) = \mathrm{Fix}(\Phi)$ *for each $t \in \boldsymbol{I}$.*

PROOF. Define inductively a descending sequence $C_1 \supset C_2 \supset \cdots$ of closed convex sets by setting

$$C_1 = \mathrm{Conv}(\Phi(\overline{U})), \qquad C_n = \mathrm{Conv}(\Phi(\overline{U} \cap C_{n-1})).$$

Letting $C_\infty = \bigcap_{n=1}^{\infty} C_n$ and using $\Phi(\overline{U} \cap C_{n-1}) \subset C_n$ gives $\Phi(\overline{U} \cap C_\infty) \subset C_\infty$. Because Φ is k-set contractive, $\alpha(C_n) \leq k\alpha(C_{n-1}) \leq k^{n-1}\alpha(C_1)$, and as $k \in (0, 1)$, (I) shows that C_∞ is compact.

Now, $\Phi : \overline{U} \cap C_\infty \to C_\infty$ extends to $F : \overline{U} \to C_\infty$, since C_∞ is convex, and F is compact, since C_∞ is. If $x = F(x)$ for some $x \in \overline{U}$, then $x \in \overline{U} \cap C_\infty$, so $x = F(x) = \Phi(x)$, and thus $\mathrm{Fix}(F) = \mathrm{Fix}(\Phi)$. Now assume that $G : \overline{U} \to C_\infty$ were another such extension and define a compact homotopy $H_t : \overline{U} \to C_\infty$ by $H_t(x) = (1-t)F(x) + tG(x)$; if $x = H_t(x)$ for some $x \in \overline{U}$ and some $t \in \boldsymbol{I}$, then $x \in \overline{U} \cap C_\infty$, and therefore $x = F(x) = \Phi(x)$, i.e., $\mathrm{Fix}(H_t) = \mathrm{Fix}(\Phi)$ for all $t \in \boldsymbol{I}$. □

The Leray-Schauder fixed point indices $i(F, U)$, $i(G, U)$ are the same, and we define $i(\Phi, U) = i(F, U)$.

It can be shown that $i(\Phi, U)$ satisfies the main axioms for the index on the set of k-set contractive maps. In particular, it is also unique.

Let now $F : \overline{U} \to C$ be a condensing map with $\mathrm{Fix}(F) \subset U$. Take $0 < \delta < \delta_0 = \inf\{\|x - F(x)\| \mid x \in \partial U\}$ ($\delta_0 > 0$ because $x \mapsto x - F(x)$ is a closed map); select any $x_0 \in U$ and set $F_t(x) = (1-t)x_0 + tF(x)$ for $x \in \overline{U}$ and

$t \in I$. Now choosing t_0 sufficiently close to 1 so that $\|F_{t_0}(x) - F(x)\| < \delta$ for all $x \in \partial U$, note that $F_{t_0} : \overline{U} \to C$ is a k-set contraction with $\mathrm{Fix}(F_{t_0}) \subset U$.

We define $i(F, U) = i(F_{t_0}, U)$. It can be verified that the definition is independent of the particular t_0, x_0 chosen and that $i(F, U)$ satisfies the main axioms for the index on the set of condensing maps that are fixed point free on ∂U. In particular, it is also unique.

For details and more general results about the index for condensing maps (and the related notion of degree) see Nussbaum [1971], [1972a,b], [1977], Borisovich–Sapronov [1968], and also Nussbaum's lecture notes [1985] and Krawcewicz's book [1997], where further references can be found.

Index for compact Kakutani maps

The index can also be extended to compact Kakutani maps. Let C be a closed convex set in a normed linear space E, and let $U \subset C$ be open in C. As before (cf. (11.7.5)), we denote by $\mathscr{K}_{\partial U}(\overline{U}, C)$ the set of all compact maps $f : \overline{U} \to C$ that are fixed point free on ∂U; recall that $\mathscr{K}_{\partial U}(\overline{U}, C)$ is equipped with the relation $\simeq$ of homotopy, and we let $\pi_{\partial U}(\overline{U}, C)$ denote the corresponding set of homotopy classes.

Let $\mathscr{K}_{\partial U}(\overline{U}, 2^C)$ be the set of all compact Kakutani maps $S : \overline{U} \to 2^C$ that are fixed point free on ∂U. Recall that two Kakutani maps $S, T \in \mathscr{K}_{\partial U}(\overline{U}, 2^C)$ are called homotopic (written $S \sim T$) if there exists a map $H \in \mathscr{K}_{\partial U \times I}(\overline{U} \times I, 2^C)$ with $H(x, 0) = Sx$ and $H(x, 1) = Tx$ for all $x \in \overline{U}$. The homotopy relation $\sim$ is an equivalence relation in $\mathscr{K}_{\partial U}(\overline{U}, 2^C)$. We denote by $[\![S]\!]$ the equivalence class of S in $\mathscr{K}_{\partial U}(\overline{U}, 2^C)$ and by $\pi_{\partial U}(\overline{U}, 2^C)$ the corresponding set of homotopy classes.

The extension of the index to compact Kakutani maps relies on the following theorem:

THEOREM. *Let U be an open subset of C. Then the map $\tau : \pi_{\partial U}(\overline{U}, C) \to \pi_{\partial U}(\overline{U}, 2^C)$ defined by $[f] \mapsto [\![f]\!]$ is a bijection.*

The proof of this theorem can be carried out with the aid of the approximation results (7.8.2) and (7.8.3). Starting off with $S \in \mathscr{K}_{\partial U}(\overline{U}, 2^C)$, we select a map $f \in \mathscr{K}_{\partial U}(\overline{U}, C)$ satisfying $[\![f]\!] = [\![S]\!]$ and define

$$\mathrm{ind}(S, U) = i(f, U).$$

By the above theorem, this is independent of the choice of f and turns out to satisfy analogues of the index axioms; again, they uniquely characterize this index.

For details about the index for compact Kakutani maps (and the related notion of degree) see Granas [1959], Hukuhara [1967], Cellina–Lasota [1969], Borisovich et al. [1969], Ma [1972], and Gęba et al. [1959].

§13. Further Results and Applications

This paragraph is devoted entirely to illustrating various uses of the fixed point index. We begin with two direct and significant applications of the index, first to establish some general bifurcation results in ANRs, and second to nonlinear partial differential equations. Then, among other applications, we derive the Leray Schauder degree theory for compact fields in normed linear spaces, and using the degree, we establish some generalizations of the Borsuk theorem. The last section develops the Leray–Schauder index for compact maps in locally convex spaces.

1. Bifurcation Results in ANRs

Numerous questions, ranging from nonlinear Sturm Liouville problems in ordinary differential equations to eigenvalue problems for elliptic partial differential equations, reduce in a natural way to the study of solutions of equations of the form

$$(*) \qquad x = F(x, \lambda),$$

where x is an element of a Banach space E, λ is a real parameter, and $F : E \times \boldsymbol{R} \to E$ is a completely continuous map with $F(0, \lambda) = 0$; note that $(*)$ has the line $\{(0, \lambda) \mid \lambda \in \boldsymbol{R}\}$ of solutions, which are called the *trivial solutions* of the problem. The solutions of interest are those in the set $\mathcal{N} = \{(x, \lambda) \mid x = F(x, \lambda) \text{ and } x \neq 0\}$; a point $(0, \lambda_0)$ is called a *bifurcation point* of the problem if it belongs to the closure of $\mathcal{N}$ in $E \times \boldsymbol{R}$. In general terms, bifurcation results are concerned with changes in the structure of the sets of solutions of $x = F(x, \lambda)$ as the parameter λ varies.

In the first part of this section our aim is to present the geometric part of the theory, leaving analytic aspects aside; in the framework of ANRs, with the Leray-Schauder index as a basic tool, some local and global bifurcation results are developed. The section ends with applications to bifurcation results in Banach spaces.

We begin with some notation and terminology.

Let X be a space, $J \subset \boldsymbol{R}$ an interval, and let $F : X \times J \to X$ be a map. For each $t \in J$ we let $F_t : X \to X$ be given by $F_t(x) = F(x, t)$; we shall repeatedly use the fact that F determines the family $\{F_t : X \to X\}_{t \in J}$ of maps and vice versa. Throughout the entire section, by $\{(X, d); p_0\}$ we denote a noncompact based ANR with a "base point" $p_0 \in X$, and by $J = (\alpha, \beta)$ a fixed open interval in $\boldsymbol{R}$. For $r > 0$, we let $B_r = B(p_0, r)$ and $K_r = K(p_0, r)$ be open and closed balls in X, respectively. In the product $X \times J$ we introduce a metric ϱ by

$$\varrho((x, s), (y, t)) = d(x, y) + |s - t|;$$

if $U \subset X \times J$ and $t \in J$, then the *t-slice* of U, written U_t, is defined by $U_t = \{x \in X \mid (x,t) \in U\}$.

(1.1) DEFINITION. Let $\{(X,d);p_0\}$ be a based ANR, $J = (\alpha, \beta) \subset \boldsymbol{R}$ open, and let $F: X \times J \to X$ be a completely continuous map.

(a) A subset $\Lambda \subset J$ is called a *singular set associated with* F if:

(i) Λ is finite or countable,

(ii) for any closed $J_0 = [a,b] \subset J$, the intersection $\Lambda \cap J_0$ is finite,

(iii) for any closed $J_0 = [a,b] \subset J - \Lambda$, there is a ball $B_\varepsilon = B(p_0, \varepsilon)$, with a sufficiently small radius $\varepsilon = \varepsilon(J_0) > 0$, for which

$$\operatorname{Fix}(F_t) \cap B_\varepsilon = \operatorname{Fix}(F_t|B_\varepsilon) = \{p_0\} \quad \text{for all } t \in J_0.$$

(b) The map F (or, equivalently, the family $\{F_t : X \to X\}_{t \in J}$ determined by F) is said to be *allowable* if it is equipped with a singular set $\Lambda = \Lambda_F$.

From now on, we assume that $((X,d);p_0)$ is a based ANR and that $F: X \times J \to X$ is a completely continuous allowable map with singular set Λ_F; we call elements of Λ_F *singular points* for F.

(1.2) PROPOSITION. *Let $\lambda_0 \in \Lambda_F$, and let $J_0 = (a,b) \subset J$ be such that $\Lambda_F \cap J_0 = \{\lambda_0\}$. Then:*

(i) *for each $t \in J_0 - \{\lambda_0\}$, there exists an $r(t) > 0$ such that* $\operatorname{Fix}(F_t|B_{r(t)}) = \{p_0\}$,

(ii) *$i(F_t, B_{r(t)})$ is constant for $a < t < \lambda_0$,*

(iii) *$i(F_t, B_{r(t)})$ is constant for $\lambda_0 < t < b$.*

PROOF. (i) is evident; with the aid of the general homotopy invariance and excision property of the index, we see that (ii) and (iii) follow at once from Definition (1.1). □

We now consider the equation

$$(*) \qquad F_t(x) = F(x,t) = x$$

and look for its solutions $(x,t) \in X \times J$; note that $F_t(p_0) = p_0$ for each $t \in J$, so the points $(x,t) \in \mathcal{T}_F = \{p_0\} \times J$ form the set of *trivial solutions* of $(*)$.

(1.3) DEFINITION. A point (p_0, λ_0) is called a *bifurcation point* of $(*)$ provided for any open nbd U of (p_0, λ_0) in $X \times J$ there exists $(x,t) \in U$ such that $x \in \operatorname{Fix}(F_t) - \{p_0\}$. The set of bifurcation points is denoted by $\mathcal{B}_F$; clearly, if $(p_0, \lambda_0) \in \mathcal{B}_F$, then $\lambda_0 \in \Lambda_F$.

We let $\mathcal{N}_F = \{(x,t) \in X \times J \mid x \in \operatorname{Fix}(F_t) - \{p_0\}\}$ be the set of *nontrivial solutions* of $(*)$, and let $\mathcal{S}_F$ be the closure of $\mathcal{N}_F$; note

that $\mathcal{B}_F = \mathcal{T}_F \cap \mathcal{S}_F$, and thus the only trivial solutions in $\mathcal{S}_F$ are bifurcation points.

(1.4) PROPOSITION. *If $\lambda_0 \in \Lambda_F$ and $(p_0, \lambda_0) \notin \mathcal{B}_F$, then the fixed point index $i(F_t, B_{r(t)})$ is constant in a certain nbd of λ_0.*

PROOF. Indeed, if $(p_0, \lambda_0) \notin \mathcal{B}_F$, then for some nbd V of (p_0, λ_0) in $X \times J$ we have $V \cap \mathcal{N}_F = \emptyset$; from this, by (1.2) and the general homotopy invariance of the index, our assertion follows. □

(1.5) DEFINITION. Assume that $\lambda_0 \in \Lambda_F$, $J_0 = (a, b) \subset J$ and $r(t) > 0$ are as in (1.2). Select t_1, t_2 such that $a < t_1 < \lambda_0$ and $\lambda_0 < t_2 < b$. We define the *bifurcation index* $\Gamma(\lambda_0)$ of the family $\{F_t\}_{t \in J}$ at λ_0 by

$$\Gamma(\lambda_0) = i(F_{t_1}, B_{r(t_1)}) - i(F_{t_2}, B_{r(t_2)}).$$

In view of (1.2), it is clear that the definition of $\Gamma(\lambda_0)$ is independent of the t_1 and t_2 chosen as above. Furthermore, $r(t_i)$, $i = 1, 2$, in the definition can be replaced by any $\theta > 0$ such that $\mathrm{Fix}(F_{t_i} | B_\theta) = \{p_0\}$, $i = 1, 2$.

A sufficient condition for a point to be a bifurcation point is given in the following:

(1.6) THEOREM (Local bifurcation). *If $\lambda_0 \in \Lambda_F$ and $\Gamma(\lambda_0) \neq 0$, then (p_0, λ_0) is a bifurcation point for $(*)$.*

PROOF. This is an immediate consequence of (1.4) and (1.5). □

Let $C \subset X \times J$ be a compact component of $\mathcal{S}_F$ with $C \cap \mathcal{B}_F \neq \emptyset$. By the compactness of C, we see that $C \cap \mathcal{B}_F$ consists of a finite number of bifurcation points $(p_0, \lambda_1), \ldots, (p_0, \lambda_k)$. We let $\Lambda_0 = \{\lambda_1, \ldots, \lambda_k\}$ and assume $\lambda_1 < \ldots < \lambda_k$.

(1.7) DEFINITION. Let $C \subset X \times J$ be a compact component of $\mathcal{S}_F$ with $C \cap \mathcal{B}_F = \{(p_0, \lambda_1), \ldots, (p_0, \lambda_k)\}$. A bounded open subset U of $X \times J$ is called a *special neighborhood* of C provided:

(i) $C \subset U$,

(ii) there exists a positive $\eta = \eta_C$ such that for each $\lambda_i \in \Lambda_0 = \{\lambda_j \mid j \in [k]\}$, we have $\Lambda_F \cap (\lambda_i - \eta, \lambda_i + \eta) = \{\lambda_i\}$,

(iii) $\mathcal{T}_F \cap U \subset \bigcup_i \{p_0\} \times (\lambda_i - \eta, \lambda_i + \eta)$,

(iv) if $x = F(x, t)$ and $(x, t) \in \partial U$, then $x = p_0$ and $t \in (\lambda_i - \eta, \lambda_i + \eta)$ for some $\lambda_i \in \Lambda_0$.

The proof of the main result of this section is based on the following

(1.8) LEMMA. *Let $C \subset X \times J$ be a compact component of $\mathcal{S}_F$ with bifurcation points $(p_0, \lambda_1), \ldots, (p_0, \lambda_k)$. Then C admits a special nbd U satisfying:*

(i) *there exist* $\eta^*, r > 0$ *such that* $\eta^* < \eta$ *and*

$$\bigcup_{i=1}^{k}(K_r \times [\lambda_i - \eta^*, \lambda_i + \eta^*]) \subset U,$$

(ii) $\mathrm{Fix}(F_t|K_r) = \{p_0\}$ *whenever* $\eta^* < |t - \lambda_i| < \eta$,

(iii) *there exists a closed interval* $[a, b] \subset J$ *such that if* $(x, t) \in U$, *then* $a < t < b$.

PROOF. We first select a closed interval $[a, b] \subset J$ such that $(x, t) \in C$ implies $t \in (a, b)$, and let

$$\begin{aligned}
\varepsilon_0 &= \mathrm{dist}(\Lambda_0, \Lambda_F - \Lambda_0),\\
\varepsilon_1 &= \min\{\min\{t \in \boldsymbol{R} \mid (x,t) \in C\} - a,\ b - \max\{t \in \boldsymbol{R} \mid (x,t) \in C\}\},\\
\varepsilon_2 &= \min\{\lambda_{i+1} - \lambda_i \mid i \in [k-1]\}.
\end{aligned}$$

Let $0 < \eta < \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$ and

$$B_i = \{(x,t) \in X \times J \mid \varrho((x,t), (p_0, \lambda_i)) < \eta/2\} \quad \text{for } i \in [k],$$

and let

$$\Omega_\delta = \{(x,s) \in X \times J \mid \varrho((x,s), C) < \delta\}$$

be the open δ-neighborhood of C in $X \times J$ with

$$\delta < \min\Big\{\eta/2, \mathrm{dist}\Big(C - \bigcup_{i=1}^{k} B_i, \{\lambda_0\} \times \boldsymbol{R}\Big)\Big\}.$$

We make the following observations:

(a) $\Lambda_0 = \{\lambda_i \mid i \in [k]\} \subset (a, b)$,

(b) $\Lambda_F \cap \bigcup_{i=1}^{k}[\lambda_i - \eta, \lambda_i + \eta] = \Lambda_0$,

(c) $\mathcal{T}_F \cap \overline{\Omega}_\delta \subset \bigcup_{i=1}^{k}\{\{p_0\} \times (\lambda_i - \eta, \lambda_i + \eta)\}$,

(d) $(x, t) \in \Omega_\delta$ implies $t \in (a, b)$.

We now apply the Kuratowski–Mazurkiewicz separation theorem (12.6.4). Consider the compact subset $K = \overline{\Omega}_\delta \cap \mathcal{S}_F$ of $X \times J$ and its closed disjoint subsets $A = C$ and $B = \partial\Omega_\delta \cap \mathcal{S}_F$. Because $\partial\Omega_\delta \cap C = \emptyset$, there is no connected subset of K joining A and B, and hence K is disconnected between A and B. By the Kuratowski–Mazurkiewicz theorem we can find two disjoint closed subsets K_A and K_B of K such that

$$A = C \subset K_A, \quad B = \partial\Omega_\delta \cap \mathcal{S}_F \subset K_B, \quad K = K_A \cup K_B.$$

Because K_A and K_B are compact and disjoint, there exists an open nbd U of C in $X \times J$ such that $U \subset \Omega_\delta$, $U \cap \mathcal{S}_F = \emptyset$, and $a < t < b$ for $(x, t) \in U$. By construction it is clear that U is the desired special nbd of C. □

We are now ready to establish the global bifurcation theorem:

(1.9) THEOREM (Rabinowitz Nussbaum). *Let* $\{(X, d); p_0\}$ *be a based ANR and* $F : X \times J \to X$ *a completely continuous allowable map. Let* C *be a compact component of* S_F *containing a bifurcation point of* $(*)$. *Then:*

(A) *if* $(p_0, \lambda) \in C \cap \mathcal{B}_F$ *and* $\Gamma(\lambda) \neq 0$, *then* C *contains another bifurcation point of* $(*)$,

(B) *if* $C \cap \mathcal{B}_F = \{(p_0, \lambda_1), \ldots, (p_0, \lambda_k)\}$, *then* $\sum_{i=1}^{k} \Gamma(\lambda_i) = 0$.

PROOF. Because (B)⇒(A), it is enough to prove (B). We first select a special bounded nbd U of C satisfying (i)–(iii) of (1.8). The proof will be carried out in a few steps:

STEP 1. We fix $\lambda_i \in \Lambda_0$ and let $\mathcal{D}_i = X \times [\lambda_i - \eta, \lambda_i + \eta]$ be a "strip" in $X \times J$ determined by (λ_i, η). We define an open subset $O \subset \mathcal{D}_i$ as the union $O = V \cup W$, where

$$V = B_r \times [\lambda_i - \eta, \lambda_i + \eta] \quad \text{and} \quad W = \mathcal{D}_i \cap U.$$

We claim that

$$\text{(1)} \qquad \mathrm{Fix}(F_t|(\widehat{\partial} O)_t) = \emptyset \quad \text{for each } t \in [\lambda_i - \eta, \lambda_i + \eta].$$

Supposing the contrary, i.e., $F_t(x) = x$ for some $x \in (\widehat{\partial} O)_t$, we may have either (a) $x \in (\partial U)_t$, or (b) $x \in \partial B_r$ and $t \in [\lambda_i - \eta, \lambda_i + \eta]$. In the first case, x must coincide with p_0, implying $x \in O_t$, which is impossible; and in the second case, from $\eta^* \leq |t - \lambda_i| \leq \eta$ and the construction of U, it follows that $x = p_0$, and from $|t - \lambda_i| \leq \eta^*$ that $x \in (\partial U)_t$; thus our assertion (1) is established.

Because of (1), by applying the general homotopy invariance of the index on $[\lambda_i - \eta, \lambda_i + \eta]$, we have

$$\text{(2)} \qquad i(F_{\lambda_i - \eta}, O_{\lambda_i - \eta}) = i(F_{\lambda_i + \eta}, O_{\lambda_i + \eta}).$$

Next, by the additivity property we get

$$\text{(3)} \qquad \begin{aligned} i(F_{\lambda_i - \eta}, O_{\lambda_i - \eta}) &= i(F_{\lambda_i - \eta}, U_{\lambda_i - \eta}) + i(F_{\lambda_i - \eta}, B_r), \\ i(F_{\lambda_i + \eta}, O_{\lambda_i + \eta}) &= i(F_{\lambda_i + \eta}, U_{\lambda_i + \eta}) + i(F_{\lambda_i + \eta}, B_r). \end{aligned}$$

Now from (2), (3), and Definition (1.3) we obtain

$$\text{(4)} \qquad \Gamma(\lambda_i) = i(F_{\lambda_i + \eta}, U_{\lambda_i + \eta}) - i(F_{\lambda_i - \eta}, U_{\lambda_i - \eta}).$$

Because λ_i was fixed arbitrarily, (4) holds for any $i \in [k]$.

STEP 2. We now consider a pair of consecutive elements $\lambda_j < \lambda_{j+1}$ of Λ_0. Because $(\lambda_j, \lambda_{j+1}) \cap \Lambda_0 = \emptyset$, we have

$$\text{(5)} \qquad \mathrm{Fix}(F_t|(\partial U)_t) = \emptyset \quad \text{for } t \in [\lambda_j + \eta, \lambda_{j+1} - \eta].$$

It follows from (5) and the general homotopy invariance of the index that

(6) $$i(F_{\lambda_{j+1}-\eta}, U_{\lambda_{j+1}-\eta}) = i(F_{\lambda_j+\eta}, U_{\lambda_j+\eta}).$$

Because the pair $\lambda_j < \lambda_{j+1}$ was fixed arbitrarily, (6) holds for any $j \in [k-1]$.

For convenience, we now set $a_i = i(F_{\lambda_i+\eta}, U_{\lambda_i+\eta})$, $b_i = i(F_{\lambda_i-\eta}, U_{\lambda_i-\eta})$ and observe that from (4) and (6) we obtain

$$\begin{aligned} a_i - b_i &= \Gamma(\lambda_i) \quad \text{for each } i \in [k], \\ a_j &= b_{j+1} \quad \text{for each } j \in [k-1], \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{i=1}^{k} \Gamma(\lambda_i) &= \sum_{i=1}^{k} (a_i - b_i) = a_k - b_1 \\ &= i(F_{\lambda_k+\eta}, U_{\lambda_k+\eta}) - i(F_{\lambda_1-\eta}, U_{\lambda_1-\eta}). \end{aligned}$$

STEP 3. To complete the proof we only need to show that $a_k = 0$ and $b_1 = 0$. To show that $a_k = 0$, we apply the general homotopy invariance property to the set $U \cap (X \times [\lambda_k + \eta, b])$. Since there are no solutions of $(*)$ on $\partial U \cap (X \times [\lambda_k + \eta, b])$ by (1.7)(iv) and $U_b = \emptyset$ by (1.8)(iii), it follows by the general homotopy invariance that $0 = i(F_{\lambda_k+\eta}, U_{\lambda_k+\eta}) = a_k$. The argument for $b_1 = 0$ is similar, and thus the proof of the theorem is complete. □

We now consider two examples:

EXAMPLE 1. Let E be an infinite-dimensional Banach space, and let $F : E \times \boldsymbol{R} \to E$ be a completely continuous map with the following properties:

(A.1) $F(x, \lambda) = \lambda T x + \omega(x, \lambda)$ for $(x, \lambda) \in E \times \boldsymbol{R}$,
(A.2) T is linear and completely continuous,
(A.3) $\lim_{\|x\| \to 0} \|\omega(x, \lambda)\| / \|x\| = 0$ uniformly in each $[a, b] \subset \boldsymbol{R}$,
(A.4) $F(0, \lambda) = 0$ for $\lambda \in \boldsymbol{R}$.

Let $r(T)$ be the set of characteristic values of T. Using the terminology of (1.1), we show that F with $(X; p_0) = (E; 0)$, $J = \boldsymbol{R}$, and $\Lambda_F = r(T)$ is allowable. To see this, it is enough to observe that $r(T)$ satisfies (a) of (1.1): (i) and (ii) of (a) are obvious, and since for every $\mu \notin r(T)$, $(I - \mu T)^{-1}$ exists, it is easily seen that properties (A.1)–(A.3) of F imply that (iii) of (a) is also satisfied, so the assertion follows.

EXAMPLE 2. Let F be as in Example 1. We show that if $\nu \in r(T)$ is a characteristic value with algebraic multiplicity m_ν, then the bifurcation index $\Gamma(\nu)$ of $\{F_t\}_{t \in \boldsymbol{R}}$ at ν is given by

$$\Gamma(\nu) = \begin{cases} 0 & \text{if } m_\nu \text{ is even}, \\ \pm 2 & \text{if } m_\nu \text{ is odd}. \end{cases}$$

To see this, take a small interval $J_0 = (\nu-\eta, \nu+\eta)$ with $J_0 \cap r(T) = \{\nu\}$ and choose a $\delta \in (0, \eta)$. By (1.2), for each $t \in J_0 - \{\nu\}$, there exists an $r(t) > 0$ such that

$$\begin{cases} \mathrm{Fix}\{F_t|B_{r(t)}\} = \{0\}, \\ J(F_{\nu-\delta}, 0) = i(F_t, B_{r(t)}) \quad \text{for all } t \in (\nu - \eta, \nu), \\ J(F_{\nu+\delta}, 0) = i(F_t, B_{r(t)}) \quad \text{for all } t \in (\nu, \nu + \eta). \end{cases}$$

This, in view of Definition (1.5) and (12.8.5), gives

$$\Gamma = J(F_{\nu-\delta}, 0) - J(F_{\nu+\delta}, 0) = J((\nu - \delta)T, 0) - J((\nu + \delta)T, 0),$$

and now the desired assertion follows from (12.8.3).

We are now in a position to establish the following basic

(1.10) THEOREM (Krasnosel'skiĭ–Rabinowitz). *Let E be a Banach space and $F : E \times \boldsymbol{R} \to \boldsymbol{R}$ be completely continuous such that*
 (i) $F(x, \lambda) = \lambda Tx + \omega(x, \lambda)$ *for* $(x, \lambda) \in E \times \boldsymbol{R}$,
 (ii) $T : E \to E$ *is linear and completely continuous,*
 (iii) $\omega(x, \lambda) = o(\|x\|)$ *as* $x \to 0$ *uniformly in* λ *in each interval* $[a, b]$.
Assume that $\nu \in r(T)$ has odd algebraic multiplicity. Then:
 (a) $(0, \nu)$ *is a bifurcation point,*
 (b) *if C is a component of S_F with $(0, \nu) \in C$, then either C is unbounded or C contains another bifurcation point of* $(*)$,
 (c) *if C is compact and if $C \cap \mathcal{B}_F = \{(0, \nu_1), \ldots, (0, \nu_k)\}$, then* $\sum_{i=1}^{k} \Gamma(\nu_i) = 0$,
 (d) *the number of characteristic values $\nu_i \in r(T)$, $i \in [k]$, appearing in* (c) *and having odd algebraic multiplicity is even.*

PROOF. In view of Examples 1 and 2, assertions (a)–(c) are immediate consequences of (1.6) and (1.9).

(d) Because in the sum $\sum_{i=1}^{k} \Gamma(\nu_i) = 0$, the nontrivial terms $\Gamma(\nu_i) = \pm 2$ involve only the $\nu_i \in r(T)$ having odd algebraic multiplicity, their number must be even. □

2. Application of the Index to Nonlinear PDEs

Partial differential equations of elliptic type (both linear and nonlinear) arise naturally in physics, geometry, and various other branches of mathematics; many basic questions in those fields appear as Dirichlet problems, and the existence proofs combine analytical and topological techniques. The analytical parts of these proofs are usually very complicated, and will not be treated here; we concentrate only on the topological aspects.

a. Nonlinear Poisson equation

Let Ω be a bounded region in $\boldsymbol{R}^n$ having a smooth boundary, $D_i = \partial/\partial x_i$, and let $\Delta = \sum_{i=1}^n D_i^2$ be the Laplace operator. Consider the homogeneous boundary value problem

$$(\mathscr{P}_\Delta) \qquad \begin{cases} \Delta u = f(x, u, Du), \\ u|\partial\Omega = 0. \end{cases}$$

It is well known that the Banach spaces $C^k(\overline{\Omega})$ are not sufficient for studying this problem. Indeed, let $C_0^2 = \{u \in C^2(\overline{\Omega}) \mid u|\partial\Omega = 0\}$; then, although the Laplace operator $\Delta : C_0^2 \to C(\overline{\Omega})$ is continuous, it is not surjective: for a given $f \in C(\overline{\Omega})$, the solution of the linear Poisson equation $\Delta u = f$, $u|\partial\Omega = 0$, may not exist in $C^2(\overline{\Omega})$.

The function spaces that provide an appropriate setting for this and more general problems are the Hölder spaces. To recall the definition we introduce the notation: for each ordered set $p = (p_1, \ldots, p_n)$ of nonnegative integers, put

$$D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}};$$

$|p| = \sum_{i=1}^n p_i$ is called the *order* of D^p. With this notation, the *Hölder space* $C^{k+\alpha}(\overline{\Omega})$, where $k \geq 0$ is an integer and $0 \leq \alpha < 1$, is the Banach space of all $u \in C^k(\overline{\Omega})$ for which the *Hölder norm*

$$\|u\|_{k+\alpha} = \sup_{|p| \leq k,\, x \in \overline{\Omega}} |D^p u(x)| + \sup_{\substack{|p|=k,\, x \neq y \\ x, y \in \overline{\Omega}}} \frac{|D^p u(x) - D^p u(y)|}{|x - y|^\alpha}$$

is finite. For example, $C^\alpha(\overline{\Omega})$ consists simply of the Hölder continuous functions with exponent α on $\overline{\Omega}$ and with the norm

$$\|u\|_0 + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

where $\| \, \|_0$ is the supremum norm on $\overline{\Omega}$.

The Hölder spaces are related as follows: if $k + \alpha > l + \beta$, then the embedding $j : C^{k+\alpha} \to C^{l+\beta}$ is completely continuous (i.e., the image of the ball $\|u\|_{k+\alpha} \leq M$ in $C^{k+\alpha}$ is relatively compact in $C^{l+\beta}$ for each $M > 0$).

Moreover, the new setting overcomes the above difficulty with the linear Poisson equation: on the space $C_0^{2+\alpha} = \{u \in C^{2+\alpha}(\overline{\Omega}) \mid u|\partial\Omega = 0\}$ the Laplace operator $\Delta : C_0^{2+\alpha} \to C^\alpha$ is a bijective linear map for each $\alpha \in (0, 1)$, and hence by the inverse mapping theorem it is an isomorphism.

(2.1) THEOREM. *Assume that $u \mapsto f(x, u, Du)$ determines a continuous map $F : C^{1+\alpha} \to C^\alpha$ such that $\|Fu\|_\alpha \leq M$ for all $u \in C^{1+\alpha}$ and*

some $M < \infty$. Then the boundary value problem $(\mathscr{P}_\Delta)$ has at least one solution $u \in C_0^{2+\alpha}$

PROOF. Starting with $\Delta : C_0^{2+\alpha} \to C^\alpha$ and the embedding $j : C_0^{2+\alpha} \to C^{1+\alpha}$, we have to show that $\Delta u = Fju$ has a solution; we convert this to the equivalent fixed point problem for the operator $Fj\Delta^{-1} : C^\alpha \to C^\alpha$. Since j is completely continuous and F is bounded, we see that the map $j\Delta^{-1}F : C^{1+\alpha} \to C^{1+\alpha}$ is compact; so by Schauder's theorem, it has a fixed point. This clearly implies that $Fj\Delta^{-1}$ also has a fixed point, and thus the desired conclusion follows. □

b. The equation $Lu = f(x, u, Du)$

Bounds for the solutions of a differential equation (linear or nonlinear) which are obtained under the sole assumption that the solution exists are called *a priori bounds*. Following the pioneering work of Bernstein, Schauder developed a technique for solving the Dirichlet problem for a linear elliptic equation that involves establishing a priori bounds in Hölder spaces which permit application of the invertibility theorem (4.3.4). We now illustrate the use of the Schauder a priori bounds in the nonlinear context.

Consider the homogeneous boundary value problem

$$(\mathscr{P}_L) \qquad \begin{cases} Lu = f(x, u, Du), \\ u|\partial\Omega = 0, \end{cases}$$

where

$$L = \sum_{i,j} a_{ij}(x)D_iD_j + \sum_i b_i(x)D_i + c(x)$$

is a differential operator with coefficients a_{ij}, b_i, c belonging to $C^\alpha(\overline{\Omega})$; we assume that the operator L is *uniformly elliptic* in $\overline{\Omega}$, i.e., $m\sum_{i=1}^n \xi_i^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le M\sum_{i=1}^n \xi_i^2$ for positive constants m, M independent of $x \in \Omega$ and for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. We assume also that $u \mapsto f(x, u, Du)$ determines a continuous map

$$F : C^{1+\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega}).$$

Starting with the linear operator $L : C_0^{2+\alpha} \to C^\alpha$ we have as before the problem $Lu = Fj(u)$; so we consider the possibility of inverting L. Under the hypothesis that $c(x) \le 0$ (which implies the injectivity of L) Schauder showed that L^{-1} exists; the proof depends on the invertibility of the Laplace operator, and proceeds as follows: consider the family of operators $L_t = (1-t)\Delta + tL : C_0^{2+\alpha} \to C^\alpha$, where $0 \le t \le 1$. Schauder established the fundamental "a priori estimates" $\|u\|_{2+\alpha} \le K\|L_tu\|_\alpha$ for any $u \in C_0^{2+\alpha}$, where K is a constant independent of u and $t \in [0, 1]$; now applying (4.3.4) one concludes that L is invertible.

(2.2) THEOREM. *Assume that* $u \mapsto f(x,u,Du)$ *determines a continuous operator* $F : C^{1+\alpha} \to C^{\alpha}$ *such that* $\|Fu\|_{\alpha} \le M$ *for all* $u \in C^{1+\alpha}$. *Then the problem* $(\mathscr{P}_L)$ *has at least one solution* $u \in C_0^{2+\alpha}$.

The proof is strictly analogous to that of (2.1).

c. Quasi-linear differential equations

We now give an application of the Leray–Schauder index to quasi-linear elliptic differential equations. The approach presented here was first introduced in the context of degree theory for compact fields by Leray and Schauder, and has proved to be a powerful tool in the treatment of various nonlinear problems.

Let $\Omega \subset \boldsymbol{R}^n$ be a bounded region with a smooth boundary. We now consider the problem of solvability of an elliptic quasi-linear differential equation

$$\sum_{i,j=1}^{n} a_{ij}(x,u,Du)D_iD_ju = f(x,u,Du)$$

with Dirichlet boundary condition

$$u|\partial\Omega = \varphi.$$

The Leray–Schauder method for finding a solution to this type of problem is based on a combination of the technique of a priori bounds with the fixed point index theory for compact operators. The analytical details for establishing the a priori bounds are complicated even for relatively simple equations in $\boldsymbol{R}^2$. Here we sketch only the topological aspect using the notation given at the beginning of this section.

Consider the "auxiliary" family of problems

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}(x,u,Du)D_iD_ju = \lambda f(x,u,Du), \\ u|\partial\Omega = \lambda\varphi, \end{cases}$$

depending on a parameter λ, $0 \le \lambda \le 1$.

For each choice of $u_0 \in C^{1+\alpha}$, we obtain a "linearized" family

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}(x,u_0,Du_0)D_iD_ju = \lambda f(x,u_0,Du_0), \\ u|\partial\Omega = \lambda\varphi, \end{cases}$$

which is easier to solve. We shall say that the original problem is *well posed* if (a) the linearized problem has a unique solution $H_\lambda(u_0) = u \in C^{2+\alpha}$ for each choice of u_0 and λ, and (b) the assignment $(u_0,\lambda) \mapsto H_\lambda(u_0)$ defines a homotopy

$$H : C^{1+\alpha} \times [0,1] \to C^{2+\alpha}$$

that maps bounded sets into bounded sets.

For well-posed problems, the existence of a solution of the given equation depends on "a priori bounds" for the solutions of the auxiliary family of equations; that is, under the (a priori) assumption that any solution of any one of the auxiliary equations is sufficiently smooth, it is then suitably majorized. The proof of the following theorem illustrates the use of the Leray–Schauder index to establish existence theorems for well-posed problems.

(2.3) THEOREM (Leray–Schauder). *Assume that*

$$\sum_{i,j=1}^{n} a_{ij}(x,u,Du)D_iD_ju = f(x,u,Du), \qquad u|\partial\Omega = \varphi,$$

is well posed, and that if $u \in C^{2+\alpha}(\overline{\Omega})$ is a solution of any of the auxiliary equations

$$\sum_{i,j=1}^{n} a_{ij}(x,u,Du)D_iD_ju = \lambda f(x,u,Du), \qquad u|\partial\Omega = \lambda\varphi,$$

then $\|u\|_{1+\alpha} \leq M$, where M is independent of u and λ. Then the given problem has at least one solution $u \in C^{2+\alpha}$.

PROOF. By hypothesis, for each $u_0 \in C^{1+\alpha}$ and $\lambda \in I$, the problem

$$\begin{cases} \sum_{i,j=1}^{n} a_{ij}(x,u_0,Du_0)D_iD_ju = \lambda f(x,u_0,Du_0), \\ u|\partial\Omega = \lambda\varphi, \end{cases}$$

has a unique solution $H_\lambda(u_0) \in C^{2+\alpha}$. The rule $u_0 \mapsto H_\lambda u_0$ therefore defines a homotopy $H_\lambda : C^{1+\alpha} \to C^{2+\alpha}$ which for each λ maps bounded sets into bounded sets. Let $j : C^{2+\alpha} \to C^{1+\alpha}$ be the (completely continuous) embedding, so that we have a completely continuous homotopy $jH_\lambda : C^{1+\alpha} \to C^{1+\alpha}$ This homotopy is fixed point free on the boundary of the ball $\|u\|_{1+\alpha} \leq M+1$ because of the a priori estimate on the solutions of the auxiliary system. Clearly, $jH_0 : C^{1+\alpha} \to C^{1+\alpha}$ is the constant mapping $u \mapsto 0$, by unicity of the solutions of the linearized system, so the Leray–Schauder fixed point index $i(jH_0, B(0, M+1))$ is 1. By homotopy invariance of the index, because jH_0 and jH_1 are compactly homotopic on $\overline{B}(0, M+1)$ without fixed points on $\partial B(0, M+1)$, we get $i(jH_1, B(0, M+1)) = 1$, so there exists at least one u with $u = jH_1(u)$; any such u is a solution of the given problem. □

3. The Leray–Schauder Degree

In this section we return to the study of compact fields in normed linear spaces. Having the index in ANRs at our disposal, we can now acquire the

Leray–Schauder degree for compact fields directly from the index, the point being that normed linear spaces are ANRs and that the degree and index are related via $f \leftrightarrow I - f$ [1].

Let E be an infinite-dimensional normed linear space. For any open subset U of E and b in E we denote by $\mathfrak{C}_b(\overline{U})$ the set of all compact fields $f : (\overline{U}, \partial U) \to (E, E - \{b\})$, and let $\mathscr{K}_{\partial U}(\overline{U}, E)$ be the set of all compact maps from $\overline{U}$ to E that are fixed point free on ∂U.

(3.1) Definition. Let U be an open subset of E, let b be in E, and let $g \in \mathfrak{C}_0(\overline{U})$ and $f \in \mathfrak{C}_b(\overline{U})$ be compact fields.

(i) The *Leray–Schauder degree* of g is

$$d(g, U) = i(I - g, U),$$

where $i(I - g, U)$ is the index of the associated compact map $I - g \in \mathscr{K}_{\partial U}(\overline{U}, E)$.

(ii) The *Leray–Schauder degree of f with respect to b* is the integer $d(f, U, b)$ defined by

$$d(f, U, b) = d(g, U),$$

where the right hand side is the Leray–Schauder degree of the field $g = f - b \in \mathfrak{C}_0(\overline{U})$.

As an immediate consequence of the properties of the index for compact maps in $\mathscr{K}_{\partial U}(\overline{U}, E)$, we obtain the following:

(3.2) Theorem. *The Leray–Schauder degree $f \mapsto d(f, U, b)$ for $f \in \mathfrak{C}_b(\overline{U})$ has the following properties*:

(I) (Normalization) *If $f(x) = x - b$, then $d(f, U, b) = 1$ or 0, depending on whether $b \in U$ or not.*

(II) (Additivity) *For any pair of disjoint open $V_1, V_2 \subset U$, if $b \notin f(\overline{U} - (V_1 \cup V_2))$, then $d(f, U, b) = d(f, V_1, b) + d(f, V_2, b)$.*

(III) (Homotopy) *If $h_t : (\overline{U}, \partial U) \to (E, E - \{b\})$ is a homotopy of compact fields, then $d(h_0, U, b) = d(h_1, U, b)$.*

(IV) (Existence) *If $d(f, U, b) \neq 0$, then $f(U)$ is a nbd of b in E.*

(V) (Excision) *If V is an open subset of U and $f \in \mathfrak{C}_b(\overline{U})$ satisfies $b \notin f(U - V)$, then $d(f, V, b) = d(f, U, b)$.* □

One of the important consequences of the homotopy property in (3.2) is that $d(f, U, b)$ depends only on the boundary values $f|\partial U$. Precisely, we have

[1] Since locally convex metrizable linear topological spaces are ANRs, the results of this section that do not refer to the notion of norm are also valid in such spaces.

(3.3) THEOREM (Dependence on boundary values). *If $U \subset E$ is open and $f, g \in \mathfrak{C}_b(U)$ satisfy $f|\partial U = g|\partial U$, then $d(f, U, b) = d(g, U, b)$.*

PROOF. Define $h : \overline{U} \times \boldsymbol{I} \to E$ by

$$h(x, t) = (1-t)f(x) + tg(x) = x - [(1-t)F(x) + tG(x)].$$

Since $h_t|\partial U = f|\partial U = g|\partial U$ for all t, we conclude that $\{h_t\}$ is a homotopy of compact fields in $\mathfrak{C}_b(\overline{U})$ joining f and g, and so $d(f, U, b) = d(g, U, b)$. □

Another useful property relating to boundary behavior is

(3.4) THEOREM (Rouché). *Let $f, g : \overline{U} \to E$ be two compact fields satisfying:*
(i) $\|f(x) - g(x)\| < \|g(x) - b\|$ *for all* $x \in \partial U$.
(ii) $g \in \mathfrak{C}_b(\overline{U})$.
Then $f \in \mathfrak{C}_b(\overline{U})$ *and* $d(f, U, b) = d(g, U, b)$.

PROOF. Let $h_t(x) = (1-t)g(x) + tf(x)$ for $(x, t) \in \overline{U} \times \boldsymbol{I}$. Then $h_t(x) - b = g(x) - b + t(f(x) - g(x))$, and hence $\|h_t(x) - b\| \geq \|g(x) - b\| - \|f(x) - g(x)\|$. For $x \in \partial U$, the right hand side is positive by assumption, and therefore $b \notin h_t(\partial U)$ for all $t \in \boldsymbol{I}$. Since by homotopy invariance $d(h_t, U, b)$ is independent of t, the desired conclusion follows. □

As another consequence of the homotopy invariance, we now give a simple criterion implying that the degree of a given compact field is zero. To formulate the result, we need some terminology.

Let b be a point in E. By a *ray R_b originating at b* is meant any set $[b, y]^{\rightarrow}$ given by

$$[b, y]^{\rightarrow} = \{x \in E \mid x = (1-\lambda)b + \lambda y,\ 0 \leq \lambda < \infty\},$$

where $y \neq b$ is in E.

(3.5) THEOREM. *Let U be an open bounded subset of E and $f \in \mathfrak{C}_b(\overline{U})$. Assume that there exists a ray R_b originating at b such that $R_b \cap f(\partial U) = \emptyset$. Then $d(f, U, b) = 0$.*

PROOF. Since $f(\overline{U})$ is bounded, there is a $\varrho > 0$ such that $y \notin f(\overline{U})$ for all y with $\|y\| \geq \varrho$. Choose now a point $b^* \in R_b$ with $\|b^*\| \geq \varrho$ such that $d(f, U, b^*) = 0$ (by (3.2)(IV)), and let

$$h_t(x) = f(x) - [(1-t)b + tb^*] \qquad \text{for } (x, t) \in \overline{U} \times \boldsymbol{I}.$$

Since $R_b \cap f(\partial U) = \emptyset$ and the segment $[b, b^*]$ is contained in R_b, it follows that $h_t(x) \neq 0$ for all $(x, t) \in \partial U \times \boldsymbol{I}$. Thus, $h_t : (\overline{U}, \partial U) \to (E, E - \{0\})$ is a homotopy of compact fields joining $f - b$ and $f - b^*$ in $\mathfrak{C}_0(\overline{U})$. By homotopy invariance of the degree, we have

$$d(f, U, b) = d(f - b, U) = d(f - b^*, U) = d(f, U, b^*). \qquad \square$$

Connectedness results

The Leray–Schauder degree can be used to determine whether the solution set of a nonlinear equation is connected.

Before stating the first result of this nature, we recall some terminology and notation.

Let $M \subset E \times [a, b]$ and $f : M \to E$. For $\lambda \in [a, b]$, we let $M_\lambda = \{x \in E \mid (x, \lambda) \in M\}$ be the λ-*slice* of M; we denote by $\{f_\lambda : M_\lambda \to E\}_{\lambda \in [a,b]}$ the family of maps determined by f, where $f_\lambda(x) = f(x, \lambda)$ for $(x, \lambda) \in M$.

The general homotopy invariance property of the index (see (12.6.3)) implies the following:

(3.6) THEOREM (Generalized homotopy invariance). *Let U be an open subset of $E \times [a, b]$, and $f : \overline{U} \to E$ a compact field with $f = I - F$ If $z \in E - f(\widehat{\partial} U)$, then the degree $d(f_\lambda, U_\lambda, z)$ is independent of $\lambda \in [a, b]$, and so in particular, $d(f_a, U_a, z) = d(f_b, U_b, z)$.* □

The first connectedness result in the context of the Leray–Schauder degree is the following special case of Theorem (12.6.5):

(3.7) THEOREM (Continuation principle in normed linear spaces). *Let $U \subset E \times [a, b]$ be open with vertical boundary $\widehat{\partial} U$, and let $f : \overline{U} \to E$ be a compact field with $f = I - F$. Let*

$$S = \{(x, \lambda) \in \overline{U} \mid f(x, \lambda) = 0\} = \{(x, \lambda) \in \overline{U} \mid F(x, \lambda) = x\}$$

and assume that:
(i) $S \cap \widehat{\partial} U = \emptyset$,
(ii) $d(f_a, U_a) \neq 0$.
Then there exists a continuum $C \subset S$ joining $S_a \times \{a\}$ to $S_b \times \{b\}$. □

The following result of Krasnosel'skiĭ–Perov is frequently used to show that the set of solutions of a differential equation is connected.

(3.8) THEOREM. *Let $U \subset E$ be open, and $f : (\overline{U}, \partial U) \to (E, E - \{0\})$ a compact field $f(x) = x - F(x)$ with $d(f, U) \neq 0$ such that $F : \overline{U} \to E$ satisfies the following condition:*

(∗) *for each $\varepsilon > 0$, there exists a compact ε-approximation $F_\varepsilon : \overline{U} \to E$ of F such that for each $b \in E$ with $\|b\| \leq \varepsilon$ the equation*

$$x = F_\varepsilon(x) + b$$

has at most one solution in $\overline{U}$.

Then the zero set $Z(f) = \{x \in U \mid f(x) = 0\}$ is connected.

PROOF. Since $d(f, U) \neq 0$, the compact set $Z = Z(f)$ is not empty. Suppose to the contrary that Z is not connected. Then $Z = Z_1 \cup Z_2$ where Z_1, Z_2

are nonempty, disjoint, and compact. Choose open sets $U_1, U_2 \subset U$ with disjoint closures such that $Z_1 \subset U_1$, $Z_2 \subset U_2$, and observe that

$$(**) \qquad d(f,U) = d(f,U_1) + d(f,U_2)$$

by the additivity and excision properties of degree. To get a contradiction, we show that $d(f,U_1) = d(f,U_2) = 0$. Let $h = F(h)$ be in Z_1, and for given $\varepsilon > 0$ set

$$g_\varepsilon(x) = x - F_\varepsilon(x) - (h - F_\varepsilon(h)) \qquad \text{for } x \in \overline{U}.$$

Consider the family of compact fields

$$h_t(x) = (1-t)f(x) + tg_\varepsilon(x) \qquad \text{for } (x,t) \in \overline{U} \times \boldsymbol{I}.$$

Let $\alpha = \inf\{\|f(x)\| \mid x \in \overline{U} - (U_1 \cup U_2)\}$ ($\alpha > 0$ by assumption) and assume $\varepsilon \leq \alpha/4$. Since for any $x \in \overline{U} - (U_1 \cup U_2)$,

$$\begin{aligned}
\|h_t(x)\| &= \|f(x) + t(F(x) - F_\varepsilon(x)) - t(h - F_\varepsilon(h))\| \\
&\geq \|f(x)\| - t\|F(x) - F_\varepsilon(x)\| - t\|h - F_\varepsilon(h)\| \\
&\geq \|f(x)\| - 2\varepsilon \geq \alpha - 2\varepsilon \geq \alpha/2 > 0,
\end{aligned}$$

we see that $h_t : (\overline{U}, \partial U) \to (E, E - \{0\})$ is a homotopy of compact fields, and hence the degree $d(h_t, U)$ is constant on $\boldsymbol{I}$; thus

$$d(g_\varepsilon, U) = d(f,U) \neq 0.$$

Because in view of $(*)$, $g_\varepsilon(x) = x - F_\varepsilon(x) - b$ with $b = h - F_\varepsilon(h)$ has at most one zero and $h \in Z_1 \subset U_1$, we see that g_ε has no zeros on U_2, and therefore $d(g_\varepsilon, U_2) = d(f, U_2) = 0$. If $h \in Z_2$, a similar argument shows that $d(g_\varepsilon, U_1) = d(f, U_1) = 0$. Thus, in view of $(**)$, we obtain a contradiction, and the proof is complete. □

4. Extensions of the Borsuk and Borsuk–Ulam Theorems

Using the Leray–Schauder degree, we now establish some extensions of the Borsuk and Borsuk–Ulam theorems. Let $S \subset E$ be centrally symmetric (i.e., $-S = S$). We recall that a map $f : S \to E$ is *odd* if $f(x) = -f(-x)$ for all $x \in S$.

(4.1) THEOREM (Antipodal theorem). *Let $U \subset E$ be a bounded centrally symmetric domain and $f : (\overline{U}, \partial U) \to (E, E - \{0\})$ be a compact field such that:*
(i) $0 \in U$,
(ii) *$f|\partial U$ is an odd function.*
Then $d(f,U)$ is odd.

PROOF. This follows at once from the definition of degree and (12.6.6). □

(4.2) Corollary. *Let $U \subset E$ be a bounded centrally symmetric domain and $f : (\overline{U}, \partial U) \to (E, E - \{0\})$ be a compact field such that*
(i) $0 \in U$,
(ii) $f(x) \neq \lambda f(-x)$ *for all* $x \in \partial U$ *and* $\lambda > 0$ *(i.e., the vectors* $f(x)$ *and* $f(-x)$ *have the same direction for no* $x \in \partial U$*).*
Then $d(f, U)$ *is odd.*

Proof. Let

$$g(x) = \frac{f(x) - f(-x)}{2} = x - \frac{F(x) - F(-x)}{2} \qquad \text{for } x \in \overline{U}$$

and

$$\begin{aligned} h_t(x) &= tf(x) + (1-t)g(x) \\ &= x - \frac{(1+t)F(x) - (1-t)F(-x)}{2} \qquad \text{for } (x,t) \in \overline{U} \times I. \end{aligned}$$

We claim that $h_t(\partial U) \subset E - \{0\}$ for all $t \in I$. Supposing the contrary, we would have $tf(x) + (1-t)g(x) = 0$ for some $x \in \partial U$ and some $t \in I$, giving

$$f(x) = \frac{1-t}{1+t} f(-x),$$

which for $t \in [0,1)$ contradicts (ii) and for $t = 1$ contradicts the assumption that f has no zeros on ∂U. Thus, $h_t : (\overline{U}, \partial U) \to (E, E - \{0\})$ is a homotopy of compact fields joining g and f. Because g is odd on ∂U, $d(g, U)$ is odd by (4.1), and the conclusion follows from the homotopy property of degree. □

(4.3) Theorem. *Let $U \subset E$ be a bounded centrally symmetric domain with $0 \in U$, and $f : \partial U \to E_0$ an odd compact field, where $E_0 \subset E$ is a closed linear subspace of codimension 1. Then $f(x_0) = 0$ for some $x_0 \in \partial U$.*

Proof. Suppose to the contrary that $f(x) \neq 0$ for $x \in \partial U$. Write $f(x) = x - F(x)$ and extend F over $\overline{U}$ to a compact $\widehat{F} : \overline{U} \to E$. Letting

$$\widehat{f}(x) = x - \widehat{F}(x) \qquad \text{for } x \in \overline{U},$$

observe that since $\widehat{f}|\partial U = f$, we have $\widehat{f} : (\overline{U}, \partial U) \to (E, E - \{0\})$, and hence by (4.1), $d(\widehat{f}, U)$ is odd.

On the other hand, since there obviously exists a ray R_0 originating at 0 with $R_0 \cap \widehat{f}(\partial U) = \emptyset$, we have $d(\widehat{f}, U) = 0$ by (3.5); this contradiction completes the proof. □

We now establish an extension of the Borsuk–Ulam theorem:

(4.4) Theorem. *Let $U \subset E$ be a bounded centrally symmetric domain with $0 \in E$, and $f : \partial U \to E_0$ a compact field, where $E_0 \subset E$ is a closed linear subspace of codimension 1. Then $f(x_0) = f(-x_0)$ for some $x_0 \in \partial U$.*

PROOF. Write $f(x) = x - F(x)$ and let

$$g(x) = \frac{f(x) - f(-x)}{2} = x - \frac{F(x) - F(-x)}{2} \qquad \text{for } x \in \partial U.$$

Since g is a compact field with $g(\partial U) \subset E_0$ and g is odd on ∂U, the conclusion follows from (4.3). □

5. The Leray–Schauder Index in Locally Convex Spaces

This section develops the Leray–Schauder fixed point index for compact maps in locally convex spaces. Throughout this section, E will denote a locally convex linear topological space and $\mathscr{V}$ the local base of convex symmetric nbds of the origin in E. Given $V \in \mathscr{V}$ and $x \in E$, we let $V(x) = x + V$ and denote by p_V the seminorm determined by V

(5.1) DEFINITION. Let $N = \{c_1, \dots, c_n\}$ be a finite subset of E, V a convex symmetric nbd of 0 in E, and

$$N_V = \bigcup_{i=1}^{n} V(c_i).$$

For each $i \in [n]$, define $\mu_i : N_V \to \boldsymbol{R}$ by $x \mapsto \max\{0, 1 - p_V(x - c_i)\}$, and let $\mu(x) = \sum_{i=1}^{n} \mu_i(x)$. The *Schauder projection* $\pi_V : N_V \to \operatorname{conv} N$ is given by

$$\pi_V(x) = \frac{1}{\mu(x)} \sum_{i=1}^{n} \mu_i(x) c_i.$$

Note that π_V is well defined, because each $x \in N_V$ belongs to some $V(c_i)$ and $\mu_i(x) \neq 0 \Leftrightarrow x \in V(c_i)$, so that $\mu(x) \neq 0$.

(5.2) PROPOSITION. *Let $c_1, \dots, c_n$ belong to some convex $C \subset E$, $V \in \mathscr{V}$ and π_V be the Schauder projection. Then:*
(a) *$\pi_V : N_V \to \operatorname{conv} N \subset C$ is a compact map,*
(b) *$\pi_V(x) \in V(x)$ for all $x \in N_V$.*

PROOF. (a) is obvious; to prove (b), note that letting $N_x = \{c_i \mid c_i \in V(x)\}$, we see that $\pi_V(x) \in \operatorname{conv} N_x \subset V(x)$ for each $x \in N_V$, and thus the proof is complete. □

This leads to the basic

(5.3) THEOREM (Approximation theorem). *Let X be a space, C a convex subset of a locally convex space E, and $f : X \to C$ a compact map. Then for each $V \in \mathscr{V}$ there exists a finite set $N = \{c_1, \dots, c_n\} \subset f(X) \subset C$ and a finite-dimensional map $f_V : X \to C$ such that:*

(a) $f_V(x) - f(x) \in V$ *for all* $x \in X$,
(b) $f_V(X) \subset \operatorname{conv} N \subset C$,
(c) *the maps* f *and* f_V *are* V*-homotopic.*

PROOF. As $f(X)$ is relatively compact, there is a finite set $N = \{c_1, \dots, c_n\} \subset f(X)$ with $f(X) \subset N_V$. Define $f_V : X \to C$ by $x \mapsto \pi_V \circ f(x)$, where $\pi_V : N_V \to \operatorname{conv} N$ is the Schauder projection; according to (5.2), the map f_V has the required properties (a) and (b). Since for any $x \in X$ the interval $(1-t)f(x) + t\pi_V f(x)$ is entirely contained in $V(f(x))$, the maps f and f_V are clearly V-homotopic. □

Let $U \subset E$ be open and $\mathscr{K}_{\partial U}(\overline{U}, E)$ be the set of all compact maps $f : \overline{U} \to E$ with $\mathrm{Fix}(f) \subset U$. Then, since $\mathrm{Fix}(f)$ is compact, there exists a continuous seminorm p such that $\eta(f) = \inf\{p(x - f(x)) \mid x \in \partial U\} > 0$.

Using Schauder projections, for each $\varepsilon < \frac{1}{2}\eta(f)$ we find a finite-dimensional map $f_\varepsilon \in \mathscr{K}_{\partial U}(\overline{U}, E)$ with $p(f_\varepsilon(x) - f(x)) < \varepsilon$ for all $x \in \overline{U}$ and such that for any two such approximations f_ε, $f_{\varepsilon'}$, the family of maps $H_t(x) = (1-t)f_\varepsilon(x) + tf_{\varepsilon'}(x)$ is a compact homotopy in $\mathscr{K}_{\partial U}(\overline{U}, E)$ joining f_ε and $f_{\varepsilon'}$. This leads to the following

(5.4) DEFINITION. Let $f \in \mathscr{K}_{\partial U}(\overline{U}, E)$, and let p be a continuous seminorm on E such that $\eta(f) = \inf\{p(x - fx) \mid x \in \partial U\} > 0$. Choose any finite-dimensional approximation $g \in \mathscr{K}_{\partial U}(\overline{U}, E)$ of f with $p(f(x) - g(x)) < \eta/2$ for $x \in \overline{U}$ and $g(\overline{U}) \subset L$, where L is a finite-dimensional linear subspace of E. The *Leray–Schauder index* of f on U is

$$i(f, U) = i(g|U \cap L, U \cap L).$$

This definition is independent of the choice of the finite-dimensional approximation. For if $g_i : U \to E$, $i = 1, 2$, are two such approximations with $g_i(\overline{U}) \subset L_i$, then taking a finite-dimensional subspace $L \supset L_1 \cup L_2$, by the contraction property we find first that

$$i(g_i|U \cap L, U \cap L) = i(g_i|U \cap L, U \cap L_i), \qquad i = 1, 2,$$

and then, because g_1 and g_2 are homotopic, that they have the same index on $U \cap L$:

$$i(g_1|U \cap L, U \cap L) = i(g_2|U \cap L, U \cap L).$$

Using the properties of the index in finite-dimensional spaces, we obtain

(5.5) THEOREM. *Let* $U \subset E$ *be open. Then the integer-valued function* $f \mapsto i(f, U)$ *for* $f \in \mathscr{K}_{\partial U}(\overline{U}, E)$ *has the following properties:*

(I) (Normalization) *If* $f \in \mathscr{K}_{\partial U}(\overline{U}, E)$ *is a constant map* $u \mapsto u_0$, *then* $i(f, U) = 1$ *or* 0, *depending on whether or not* $u_0 \in U$.

(II) (Additivity) *If* $f \in \mathscr{K}_{\partial U}(\overline{U}, E)$ *and* $\mathrm{Fix}(f) \subset U_1 \cup U_2 \subset U$ *with* U_1, U_2 *open and disjoint, then* $i(f, U) = i(f, U_1) + i(f, U_2)$.

(III) (Homotopy) *If* $h_t : \overline{U} \to E$ *is an admissible compact homotopy in* $\mathscr{K}_{\partial U}(\overline{U}, E)$, *then* $i(h_0, U) = i(h_1, U)$.
(IV) (Existence) *If* $i(f, U) \neq 0$, *then* $\mathrm{Fix}(f) \neq \emptyset$.
(V) (Excision) *If* V *is an open subset of* U *and* $f \in \mathscr{K}_{\partial U}(\overline{U}, E)$ *has no fixed points in* $U - V$, *then* $i(f, U) = i(f, V)$. □

Next we define the index for compact and compactly fixed maps:

(5.6) DEFINITION. Let $U \subset E$ be open, let $\mathscr{F}(U, E)$ be the set of all compactly fixed maps $f : U \to E$, and let $f \in \mathscr{F}(U, E)$ be compact. We define the *Leray Schauder index* $I(f) = I(f, U)$ of f by

$$I(f, U) = i(f, V),$$

where V is any nbd of $\mathrm{Fix}(f)$ such that $V \subset \overline{V} \subset U$.

By the excision property of i, it follows that this definition is independent of the choice of V, and thus $I(f)$ is well defined.

(5.7) THEOREM. *Let* C *be the category of locally convex spaces, and let* $\mathscr{F}$ *be the class of all compact maps* $f \in \mathscr{F}(U, E)$ *for all* $E \in C$ *and all* $U \subset E$ *open. Then the Leray-Schauder index* $I : \mathscr{F} \to Z$ *has all the properties* (I)–(VII) *of* (12.2.1), *provided that in* (III) *all homotopies are assumed to be compact and in* (VII) *it is required that either both* f *and* gf *be compact, or both* g *and* fg *be compact.*

PROOF. Properties (I)–(V) follow in a straightforward manner from the corresponding properties of the index function i. The multiplicativity of I follows from the corresponding property of i in (12.2.2). The proof of the commutativity consists of two parts as in Section 4 of §12. The proof of the first part is strictly analogous to that in §12; the proof of the second part also follows (with appropriate modifications) the steps of the second part of the proof in §12; because it is long and technical, it is omitted. □

We now show that still more generally, the Leray–Schauder index can be extended to compact and compactly fixed maps $f : U \to X$, where $U \subset X$ is open and X is a nbd retract of a locally convex space E. Let $r : V \to X$ be a retraction, where V is open in E, and let $f \in \mathscr{F}(U, X)$. Consider the composition

$$r^{-1}(U) \xhookrightarrow{r} U \xrightarrow{f} X \xhookrightarrow{j} V,$$

where $j : X \hookrightarrow V$ is the inclusion, and define

$$I(f, U) = I(jfr, r^{-1}(U)).$$

This is independent of the particular V and r selected: if $\varrho : W \to X$ is a retraction of another neighborhood onto X, and $\eta : X \hookrightarrow W$ is the inclusion,

then by commutativity,

$$\begin{aligned} I(jfr, r^{-1}(U)) &= I((j\varrho)(\eta fr), r^{-1}(U)) = I((\eta fr)(j\varrho), (j\varrho)^{-1}r^{-1}(U)) \\ &= I(\eta f\varrho, \varrho^{-1}(U)), \end{aligned}$$

because $\varrho\eta = rj = \mathrm{id}$. (We have used the fact that ηfr and $(j\varrho)(\eta fr)$ are compact, so that the commutativity property could be applied.)

We are now ready to state the main result of this section:

(5.8) THEOREM. *Let C be the category of spaces that are r-dominated by open sets in locally convex linear topological spaces, and let $\mathscr{F}$ be the class of all compact maps $f \in \mathscr{F}(V, X)$ for all $X \in C$ and $V \subset X$ open. Then the fixed point index $I : \mathscr{F} \to Z$ defined by* $(*)$ *has all the properties* (I)–(VII) *of* (12.2.1), *provided that in* (III) *it is assumed that all homotopies are compact and in* (VII) *it is required that either f and gf be compact, or g and fg be compact.* □

A compact space X is said to be an *ANR for normal spaces* if X embedded in any normal space Y is a nbd retract of Y. Let X be a compact ANR for normal spaces. Since any such X can be embedded in a locally convex space, we may assume that X is a subset of a locally convex space E. It can be proved (see (17.4.4)) that the linear span of X in E is normal, and therefore X is r-dominated by an open set in a locally convex space.

As a corollary of Theorem (5.8) we obtain:

(5.9) THEOREM (Index for compact nonmetrizable ANRs). *Let C be the category of compact ANRs for normal spaces, and let $\mathscr{F}$ be the class of all maps $f \in \mathscr{F}(U, X)$ for all $X \in C$ and $U \subset X$ open. Then there exists a fixed point index function $I : \mathscr{F} \to Z$ with properties* (I)–(VII). □

6. Miscellaneous Results and Applications

A. Kronecker characteristic

Throughout this subsection we let $\mathfrak{C}_0$ denote the class of maps defined by the condition: $f \in \mathfrak{C}_0$ if and only if $f : \partial U \to E - \{0\}$ is a compact field, where U is an open bounded subset of a normed linear space E. If $f \in \mathfrak{C}_0$, the *Kronecker characteristic* of f is the integer $\gamma(f, U)$ defined by $\gamma(f, U) = d(\widehat{f}, U)$, where $\widehat{f} \in \mathfrak{C}(\overline{U})$ is a compact field such that $\widehat{f}|\partial U = f$.

(A.1) Establish the following properties of the function $f \mapsto \gamma(f, U)$ for $f \in \mathfrak{C}_0$:

(I) (*Normalization*) If $x_0 \notin \partial U$ and $f(x) = x - x_0$ for $x \in \partial U$, then $\gamma(f, U) = 1$ or 0, depending on whether $x_0 \in U$ or not.

(II) (*Additivity*) Let $U \subset E$ be open bounded, and let $U_1, \dots, U_k$ be disjoint open subsets of U. Let $\widehat{f} : \overline{U} \to E$ be a compact field with $\widehat{f}(\overline{U} - \bigcup_{i=1}^{k} U_i) \subset E - \{0\}$. Then $\gamma(f, U) = \sum_{i=1}^{k} \gamma(\widehat{f}|\partial U_i, U_i)$.

(III) (*Homotopy*) If $h_t : \partial U \to E - \{0\}$ is a homotopy of compact fields, then $\gamma(h_t, U)$ is independent of $t \in I$.

(A.2) Let $f : \partial U \to E - \{0\}$ be a compact field with $\gamma(f, U) \neq 0$. Prove: Each ray R_0 in E originating at zero intersects $f(\partial U)$.

(A.3) A compact field $f : \partial U \to E - \{0\}$ is called *inessential* if it can be extended over $\overline{U}$ to a compact field $\widehat{f}$ $\overline{U} \to E - \{0\}$; otherwise, f is called *essential*. Prove: If U is connected, then f is inessential if and only if $\gamma(f, U) = 0$.

(A.4) (*Antipodal theorem*) Let U be a bounded centrally symmetric domain with $0 \in U$. Show: If $f : \partial U \to E - \{0\}$ is an odd compact field, then $\gamma(f, U)$ is odd.

(A.5) (*Generalized Borsuk-Ulam theorem*) Let U be as in (A.4), and let $f : \partial U \to E_0$ be a compact field, where $E_0 \subset E$ is a closed linear subspace of codimension 1. Prove: f sends at least one pair of antipodal points to the same point, i.e., $f(x) = f(-x)$ for some $x \in \partial U$.

(A.6) (*Hopf Rothe theorem*) Let $U = B(0, 1)$ be the open unit ball in E and $S = \partial U$. Let $f, g : S \to E - \{0\}$ be two compact fields with $\gamma(f, U) = \gamma(g, U)$. Prove: The compact fields f and g are homotopic.

[(i) Assuming first that f and g are finite-dimensional, let $F : S \to E$ and $G : S \to E$ be the associated finite-dimensional maps. Take a linear subspace $E_0 \subset E$ with $\dim E_0 < \infty$, $F(S), G(S) \subset E_0$, let $U_0 = U \cap E_0$, $\partial U_0 = S_0$ and $f_0 = f|S_0$, $g_0 = g|S_0$; note that for $f_0, g_0 : S_0 \to E_0 - \{0\}$, $\gamma(f_0, U_0) = \gamma(g_0, U_0)$.

(ii) Using the Hopf theorem (9.8.3), show that the maps $f_0, g_0 : S_0 \to E_0 - \{0\}$ are homotopic. This implies that if we write $f_0(x) = x - F_0(x)$, $g_0(x) = x - G_0(x)$, then there exists a fixed point free homotopy $\widehat{H} : S_0 \times I \to E_0$ such that $f_0(x) = \widehat{H}(x, 0)$, $G_0(x) = \widehat{H}(x, 1)$ for $x \in S_0$.

(iii) Letting $\pi_0 : E \to E$ be a linear projection onto E_0, define $h : S \times I \to E - \{0\}$ by

$$h(x, t) = \begin{cases} x - \|\pi_0(x)\| \widehat{H}(\|\pi_0(x)\|^{-1} \pi_0(x), t) & \text{if } \pi_0(x) \neq 0, \\ x & \text{otherwise.}] \end{cases}$$

B. Some geometric applications of the Leray Schauder degree

In this subsection E stands for a normed linear space with $\dim E > 1$, and X, Y for proper (i.e., nonempty and different from the entire space) subsets of E. We denote by **C** the category whose objects are closed proper subsets X of E and whose morphisms are compact fields f $X \to Y$; for the terminology concerning compact fields, see §6. By a *special compact field* is meant a field $f : X \to E$ such that for some finite-dimensional linear subspace $E_0 \subset E$, there exists a continuous linear projection $p : E \to E_0$ and a compact $F_0 : E_0 \to E_0$ satisfying $f(x) = x - F_0(p(x))$ for $x \in X$.

(B.1) Let $X \subset E$ be closed, and let Z be a compact subset of a finite-dimensional linear subspace of E. Show: For each compact field $f : X \to E - Z$ there exists a special compact field $f_0 : X \to E - Z$ such that f_0 is homotopic to f.

(B.2) Let $X \subset E$ be closed, and let $f : E \to E$ be a compact field such that $f|X = f' : X \to E - \{0\}$. Show: There is a compact field $g : E \to E$ such that:

(i) $g^{-1}(0) = f^{-1}(0)$ and $g(x) = x$ outside a bounded set,

(ii) $g|X = g' . X \to E - \{0\}$,

(iii) g' is homotopic to f'

(B.3)* Let E be finite-dimensional, and let $X \subset E$ be compact such that $S = \{x \in E \mid \|x\| = r\} \subset X$. Show: If $f, g : X \to E - \{0\}$ are homotopic and $f(x) = g(x) = x$ for all $x \in S$, then there exists a homotopy $h : X \times I \to E - \{0\}$ joining f and g such that $h(x,t) = x$ for $(x,t) \in S \times I$.

[Use the standard retraction of $E - \{0\}$ on S and the Borsuk homotopy extension theorem for a suitable subset of $X \times I \times I$.]

(B.4) Let $U \subset E$ be open, $X \subset E$ closed, $b \in E$, and let $g : \partial U \to X$, $f : X \to E - \{b\}$ be two compact fields.

(a) Let K be a connected open set in $E - g(\partial U)$. For every $y \in K$ let $\hat{g} : E \to E$ be a compact field extending $g : \partial U \to E - \{y\}$ over E, and let $d(g, U, y) = d(\hat{g}, U, y)$. Show: The integer-valued function $y \mapsto d(g, U, y)$, $y \in K$, is well defined and is constant on K.

(b) Let $\mathcal{K}$ be the set of components K of $E - X$ and $\mathcal{K}_0 = \{K \in \mathcal{K} \mid d(f, K, b) \neq 0\}$. Show:

(i) The set $\mathcal{K}_0$ is finite and $\sum_{K \in \mathcal{K}_0} d(f, K, b) = 1$.

(ii) $d(fg, U, b) = \sum_{K \in \mathcal{K}_0} d(g, U, K) d(f, K, b)$, where $d(g, U, K)$ is the value of the constant function defined in (a).

[Use (10.10.A.7).]

(B.5) (*The functor* π) For X in $\mathbf{C}$, let $\pi(X)$ be the set of homotopy classes $[u]$ of fields $u : X \to E - \{0\}$, and for $f \in \mathbf{C}(X, Y)$ let $\pi(f) : \pi(Y) \to \pi(X)$ be given by

$$\pi(f)([v]) = [vf] \qquad \text{for every field } v : X \to E - \{0\}.$$

Show: π is a contravariant functor from $\mathbf{C}$ to the category **Ens**.

(B.6) (*The functor* σ) For X, Y in $\mathbf{C}$, let $\mathcal{K}$ be the set of components of $E - X$, and $\mathcal{L}$ the set of components of $E - Y$. Let $\sigma(X) = \{\chi : \mathcal{K} \to Z \mid \chi(K) \neq 0$ only for a finite number of $K \in \mathcal{K}$ and $\sum_{K \in \mathcal{K}} \chi(K) = 1\}$. For $f \in \mathbf{C}(X, Y)$, we define $\sigma(f) : \sigma(Y) \to \sigma(X)$ as follows: for $\eta \in \sigma(Y)$, we let $\chi = \sigma(f)(\eta)$ be given by

$$\chi(K) = \sum_{L \in \mathcal{L}} d(f, K, L) \eta(L) \qquad \text{for } K \in \mathcal{K}.$$

Show: (a) $\sigma(f)$ is well defined, (b) σ is a contravariant functor from $\mathbf{C}$ to **Ens**.

[Use (B.4).]

(B.7) (*The natural transformation* μ) For X in $\mathbf{C}$, let $\mu(X) : \pi(X) \to \sigma(X)$ be the map that assigns to $[u] \in \pi(X)$ the integer-valued function $\chi = \pi(X)([u]) : \mathcal{K} \to Z$ with $\chi(K) = d(u, K)$ for a component $K \in \mathcal{K}$ of the set $E - X$. Show: $\mu : \pi \to \sigma$ is a natural transformation of functors.

[Use (B.4).]

(B.8)* Let E be of finite dimension greater than 1, let $X \subset E$ be compact, and let $\mathcal{K}$ be the set of all bounded components of $E - X$. Prove:

(a) For every function $\chi : \mathcal{K} \to Z$ with $\chi(U) \neq 0$ only for a finite number of $U \in \mathcal{K}$, there exists a map $f : X \to E - \{0\}$ such that $d(f, U) = \chi(U)$ for every $U \in \mathcal{K}$.

(b) If $f, g : X \to E - \{0\}$ are such that $d(f, U) = d(g, U)$ for every $U \in \mathcal{K}$, then f is homotopic to g.

[For (a): For every $U \in \mathcal{K}$ with $\chi(U) \neq 0$ choose an open ball $B \subset U$ and a map $f : \overline{B} \to E$ with $d(f, B) = \chi(U)$. Extend this map over E in such a way that the possible new zeros all lie outside a ball containing X.

For (b): Use the fact that the set of homotopy classes of maps $X \to E - \{0\}$ has the structure of an abelian group: if $\dim E = 2$, then $E - \{0\}$ is the multiplicative group of

complex numbers; for $\dim E \geq 3$, see Kuratowski's monograph [1968], Vol. II, Sect. 60. It is sufficient to prove that if $d(f, U) = 0$ for every $U \in \mathcal{K}$, then f is homotopic to a constant map. Such an f extends to a map $f_1 : B \to E - \{0\}$ of a ball B containing X by concentration of zeros in balls contained in $U \in \mathcal{K}$, and their elimination by the Hopf theorem; f_1 is nullhomotopic by contractibility of B.]

(B.9)* Prove: The transformation $\mu : \pi \to \sigma$ is a natural equivalence.

[To show that $\mu(X) : \pi(X) \to \sigma(X)$ is surjective, use (B.8)(a) for a suitable finite-dimensional linear subspace E_0 of E, the Borsuk homotopy extension theorem, and a linear projection of E onto E_0. To show that $\mu(X)$ is injective, choose a suitable finite-dimensional linear subspace E_1 of E and a sufficiently large ball $B_1 \subset E_1$. Use (B.2), (B.8)(b), (B.3) and properties of the degree.]

(B.10) (*Theorem on disconnection of E*) Let X be a closed subset of E. Show: X does not disconnect E if and only if any two compact fields $f, g : X \to E - \{0\}$ are homotopic.

(B.11) For $X, Y \subset E$ closed, we say that X is *h-dominated* by Y if there are compact fields $f : X \to Y$ and $g : Y \to X$ such that $gf : X \to X$ is homotopic to 1_X; in this case we say that g is a *left homotopy inverse* of f, and f is a *right homotopy inverse* of g. Prove:

(a) If X is h-dominated by Y, then the number of components of $E - X$ is not greater than the number of components of $E - Y$

(b) If a compact field $f : X \to Y$ has a left homotopy inverse g, then for every component K of $E - X$ there exists a component L of $E - Y$ such that $d(f, K, L) \neq 0$.

[Consider $d(f, K, L)$ and $d(g, L, K)$ as entries of matrices of linear maps between real linear spaces and use properties of the degree.]

(B.12) (*Generalized Leray theorem*) Closed subsets X, Y of E are called *h-equivalent* whenever X is h-dominated by Y and Y is h-dominated by X. Show: If X and Y are h-equivalent, then their complements $E - X$ and $E - Y$ have the same number of components.

(B.13) Let $X, Y \subset E$ be closed, $f : X \to Y$ a compact field having a left homotopy inverse, $\mathcal{K}_0$ a subset of the set of components of $E - X$, $D = \bigcup\{K \mid K \in \mathcal{K}_0\}$, and $A = X \cup D$. Show: If a compact field $\hat{f} : A \to E$ is an extension of f such that $\hat{f}(D)$ and Y are disjoint, then for $K \in \mathcal{K}_0$ the images $\hat{f}(K)$ are distinct components of $E - Y$ and $d(f, K, \hat{f}(K)) = \pm 1$.

(B.14) (*Generalized invariance of domain*) Let $U \subset E$ be open and connected, and let $f : \overline{U} \to E$ be a compact field such that $f|\partial U : \partial U \to E$ is injective and $f(\partial U) \cap f(U) = \emptyset$. Show: $f(U)$ is open and connected and $f(\partial U) = \partial f(U)$.

(B.15) If $U \subset E$ is open, we say that $\overline{U}$ is *regularly shrinkable* whenever there exists a deformation $d : \overline{U} \times I \to \overline{U}$ such that $d(x, t) \in U$ for $(x, t) \in \overline{U} \times (0, 1]$ and $d(\overline{U}, 1) = u_0$ for some $u_0 \in U$; the closure of an open convex set is a regularly shrinkable set. Show: If $\overline{U}$ is regularly shrinkable and $F : \overline{U} \to E$ is compact with $F(\partial U) \subset \overline{U}$, then F has a fixed point.

(Most of the results of this subsection, which are also valid in locally convex spaces, are due to Bowszyc [1970]; for (B.14) see also J.H. Michael [1957].)

C. Degree for representations of maps

In this subsection we let X denote an ANR and Y a locally connected bounded metric space.

By an *admissible class* $\mathscr{M}$ of homeomorphisms from X to Y is meant a family of homeomorphisms $h_U : \overline{U} \to \overline{h(U)}$, for U open in X, satisfying:

(a) h_U maps U homeomorphically onto $h(U)$ open in Y,

(b) for each V open in U, $h_U|\overline{V}$ is in $\mathscr{M}$.

If $\mathscr{M}$ is admissible and $U \subset X$ is open, then we denote by $\mathscr{M}_U$ the subfamily of $\mathscr{M}$ of homeomorphisms h defined on $\overline{U}$; $\mathscr{M}_U$ is equipped with the sup metric. Throughout this subsection we let $\mathscr{M}$ be an admissible family of homeomorphisms.

Let $U \subset X$ be open and $f : U \to Y$ a map. By a *representation* of f with respect to $\mathscr{M}$ is meant a pair (W, F), where $W \subset X \times X$ is open and $F : W \to Y$ has the following properties:

(i) The diagonal $\Delta_U = \{(x,x) \mid x \in U\} \subset W$ and $f(x) = F(x,x)$ for each $x \in U$.

(ii) For each $v \in X$, let $W_v = \{u \in X \mid (u,v) \in W\}$ and $F_v(u) = F(u,v)$. Then $F_v : W_v \to Y$ is a homeomorphism onto an open subset of Y, belonging to $\mathscr{M}$.

(iii) For any $(u_0, v_0) \in W$, there are nbds N_{u_0} and N_{v_0} of u_0 and v_0, respectively, in X with $\overline{N}_{u_0} \times \overline{N}_{v_0} \subset W$ such that $F_v(\overline{N}_{u_0})$ is closed in Y for each $v \in N_{v_0}$ and $v \mapsto F_v|\overline{N}_{u_0}$ is a continuous compact map from N_{v_0} to $\mathscr{M}_{N_{u_0}}$

(C.1)* Let $U \subset X$ be open, $f : U \to Y$ a map, and assume that f has a representation (W, F) with respect to $\mathscr{M}$. For any $y \in Y$ let $U_y = \{v \in U \mid y \in F_v(W_v)\}$ and define $T_y(v) = F_v^{-1}(y)$ for $v \in U_y$. Show that for each $y \in Y$:

(i) U_y is open in U,

(ii) $T_y : U_y \to X$ is continuous and locally compact,

(iii) $\mathrm{Fix}(T_y) = \{v \in U \mid f(v) = y\}$, and $\mathrm{Fix}(T_y)$ is compact if f is proper.

(C.2)* Let $U \subset X$ be open and $f : U \to Y$ be a proper map having a representation (W, F) with respect to $\mathscr{M}$. Let $y \in Y$ and let V be an open nbd of $f^{-1}(y)$ in U such that $T_y|V : V \to X$ is compact. For the pair $[f, F]$, we define a degree function by

$$\deg_{\mathscr{M}}([f,F], U, y) = I(T_y, V).$$

(a) Show: The definition of $\deg_{\mathscr{M}}([f,F], U, y)$ is independent of the choice of V

(b) Establish the following properties of $\deg_{\mathscr{M}}([f,F], U, y)$:

(i) (*Existence*) If $\deg_{\mathscr{M}}([f,F], U, y) \neq 0$, then $f(x) = y$ for some $x \in U$.

(ii) (*Additivity*) If $U_1, U_2 \subset U$ are open and $f^{-1}(y) \subset U_1 \cup U_2$, then

$$\deg_{\mathscr{M}}([f,F], U, y) = \deg_{\mathscr{M}}([f,F], U_1, y) + \deg_{\mathscr{M}}([f,F], U_2, y).$$

(iii) (*Homotopy*) Let $h_t : U \to Y$, $t \in I$, be a homotopy and $\{y_t\}_{t \in I}$ a curve in Y such that $h_t^{-1}(y_t) \in U$ for all $t \in I$. For an appropriate representation (W, H_t) of h_t with respect to $\mathscr{M}$, $\deg_{\mathscr{M}}([h_t, H_t], U, y_t)$ is independent of $t \in I$.

(iv) (*Normalization*) If the representation $(u,v) \mapsto F(u,v)$ of f is independent of v, and $y \in f(U)$, then $\deg_{\mathscr{M}}([f,F], U, y) = 1$.

(C.3)* (*Browder-Caccioppoli theorem*) Let E, $\widehat{E}$ be Banach spaces, $\mathscr{L}(E, \widehat{E})$ the Banach space of continuous linear maps from E to $\widehat{E}$, and $\mathscr{L}_c(E, \widehat{E})$ the subspace of completely continuous linear maps. Assume that $f : E \to \widehat{E}$ is a proper C^1 map such that:

(i) For any compact convex $C \subset E$ there exists a continuous map $\varphi : C \to \mathscr{L}_c(E, \widehat{E})$ such that for every $v \in C$ the map $Df_v + \varphi(v) \in \mathscr{L}(E, \widehat{E})$ is an isomorphism.

(ii) There exists a point $y_0 \in \widehat{E}$ such that $f^{-1}(y_0) = \{x_1, \ldots, x_{2k+1}\}$ has an odd number of elements and Df_{x_i} is an isomorphism for each $i \in [2k+1]$.

Show: f is surjective.

(The above results are due to Browder [1969];

D. Brouwer degree via differential forms

This subsection requires the knowledge of the theory of differential forms. We recall briefly the terminology and notation, and refer for details to the book of Bott and Tu [1982].

Let Λ^* be the associative algebra over $\boldsymbol{R}$ generated by elements $1, dx_1, \dots, dx_n$ subject to the relations: (i) 1 is a multiplicative identity, (ii) $dx_i \cdot dx_j = -dx_j \cdot dx_i$ for all i, j. The algebra Λ^* has a natural grading by the degree of monomials:

$$\Lambda^* = \bigoplus_{j=0}^{n} \Lambda^j,$$

where $\Lambda^0 = \boldsymbol{R} \cdot 1 = \boldsymbol{R}$ and each Λ^j is a finite-dimensional vector space over $\boldsymbol{R}$.

If $U \subset \boldsymbol{R}^n$ is open and $C^\infty(U) = \{f : U \to \boldsymbol{R} \mid f \text{ is of class } C^\infty\}$, the set of C^∞ differential forms on U is

$$\Lambda^*(U) = C^\infty(U) \otimes_{\boldsymbol{R}} \Lambda^*$$

If ω is such a form, then it can be uniquely written as

$$\omega = \sum_{k=0}^{n} \sum_{i_1 < \cdots < i_k} f_{i_1,\dots,i_k} dx_{i_1} \dots dx_{i_k},$$

where $f_{i_1,\dots,i_k} \in C^\infty(U)$; we write $\omega = \sum f_I dx_I$. The algebra $\Lambda^*(U) = \bigoplus_{k=0}^{n} \Lambda^k(U)$ is naturally graded; here $\Lambda^k(U)$ consists of the k-forms on U. There is a differential operator $d : \Lambda^k(U) \to \Lambda^{k+1}(U)$ given by:

(a) If $f \in \Lambda^0(U)$, then

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j.$$

(b) If $\omega = \sum f_I dx_I$, then $d\omega = \sum df_I dx_I$.

Because $d^2 = 0$, $\Lambda^*(U)$ is a cochain complex, and its cohomology groups are called the *de Rham cohomology* of U.

Let $U \subset \boldsymbol{R}^m$ and $V \subset \boldsymbol{R}^n$ be open and $f : U \to V$ be a C^∞ map: $x = (x_1, \dots, x_m) \mapsto f(x) = (y_1, \dots, y_n) = (f_1(x), \dots, f_n(x))$. The induced map $f^* : \Lambda^*(V) \to \Lambda^*(U)$ is defined by

$$f^*\left(\sum g_{i_1,\dots,i_k} dy_{i_1} \dots dy_{i_k}\right) = \sum g_{i_1,\dots,i_k} \circ f \, df_{i_1} \dots df_{i_k}.$$

In particular, if $m = n$, for an n-form $\omega = g(y) dy_1 \dots dy_n$ we have

$$f^*(\omega) = \sum_{k=1}^{n} g(f(x)) J_f(x) dx_1 \dots dx_n,$$

where $J_f(x) = \det f'_x$ is the Jacobian of f at x. Because f^* commutes with d, f^* is in fact a cochain transformation of $\Lambda^*(V)$ to $\Lambda^*(U)$.

The notions of differential form, differential operator, and induced map as well as their properties extend in a natural way to the category of manifolds.

Recall that a C^∞ manifold M of dimension n with boundary is given by an atlas $\{U_\alpha, \varphi_\alpha\}$, where φ_α is a homeomorphism of $U_\alpha \subset M$ either on $\boldsymbol{R}^n$ or on the half-space $H^n = \{(x_1, \dots, x_n) \mid x_1 \le 0\}$ such that each $\varphi_\alpha \varphi_\beta^{-1}$ is a C^∞ map. The boundary ∂M is an $(n-1)$-dimensional manifold. An oriented atlas $\{U_\alpha, \varphi_\alpha\}$ (i.e., one with all $\varphi_\alpha \varphi_\beta^{-1}$ having positive Jacobian) induces an oriented atlas on ∂M. For an n-dimensional oriented manifold M and an n-form η on M with compact support there is defined the integral

$\int_M \eta$. A basic result due to Stokes asserts that if ω is an $(n-1)$-form with compact support on an oriented manifold M of dimension n, and if ∂M is given the induced orientation, then

$$\int_M d\omega = \int_{\partial M} i^*(\omega),$$

where $i : \partial M \to M$ is the inclusion.

(D.1)* (*Brouwer degree between smooth manifolds*) In what follows, M, N stand for (paracompact) oriented C^∞ manifolds without boundary of the same dimension n; $U \subset M$ is open with compact closure $\overline{U}$; $f : \overline{U} \to N$ a C^∞ map; and $b \in N - f(\partial U)$. An open nbd V of b is called *Euclidean* if it is diffeomorphic to $\boldsymbol{R}^n$. All differential forms on M are assumed to be of class C^∞.

(a) Let η be a differential n-form with compact support on $\boldsymbol{R}^n$ (or on a Euclidean nbd V). Show: There exists a differential $(n-1)$-form ω with compact support such that $\eta = d\omega$ if and only if $\int \eta = 0$.

[For necessity, use the Stokes theorem, and for sufficiency, apply induction on n; cf. Nirenberg's lecture notes [1974], Lemma 1.3.3.]

(b) Let V be a Euclidean nbd of $b \in N - f(\partial U)$, and let η be a differential n-form with compact support contained in V such that $\int_N \eta = 1$. Prove: (i) The number $\deg(f, U, b) = \int_U f^*\eta$ does not depend on the choice of η; (ii) if $\deg(f, U, b_0) \neq 0$, then there exists $x_0 \in U$ such that $f(x_0) = b_0$.

(c) Show: (i) If b_0, b_1 belong to the same component of $N - f(\partial U)$, then $\deg(f, U, b_0) = \deg(f, U, b_1)$; (ii) if N is connected and $U = M$ is compact, then $\deg(f) = \deg(f, M, b_0)$ is independent of b_0.

(d) Prove: If η is an n-form with compact support contained in the component of $N - f(\partial U)$ containing b_0, then

$$\int_U f^*\eta = \deg(f, U, b_0) \int_N \eta.$$

(We remark that this equality provides an equivalent definition of the degree.)

[Use a C^∞ partition of unity on N.]

(D.2)* (*Properties of the degree*) (a) Let b be a *regular value* of f (i.e., if $f(x) = b$, then the derivative $Df(x)$ of f at x is an isomorphism of the appropriate tangent spaces). Show:

$$\deg(f, U, b) = \sum_j \operatorname{sgn} Df(x_j),$$

where $f^{-1}(b) = \{x_1, \ldots, x_k\}$ and $\operatorname{sgn} Df(x_j)$ is 1 or -1 according as $Df(x_j)$ preserves or reverses orientation. The Sard theorem implies that $\deg(f, U, b)$ is defined for some b.

(b) (*Additivity*) If U is the union of disjoint open subsets U_1 and U_2, and if $b \notin f(\overline{U} - (U_1 \cup U_2))$, then

$$\deg(f, U, b) = \deg(f, U_1, b) + \deg(f, U_2, b).$$

(c) (*General homotopy invariance*) Let $W \subset M \times I$ be open with compact closure, $I \ni t \mapsto b_t \in N$ a continuous path, and $F : \overline{W} \to N$ a C^∞ map. For $t \in I$ and any $S \subset M \times I$, set $S_t = \{x \in M \mid (x, t) \in S\}$, and let $F_t : (\overline{W})_t \to N$ be given by $F_t(x) = F(x, t)$. Assume that $b_t \notin F_t((\overline{W})_t - W_t)$ for every $t \in I$. Prove: $\deg(F_t, W_t, b_t)$ is independent of $t \in I$. In particular, if M is compact and N is connected, then $\deg(f, M, b)$ does not depend on $b \in N$ and is denoted simply by $\deg(f)$.

(d) Let P be an oriented $(n+1)$-dimensional C^∞ manifold with oriented boundary $\partial P = M$, V an open subset of P with compact closure and $F : \overline{V} \to N$ a C^∞ map. Set $X = V \cap M$ and $f = F|\bar{X} : \bar{X} \to N$. Assume $y_0 \in N - F(\partial V)$. Prove: $\deg(f, X, y_0) = 0$. (In particular, this may be applied when $V = P$ is compact; Milnor [1965].)

[Use the Stokes theorem.]

(e) (*Composition law*) Let M, N, P be oriented n-dimensional C^∞ manifolds without boundary such that M, N are compact and N, P are connected. Show: If $M \xrightarrow{f} N \xrightarrow{g} P$ are C^∞ maps, then $\deg(gf) = \deg(g)\deg(f)$.

[Use (d) of (D.1).]

(f) (*Multiplicativity*) Let M, M', N, N' be oriented C^∞ manifolds without boundary, $\dim M = \dim N$, $\dim M' = \dim N'$, and assume that $U \subset M$, $U' \subset M'$ are open subsets with compact closures. Show: If the maps $f : \overline{U} \to N$, $f' : \overline{U'} \to N'$ are of class C^∞ and $y_0 \in N - f(\partial U)$, $y_0' \in N' - f(\partial U')$, then

$$\deg(f \times f', U \times U', (y_0, y_0')) = \deg(f, U, y_0)\deg(f', U', y_0').$$

7. Notes and Comments

Bifurcation theory

Theorem (1.10), due to P. Rabinowitz and M. Krasnosel'skiĭ, is the basic classical result in bifurcation theory. Theorems (1.8) and (1.9), which appear in a slightly more general form in Nussbaum's lecture notes [1985], extend the Krasnosel'skiĭ-Rabinowitz theorem to maps in metric ANRs (e.g. cones).

The "local bifurcation" result (1.10)(a) is due to Krasnosel'skiĭ [1956]; (1.10)(b) was discovered by Rabinowitz [1971]; a remarkable feature of this result is that it provides information about global behavior from purely local data. The refinements of the Rabinowitz theorem given in (1.10)(c), (d) were found by Ize [1976]. All the above theorems were proved by using the Leray–Schauder degree theory.

For a direct proof of the Rabinowitz theorem using the Leray–Schauder degree, the reader may consult the books by Chow–Hale [1974], Brown [1953], and Nirenberg's lecture notes [1974]. For more general closely related results for compact maps $f : C \times J \to C$, where C is a closed convex set in a Banach space, see Dancer [1974], Nussbaum [1975], and Turner [1974].

For some applications of bifurcation results to Sturm–Louville problems and eigenvalue problems for elliptic partial differential equations the reader is referred to Rabinowitz's lecture notes [1975] and the books of Chow–Hale [1974] and Zeidler [1986], where further references can be found.

Applications to nonlinear PDEs

For (2.1) see Schauder [1927b]; the fact that the spaces $C^n(\Omega)$ are not sufficient for the problem $\Delta u = f$, $u|_{\partial\Omega} = 0$ was observed by Zaremba

[1911]; Hölder spaces in the abstract setting were considered for the first time by Schauder [1927a], [1932]. Many applications of the Schauder theorem for proving the existence of classical solutions of partial differential equations are treated in the book by Pogorzelski [1966]; for other applications and further references see Krasnosel'skiĭ's book [1964b] and Griffel [1985].

The scheme outlined at the beginning of Section 3 in §7 and used in the proof of (2.2) and (2.3) is an explicit and abstract formulation of the general approach used by Schauder [1927a], [1927b]. Application of this scheme to (2.3) (carried out by Schauder in [1933]) was an important mathematical achievement; it involved the development of the theory of so-called Schauder estimates for linear second order elliptic operators (Schauder [1932]). The same general ideas were exploited by Schauder in another important direction: for proving the existence of weak solutions of some nonlinear problems. This involved the study of the classes $W^{k,2}$, the derivation of Schauder estimates for linear differential operators in this setting and finally application of topological arguments (Schauder [1934], where the case of quasi-linear elliptic equations with continuous coefficients in $\boldsymbol{R}^2$ is outlined, and also Schauder [1935], where quasi-linear hyperbolic equations are treated).

A detailed and modern exposition of the theory of Schauder estimates and its applications to nonlinear problems can be found in the books of Gilbarg–Trudinger [1977] and Ladyzhenskaya–Ural'tseva [1964].

The Leray–Schauder method represents one of the most basic tools of nonlinear analysis; in Section 2 we indicate in a simple setting how this method may be used in the study of nonlinear partial differential equations.

The first fundamental results on quasi-linear second order equations of eliptic type were obtained by Bernstein [1912]; further significant progress was made through the work of Schauder and Leray (cf. Schauder [1932], [1933], Leray–Schauder [1934], Leray [1938], [1939]). In their joint memoir [1934] Leray and Schauder brought together a number of important ideas and developed a new method for solution of quasi-linear equations; their work, while embracing several preceding discoveries, provided also a general framework for the treatment of many other nonlinear problems. For an account of historical beginnings see the surveys by Schauder [1936a], [1936b] and Leray [1936], where a number of references to older literature can be found.

A modern presentation of the theory of second order quasi-linear elliptic equations, which has been brought to a significant level of completeness, can be found in the books of Gilbarg–Trudinger [1977] and Ladyzhenskaya–Ural'tseva [1964]. These books also present applications of the continuation method to the proofs of existence of generalized solutions of some nonlinear problems. Further literature can be found in Nirenberg's lecture notes [1974] and the survey by Serrin [1976].

The Leray–Schauder degree theory

We first remark that in an infinite-dimensional normed linear space E, there can be no degree theory applicable to all continuous maps: for if $\overline{U}$ is the closed unit ball in E, then there is a retraction $r : \overline{U} \to \partial U$; since $r|\partial U = \mathrm{id}\,|\partial U$, the normalization and homotopy properties would imply $d(r, U) = d(\mathrm{id}, U) = 1$, giving the contradiction that $r(x) = 0$ has a solution in U. The class of maps for which a degree or index theory can be defined must therefore be restricted.

In their 1934 memoir, Leray and Schauder singled out the class of compact fields $f(x) = x - F(x)$ in Banach spaces and showed that a degree (preserving all properties of degree in $\boldsymbol{R}^n$) could be defined for maps in this class. Theorems (3.6) and (3.7) were established by Leray–Schauder [1934]. Theorem (3.8) is due to Krasnosel'skiĭ–Perov [1959].

For a direct exposition of the Leray–Schauder degree theory, the reader may consult the books by Deimling [1985], Lloyd [1978], and Rothe [1986].

Fixed point index in locally convex spaces

The results of Section 5 are taken from Granas [1969], [1972]. The existence of the index for compact (nonmetrizable) ANRs given in (5.9) was established earlier by combinatorial means and in a different form by several authors (Leray [1959], Deleanu [1959], Bourgin [1955], [1956a,b], and Browder [1960]).

We remark that the Leray–Schauder degree for compact fields in locally convex spaces (Leray's survey [1950], Nagumo [1951b]) can be derived from the fixed point index as in the metrizable case (Section 3).

Extensions of the Leray–Schauder degree

We now list some of the numerous extensions of the Leray–Schauder degree theory that are not treated in this book.

(i) Degree for *condensing fields* in Banach spaces (Nussbaum [1972], Borisovich–Sapronov [1968], Sadovskiĭ's survey [1972], and Krawcewicz's book [1997]).

(ii) Degree for *A-proper maps*, motivated by the needs of the theory of monotone operators (Browder–Petryshyn [1969], Petryshyn's survey [1975], and his book [1995]).

(iii) Degree for monotone-like operators between a Banach space and its dual (Skrypnik's book [1973], Browder [1983], Browder's survey [1983b]).

(iv) Degree for *compact Kakutani fields* (Granas [1959], Hukuhara [1967], Cellina–Lasota [1969], and Ma [1972]).

Degree for generalized compact fields

Let E be a normed linear space, $X, Y \subset E$, and $f : X \to Y$ a map. We say that f is a *Rothe map* (or simply an *R-map*) if it is of the form $f(x) = \lambda(x)x - F(x)$ for $x \in X$, where $F : X \to E$ is compact and λ $X \to \boldsymbol{R}$ is a continuous function satisfying $0 < m \leq \lambda(x) \leq M$ for some some constants m, M and all $x \in X$. By an *R-homotopy* is meant a family $h_t(x) = \lambda(x,t)x + H(x,t)$, $(x,t) \in X \times \boldsymbol{I}$, such that $0 < m \leq \lambda(x,t) \leq M$ and $H . X \times \boldsymbol{I} \to E$ is compact (cf. Rothe [1939]).

Let U be a bounded open subset of E, $b \in E$, and let f $(\overline{U}, \partial U) \to (E, E - \{b\})$ be an R-map. Observe that if

$$F_0(x) = \frac{1-\lambda(x)}{\lambda(x)}\, b + \frac{1}{\lambda(x)}\, F(x), \qquad f_0(x) = x - F_0(x),$$

then $Z(f-b) = Z(f_0 - b)$. We then define the degree of f with respect to b by $d(f, U, b) = d(f_0, U, b)$. It can be verified that the degree for R-maps has the same properties as the Leray–Schauder degree for compact fields. For details, see Rothe's book [1986].

Degree for maps having a diagonal representation

If E, F are Banach spaces, a map g $E \to F$ has a *diagonal representation* if $g(x) = G(x,x)$, where $G : E \times E \to F$ is such that for each fixed $v \in E$, the equation $G(u,v) = 0$ has a unique solution $u = \Gamma(v)$ defining a completely continuous map $\Gamma : E \to E$. Maps of this type appeared first in Leray Schauder [1934]. For maps with a diagonal representation, the degree was developed by Zabreĭko Krasnosel'skiĭ [1967] and independently by Browder–Nussbaum [1968]. For details the reader may consult the book of Zabreĭko–Krasnosel'skiĭ and Browder [1968], [1976]; see also "Miscellaneous Results and Examples'

Coincidence degree

Let E, F be normed linear spaces, and L $E \to F$ a fixed linear Fredholm operator of index zero. If $U \subset E$ is open and bounded, and $N : \overline{U} \to F$ is a compact map, one can define, for maps of the form $L + N : (\overline{U}, \partial U) \to (F, F - \{b\})$, the *coincidence degree* $\deg(L + N, U, \{b\})$ with respect to U and b. This degree, developed by Mawhin [1972], retains the main properties of the Leray–Schauder degree. Extensions of the continuation theorems can also be established in this setting. For details, see Gaines Mawhin [1977].

Degree for Fredholm maps

We now describe, in a simple setting, a direct extension of the Leray–Schauder degree to Fredholm maps due to Smale [1965]. We first recall some terminology.

Let E, F be Banach spaces, U a domain in E, and $f : U \to F$ a C^1 map. An element $x \in U$ is called a *regular point* for f if $Df(x) \in \mathscr{L}(E, F)$ is surjective, and a *critical point* otherwise. The image of a critical point is a *critical value* of f. An element $y \in F$ is a *regular value* of f if it is not a critical value. The map f is called *Fredholm* if each differential $Df(x)$ is a Fredholm map, i.e., $Df(x) \in \Phi(E, F)$ for each $x \in X$. Since U is connected, the index $\operatorname{Ind} Df(x)$ is constant on U; its value (denoted by $\operatorname{ind} f$) is called the *index* of f. The set of all Fredholm C^q maps of index n is denoted by $\Phi_n C^q$.

Given a domain $U \subset E$, we say that a map f $(\overline{U}, \partial U) \to (E, E - \{0\})$ is *allowable* if f is proper and $f|U : U \to E$ is a C^1 Fredholm map of index 0. We let $\mathscr{C}_0(\overline{U}, E)$ denote the set of allowable maps f $(\overline{U}, \partial U) \to (E, E - \{0\})$. Call $f, g \in \mathscr{C}_0(\overline{U}, E)$ *homotopic* (written $f \sim g$) if there exists a proper map $h : (\overline{U}, \partial U) \times \boldsymbol{I} \to (E, E - \{0\})$ such that:

(i) $h|U \times I : U \times I \to E$ is of class C^2,
(ii) $Dh(x,t) \in \Phi_1(E \oplus R, E)$ for all $(x,t) \in U \times I$,
(iii) $h(x,0) = f(x)$, $h(x,1) = g(x)$ for all $x \in \overline{U}$.

The main tool for defining the mod 2 degree for allowable maps is the following:

THEOREM (Smale Sard). *If $f \in \mathscr{C}_0(\overline{U}, E)$, then the subset of $E - f(\partial U)$ consisting of the regular values of $f|U$ is open and dense in $E - f(U)$.*

Let $f \in \mathscr{C}_0(\overline{U}, E)$ and $y \in E$ be a regular value for $f|U$ with $\|y\| < \inf\{\|f(x)\| \mid x \in \partial U\}$. It follows that either $f^{-1}(y) = \emptyset$ or $f^{-1}(y) = \{x_1, \ldots, x_k\} \subset U$. The *mod* 2 *degree* $\deg(f; y)$ of f with respect to y is then defined to be, respectively, 0 or k mod 2. One shows that $\deg(f; y)$ does not depend on the choice of the regular value y and we set $\deg f = \deg(f; y)$. The main properties of the degree are:

(i) $\deg f \neq 0 \Rightarrow f(x) = 0$ for some $x \in U$,
(ii) $f \sim g \Rightarrow \deg f = \deg g$.

Using the mod 2 degree, one can extend the antipodal theorem to Fredholm maps: *Let $U \subset E$ be a bounded symmetric domain containing* 0 *and $f \in \mathscr{C}_0(\overline{U}, E)$ be odd (i.e., $f(x) = -f(-x)$ for $x \in \overline{U}$). Then* $\deg f \equiv 1 \pmod 2$. For details the reader is referred to Smale [1965] and Elworthy Tromba [1970]; for a proof of the antipodal theorem see Gęba-Granas [1983].

Degree for maps between Banach manifolds

Let E be a Banach space, $\mathscr{L}(E)$ the Banach algebra of bounded linear operators on E, and $\mathrm{GL}(E)$ the multiplicative subgroup of invertible elements. Let $\mathscr{K}(E) \subset \mathscr{L}(E)$ be the set of completely continuous linear operators and $\mathscr{L}_c(E)$ and $\mathrm{GL}_c(E)$ the subsets of $\mathscr{L}(E)$ and $\mathrm{GL}(E)$, respectively, of operators of the form $I + T$, where $T \in \mathscr{K}(E)$. Then $\mathrm{GL}_c(E)$ is a subgroup of $\mathrm{GL}(E)$. It is known that $\mathrm{GL}_c(E)$ has two components (Gęba [1968], where references to earlier more special results of A. Schwarz and Palais can be found; see also the survey by Borisovich Zvyagin–Sapronov [1977], Isnard [1973], and "Additional References") We denote the component containing the identity by $\mathrm{GL}_c^+(E)$ and the other by $\mathrm{GL}_c^-(E)$. Given a Banach manifold M, a *c-structure* on M is an admissible atlas $\{\varphi_i, U_i\}$ maximal with respect to the property: for any i, j, the differential $d(\varphi_j \varphi_i^{-1})$ at any point is in $\mathrm{GL}_c(E)$. The c-structure is *orientable* if it admits a subatlas for which the differentials actually lie in $\mathrm{GL}_c^+(E)$. An *orientation* is a subatlas maximal with respect to this property. A smooth map $f : M \to N$ between c-manifolds (i.e., manifolds with distinguished c-structures) is a *c-map* if for any local representative $\psi_j f \varphi_i^{-1}$ of f the differential $d(\psi_j f \varphi_i^{-1})$ at any point is in $\mathscr{L}_c(E)$. This implies that f is Fredholm of index zero.

Suppose f is a proper c-map between oriented manifolds M, N with N connected. Then the *oriented degree* of f is defined: By the Smale-Sard theorem, f has a regular value $y \in N$ and $f^{-1}(y) \subset M$ consists of a finite number of points. Counting these points with proper signs gives the degree:

$$\deg f = \sum_{x \in f^{-1}(y)} \operatorname{sgn} df_x .$$

The above sign is determined as follows: Take any local representative $\psi_j f \varphi_i^{-1}$ about x. The differential $d(\psi_j f \varphi_i^{-1})$ at $\varphi_i(x)$ is then in $\mathrm{GL}_c(E)$, since x is a regular point. Define $\operatorname{sgn} df_x$ to be 1 if the differential is in $\mathrm{GL}_c^+(E)$, and -1 otherwise. This definition of degree obviously extends the finite-dimensional one and is due to Elworthy–Tromba [1970].

V. The Lefschetz–Hopf Theory

This chapter is algebraic in character. We develop here the homological tools needed to formulate and prove some of the central results in topological fixed point theory: (i) the Lefschetz fixed point theorem for various classes of maps of non-compact spaces, and (ii) the Hopf index theorem expressing the relation between the generalized Lefschetz number and the fixed point index for compact maps of ANRs. The chapter ends with a number of applications.

§14. Singular Homology

In order to provide an appropriate framework for the main results of the chapter, we need first to extend simplicial homology to arbitrary topological spaces. One way of doing this is to approximate the space, in some sense, by polyhedra, and this method leads to what is known as "Čech homology". The other way is to discard the idea of triangulation and substitute it by properly defined "embeddings" of a standard simplex into the space. This leads to what is known as "singular homology" and it is this approach we will take.

In this paragraph we shall develop elementary parts of singular homology and only in such generality that is needed for the aims of the chapter. We remark that the proofs of several results of this paragraph are rather long and tedious; a reader primarily interested in fixed point theory may omit such proofs on a first reading and return to them if need be.

1. Singular Chain Complex and Homology Functors

From the intuitive point of view, the development of singular homology arises from the observation that topological spaces can sometimes be distinguished from each other because every closed curve in one space forms

the boundary of a surface in that space (e.g., a great circle on a sphere) while this is not true in another space (a meridian circle on a torus— a cut can be made without breaking the surface into separate pieces; on more complicated surfaces, more than two cuts can be made without breaking the surface into separate pieces). To make this more precise for algebraic treatment, the notion of an r-dimensional surface in $\boldsymbol{R}^{n+1}$ is replaced by the notion of a linear combination of singular simplices in $\boldsymbol{R}^{n+1}$. Geometrically, this is thought of as a surface built up from simplices embedded in $\boldsymbol{R}^{n+1}$ Thus, the main general idea of constructing singular homology is to transfer the algebraic notions developed in §8 to arbitrary spaces; this will be accomplished by transferring the algebraic operations with simplices onto the set of singular simplices.

Categories and functors

To formulate the above intuitive ideas in a precise formal way, we shall use the language of categories and functors.

Recall that a *category* $\mathbf{K}$ consists of a class of *objects*, and for any objects A and B, there is a set $\mathbf{K}(A, B)$ (also denoted by $[A, B]_{\mathbf{K}}$) of *morphisms* $f : A \to B$ from A to B with the following properties: given two morphisms $f : A \to B$ and $g : B \to C$, the *composite morphism* $gf : A \to C$ is defined; furthermore, the composition operation is required to be associative and to have an identity 1_A in $\mathbf{K}(A, A)$ for every object A in $\mathbf{K}$.

For example, the class $\mathbf{K}$ of all sets, with $\mathbf{K}(A, B)$ being the set of all maps of A into B, and with the customary composition law, obviously forms a category **Ens**. Similarly, the class $\mathbf{K}$ of all topological spaces, with $\mathbf{K}(A, B)$ the set of all continuous maps of A into B and with the usual composition law, forms a category **Top**. There are many categories in which all objects have some type of algebraic structure, such as the category **Ab** of abelian groups (in which all morphisms are homomorphisms) or that of vector spaces, or modules over a fixed ring, whose definitions are obvious.

For any category $\mathbf{K}$, the *opposite category* $\mathbf{K}^*$ is that having the same objects as $\mathbf{K}$, morphisms $\mathbf{K}^*(A, B) = \mathbf{K}(B, A)$, and the same composition law as in $\mathbf{K}$.

A morphism $u : A \to B$ in a category $\mathbf{K}$ is called an *isomorphism* if there is a morphism $v : B \to A$ in $\mathbf{K}$ with $uv = 1_B$ and $vu = 1_A$; the morphism v is called the *inverse* of u, and is unique. Two objects $A, B \in \mathbf{K}$ are *equivalent* (written $A \cong B$) if there is an isomorphism $u : A \to B$ in $\mathbf{K}$; the relation is indeed an equivalence relation in $\mathbf{K}$. These notions yield the customary concept of isomorphism in **Ens**, **Top**, and **Ab**.

When several morphisms between various objects in a category are considered simultaneously, it is convenient to display the morphisms as arrows

in a diagram, such as

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, "u"] & B \\ C \arrow[u, "f"] \arrow[r, "g"'] & D \arrow[u, "v"'] \end{tikzcd}$$

The diagram $\mathcal{D}$ is said to be *commutative* if $uf = vg$. The same terminology applies to more complicated diagrams.

Let $\mathbf{K}$, $\mathbf{L}$ be two categories. A *covariant functor* $T : \mathbf{K} \to \mathbf{L}$ is a rule that assigns to each object $A \in \mathbf{K}$ an object $T(A) \in \mathbf{L}$ and to each morphism $u \in \mathbf{K}(A, B)$ a morphism $T(u) \in \mathbf{L}(T(A), T(B))$ so that:

(a) $T(1_A) = 1_{T(A)}$ for each $A \in \mathbf{K}$,

(b) $T(uv) = T(u)T(v)$ whenever uv is defined.

If $\mathbf{K}^*$ is the opposite category, a functor $T : \mathbf{K}^* \to \mathbf{L}$ is called a *contravariant functor* (or briefly a *cofunctor*) on $\mathbf{K}$; it can be interpreted as a functor on $\mathbf{K}$ that reverses the direction of each morphism when sent to $\mathbf{L}$.

A functor $T : \mathbf{K} \to \mathbf{L}$ transfers commutative diagrams in $\mathbf{K}$ to commutative diagrams in $\mathbf{L}$. Moreover, if u is an isomorphism in $\mathbf{K}$, then $T(u)$ is an isomorphism in $\mathbf{L}$; the converse is not necessarily true.

We list two simple but important examples:

(i) *The functor* $\mathbf{F} : \mathbf{Ens} \to \mathbf{Ab}$. For any set A, let $\mathbf{F}(A)$ be the free abelian group generated by A. The defining property of free abelian groups implies that if $f : A \to B$ is any function, then there is a unique homomorphism $\mathbf{F}f : \mathbf{F}(A) \to \mathbf{F}(B)$ with $(\mathbf{F}f)i = jf$, where $i : A \to \mathbf{F}(A)$ and $j : B \to \mathbf{F}(B)$ are the inclusions.

(ii) *The functor* $\mathbf{S}_M : \mathbf{Top} \to \mathbf{Ens}$. For a fixed space M, we define $\mathbf{S}_M : \mathbf{Top} \to \mathbf{Ens}$ by

$$\mathbf{S}_M(X) = \mathbf{Top}(M, X), \qquad X \in \mathbf{Top},$$

and for $f : A \to B$, $\mathbf{S}_M(f) : \mathbf{Top}(M, A) \to \mathbf{Top}(M, B)$ is given by

$$[\mathbf{S}_M(f)](u) = fu, \qquad u \in \mathbf{Top}(M, A).$$

Let $S, T : \mathbf{K} \to \mathbf{L}$ be two functors. A *natural transformation* $t : S \to T$ of S into T is a function that assigns to each $A \in \mathbf{K}$ a morphism $t_A \in \mathbf{L}(S(A), T(A))$ so that for all A, B, u in $\mathbf{K}$, the diagram

$$\begin{tikzcd} S(A) \arrow[r, "S(u)"] \arrow[d, "t^A"'] & S(B) \arrow[d, "t^B"] \\ T(A) \arrow[r, "T(u)"'] & T(B) \end{tikzcd}$$

is commutative. The natural transformation $t : S \to T$ is called a *natural equivalence* of the functors S and T if each t^A is an isomorphism in $\mathbf{L}$.

In the remaining part of this section our aim is to show that the singular homology assigning to each space X an abelian group $H_n(X)$, the nth homology group of X, and to each map $f : X \to Y$ of spaces a homomorphism $H_n(f) : H_n(X) \to H_n(Y)$ of groups is, in fact, a functor, called the nth singular homology functor from **Top** to **Ab**.

Singular simplices and singular chains

We begin the description of singular homology by defining the notion of a singular simplex in a topological space. As preparation, for each dimension n, we select a definite geometric simplex in $\boldsymbol{R}^{n+1}$

Given an integer $i = 0, 1, \ldots, n$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the unit point of the ith coordinate axis of $\boldsymbol{R}^{n+1}$. The *standard Euclidean n-simplex* Δ^n in $\boldsymbol{R}^{n+1}$ is the simplex spanned by the points $e_0, \ldots, e_n$. By definition,

$$\Delta^n = \Big\{(t_0, \ldots, t_n) \in \boldsymbol{R}^{n+1} \,\Big|\, \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\Big\}.$$

For each $i = 0, 1, \ldots, n$ we let $\Delta_i^n = \{(t_0, \ldots, t_n) \in \Delta^n \mid t_i = 0\}$ be the ith face of Δ^n opposite the vertex e_i. For points $p_0, \ldots, p_n$ (not necessarily distinct) in Δ^q, we let the symbol $(p_0 \ldots p_n)$ denote the unique affine map $\Delta^n \to [p_0, \ldots, p_n]$ sending e_i into p_i. With this notation, for the identity map id_n from Δ^n to Δ^n we have $\mathrm{id}_n = (e_0 \ldots e_n)$.

(1.1) DEFINITION. Given a topological space Y, a *singular n-simplex* in Y is defined as a continuous map $T : \Delta^n \to Y$; the set $T(\Delta^n) \subset Y$ is called the *support* of T and denoted by $|T|$. A *singular n-chain* in Y is a finite formal linear combination $c = \sum n_i T_i$ of singular n-simplices in Y with coefficients in $\boldsymbol{Z}$; the set $|c| = \bigcup\{|T_i| \mid n_i \neq 0\}$ is called the *support* of c. With addition of chains defined by adding coefficients, the n-chains in Y form an abelian group $C_n(Y)$. Precisely, the group $C_n(Y)$ is the free abelian group generated by the set of all singular n-simplices in Y.

Note that a singular simplex in Y is a map into Y and not a subset of Y, although much of the development can be motivated by regarding T as a simplex embedded in Y. Intuitively, an n-chain can be thought of as a kind of n-dimensional surface embedded in Y and built up from simplices with various multiplicities.

(1.2) DEFINITION. Let X and Y be two spaces, and $f : X \to Y$ continuous. If $T : \Delta^n \to X$ is a singular n-simplex in X, then the composition $f \circ T : \Delta^n \to Y$ is a singular n-simplex in Y denoted by fT. Forming an n-chain $c = \sum n_i T_i$ in $C_n(X)$ and letting $C_n(f)(c) = \sum n_i(fT_i)$ we get a homomorphism $C_n(f) : C_n(X) \to C_n(Y)$ *induced* by f.

Because $C_n(g \circ f) = C_n(g) \circ C_n(f)$ for any map $g : Y \to Z$ and $C_n(\mathrm{id}_X) = \mathrm{id}_{C_n(X)}$, the assignments $X \mapsto C_n(X)$, $f \mapsto C_n(f)$ yield the *singular n-chain functor* C_n : **Top** $\to$ **Ab**.

Boundary homomorphism. The singular complex $C_*(X)$

The next step is to introduce an algebraic operation on chains that will represent in a natural way the geometric operation of taking the boundary of a surface. This should agree with the idea that the boundary of a surface is itself a closed surface. We already know from §8 what the boundary of the standard simplex is, so it is reasonable to take the boundary of a singular simplex T in Y to be the restrictions of T to the faces of Δ^n. Unfortunately, these are not maps of the standard $(n-1)$-simplex Δ^{n-1}, so we must transfer them so that they can be regarded as such maps. To this end, for each $i = 0, 1, \ldots, n$ we define standard affine maps $d_i : \Delta^{n-1} \to \Delta^n$ by setting

$$d_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

Note that d_i maps Δ^{n-1} onto Δ_i^n, the ith face of Δ^n opposite the vertex e_i.

(1.3) DEFINITION. If $T : \Delta^n \to X$ is a singular n-simplex, then the ith *face* of T is the $(n-1)$-dimensional singular simplex $\delta_i T = T d_i : \Delta^{n-1} \to X$. For each $n \geq 1$, the *boundary operator*

$$\partial = \partial_n : C_n(X) \to C_{n-1}(X)$$

is the homomorphism defined on each generator T of $C_n(X)$ by setting

$$\partial T = \partial_n T = \sum_{i=0}^{n} (-1)^i \delta_i T.$$

It is straightforward to verify that any composition $\partial_n \partial_{n+1}$ is zero, which is equivalent to the statement that $\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n$. Geometrically, this says that the boundary of any $(n+1)$-chain is an n-chain having no boundary. It is this fundamental property that leads to the notion of the homology group.

The kernel of $\partial_n : C_n(X) \to C_{n-1}(X)$ is denoted by $Z_n(X)$ and called the *group of singular n-cycles* of X; note that $C_0(X) = Z_0(X)$. The image of $\partial_{n+1} : C_{n+1}(X) \to C_n(X)$ is called the *group $B_n(X)$ of singular n-boundaries* of X, and the quotient group $H_n(X) = Z_n(X)/B_n(X)$ is the nth *singular homology group* of X. The elements of $H_n(X)$ are called *homology classes*, the coset $z + B_n(X)$ being the homology class of the n-cycle z. Two n-cycles z, z' belonging to the same homology class are called *homologous*; this will occur if and only if $z - z' = \partial_{n+1} c$ for some $(n+1)$-chain c. If the homology group $H_n(X)$ is finitely generated, its rank is called the nth *Betti number* of X.

For a space X, the sequence $C_*(X) = \{C_n(X), \partial_n \mid n = 0, 1, 2, \ldots\}$ is called the *singular complex* of X and the sequence

$$\{H_n(X)\} = H_*(X) = H(C_*(X)) = \{H_n(C_*(X))\}$$

is called the *graded singular homology group* of X.

For a continuous map $f : X \to Y$, let $H_*(f) = \{H_n(f) : H_n(X) \to H_n(Y)\}$ be the graded homomorphism of singular homology groups induced by the chain map $C_*(f)$. The assignments $X \mapsto H_*(X)$, $f \mapsto H_*(f)$ define the *singular homology functor* $H_* :$ **Top** $\to$ **GrAb** from the category of topological spaces to the category of graded abelian groups.

Some examples

We calculate the singular homology of some simple spaces.

EXAMPLE 1. In contrast to the simplicial case, observe that if $X \neq \emptyset$, then $C_n(X) \neq 0$ for every $n \geq 0$. However, we show that if $*$ is a one-point space, then $H_0(*) \cong Z$ and $H_i(*) = 0$ for all $i > 0$. For there is exactly one singular simplex $T^p : \Delta^p \to *$ for each $p \geq 0$. If $p > 0$ is even, then $\partial T^p = T^{p-1}$, so that there are no p-cycles, and $H_p(*) = 0$. If $p > 0$ is odd, then $\partial T^p = 0$ and $\partial T^{p+1} = T^p$, so that every cycle bounds, and again $H_p(*) = 0$. In the case $n = 0$, we have $Z_0(*) =$ cyclic group generated by T^0 and $B_0(*) = \partial C_1(*) = 0$; therefore $H_0(*) = Z_0(*) = Z$.

EXAMPLE 2. An easy modification of the proof in Example 1 shows that $H_0(X) = Z$ whenever X is path-connected.

EXAMPLE 3. Let $X \subset R^p$ be a convex set. Then $H_0(X) = Z$ and $H_i(X) = 0$ for all $i > 0$. The first statement being evident from Example 2, we concentrate on the second. We will show that $H_n(X) = 0$ for $n \geq 1$ by constructing a chain homotopy $D : C_n(X) \to C_{n+1}(X)$ (see §6); since $C_n(X)$ is the free abelian group generated by the singular n-simplices T, it is enough to define $DT \in C_{n+1}(X)$ for each generator $T : \Delta^n \to X$.

Choose a point $\xi \in X$; for each $n > 0$ and each $T : \Delta^n \to X$ set

$$DT(x_0, \ldots, x_{n+1}) = \begin{cases} x_0\xi + (1 - x_0)T\left(\dfrac{x_1}{1 - x_0}, \ldots, \dfrac{x_{n+1}}{1 - x_0}\right), & x_0 \neq 1, \\ \xi, & x_0 = 1. \end{cases}$$

This is the analogue of the cone construction with vertex ξ: on the face of Δ^{n+1} opposite the vertex $(1, 0, \ldots, 0)$, we are setting

$$DT(0, x_1, \ldots, x_{n+1}) = T(x_1, \ldots, x_{n+1}) \quad \text{and} \quad DT(1, 0, \ldots, 0) = \xi,$$

then mapping each line segment in Δ^{n+1} joining $(1, 0, \ldots, 0)$ to a point x on the face of Δ^{n+1} opposite $(1, 0, \ldots, 0)$ linearly onto the segment in X

joining ξ to Tx. The continuity of $DT : \Delta^{n+1} \to X$ is clear from this description; formally, DT is evidently continuous at all points except possibly $(1,0,\ldots,0)$, and the continuity at $(1,0,\ldots,0)$ follows easily from the boundedness of the given T.

By construction, we have

$$(\delta_0 DT)(x_0,\ldots,x_n) = DT(0,x_0,\ldots,x_n) = T(x_0,\ldots,x_n).$$

Moreover, for $i > 0$, we have

$$\begin{aligned}
&(\delta_i DT)(x_0,\ldots,x_n)\\
&\quad = DT(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_n)\\
&\quad = x_0\xi + (1-x_0)T\left(\frac{x_1}{1-x_0},\ldots,\frac{x_{i-1}}{1-x_0},0,\frac{x_i}{1-x_0},\ldots,\frac{x_n}{1-x_0}\right)
\end{aligned}$$

and

$$\begin{aligned}
&(D\delta_{i-1}T)(x_0,\ldots,x_n)\\
&\quad = x_0\xi + (1-x_0)\left[\delta_{i-1}T\left(\frac{x_1}{1-x_0},\ldots,\frac{x_n}{1-x_0}\right)\right]\\
&\quad = x_0\xi + (1-x_0)\left[T\left(\frac{x_1}{1-x_0},\ldots,\frac{x_{i-1}}{1-x_0},0,\frac{x_i}{1-x_0},\ldots,\frac{x_n}{1-x_0}\right)\right],
\end{aligned}$$

so that $\delta_i DT = D\delta_{i-1}T$ for $i > 0$. Thus,

$$\begin{aligned}
\partial DT &= \delta_0 DT - \delta_1 DT + \cdots \pm \delta_{n+1} DT\\
&= T - D\delta_0 T + \cdots \pm D\delta_n T = T - D(\partial T),
\end{aligned}$$

so $T = \partial DT + D\partial T$ and $c = \partial Dc + D\partial c$ for every $c \in C_n(X)$. If c is a cycle, then $\partial c = 0$, so $c = \partial Dc$ and every cycle bounds. This completes the proof.

Relative groups. Long exact sequence

A somewhat more general notion is needed to develop a theory, namely, that of relative homology. Here, chains on a certain subspace $A \subset X$ are identified with 0. Thus, to be a cycle of X mod A a chain must have a boundary that is a chain on A, rather than have boundary zero. The groups $Z_n(X,A)$ can be expressed directly in terms of chains in X: Let

$$Z_n(X,A) = \begin{cases} \{c \in C_n(X) \mid \partial c \in C_{n-1}(A)\}, & n > 0,\\ C_n(X), & n = 0;\end{cases}$$

this is called the *group of n-cycles of X mod A*. Let

$$\begin{aligned}
B_n(X,A) &= B_n(X) + C_n(A)\\
&= \text{the subgroup of } C_n(X) \text{ generated by } B_n(X) \text{ and } C_n(A);
\end{aligned}$$

this is called the *group of bounding n-cycles of X mod A*; clearly, $B_n(X,A) \subset Z_n(X,A)$, and the quotient group

$$H_n(X,A) = Z_n(X,A)/B_n(X,A)$$

is the nth *homology group of X mod A*. If the group $H_n(X,A)$ is finitely generated, then its rank, denoted by $b_n(X,A)$, is called the nth *Betti number* of (X,A). Let $C_n(X,A) = C_n(X)/C_n(A)$; note that if $A = \emptyset$, then $C_n(X,\emptyset) = C_n(X)$.

Let A be a subspace of X. One of the characteristic features of homology theory is a long exact sequence that relates the homologies of X, A and X mod A. To define this interrelation, let

$$i_* : H_n(A) \to H_n(X)$$

be induced by the inclusion $i : A \to X$, and

$$j_* : H_n(X) \to H_n(X,A)$$

be induced by the inclusion $j : (X,\emptyset) \to (X,A)$, specifically,

$$j_* : [z_n + B_n(X)] \mapsto [z_n + B_n(X) + C_n(A)],$$

and let $\partial_n : H_n(X,A) \to H_{n-1}(A)$ be the homomorphism

$$[c_n + B_n(X) + C_n(A)] \mapsto [\partial_n c_n + B_{n-1}(A)].$$

It is straightforward to verify

(1.4) THEOREM (Homology sequence of a pair). *Let A be a subset of X. Then the homology sequence of the pair* (X,A),

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A) \to \cdots \to H_0(A) \to H_0(X) \to H_0(X,A) \to 0$$

is exact. Moreover, a continuous map $f : (X,A) \to (Y,B)$ *induces a homomorphism of the exact sequence of* (X,A) *into that of* (Y,B), *meaning that in the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "(f|A)_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "(f|A)_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

each square is commutative.

The proof uses the fact that if $A \subset X$, then the chain complex $\{C_n(A), \partial\}$ can be identified with a subcomplex of $\{C_n(X), \partial\}$, leading to the quotient chain complex $\{C_n(X)/C_n(A), \hat{\partial}\} = \{C_n(X,A), \hat{\partial}\}$; this gives the relative

homology groups $H_*(X, A)$ and the long exact sequence of (1.4) (see Appendix, Section D).

By regarding the exact sequence of (1.4) as arising from a chain complex and a subcomplex, we are led to a generalization: Starting with spaces $A \subset B \subset X$, we have the chain complex $\{C_n(X)/C_n(A)\}$ and a subcomplex $\{C_n(B)/C_n(A)\}$; the quotient complex is $\{C_n(X)/C_n(B)\}$, so we obtain more generally the following result:

(1.5) THEOREM (Homology sequence of a triple). *Let $A \subset B \subset X$ be topological spaces. Then there is an exact sequence*

$$\cdots \to H_n(B, A) \to H_n(X, A) \to H_n(X, B) \xrightarrow{\hat{\partial}} H_{n-1}(B, A) \to \cdots \to H_0(X, B) \to 0,$$

where all unlabeled homomorphisms are induced by inclusions and $\widehat{\partial}$ is the composition

$$H_n(X, B) \xrightarrow{\partial} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(B, A).$$

Moreover, a continuous map $f : (X, B, A) \to (\widehat{X}, \widehat{B}, \widehat{A})$ induces a homomorphism of the exact sequence of the triple (X, B, A) into that of $(\widehat{X}, \widehat{B}, \widehat{A})$. □

We remark that if $A = \emptyset$, then the exact sequence of (1.5) is precisely that of (1.4).

REMARK. Within the framework of chain complexes, we can define, for a space X, the singular homology groups over an arbitrary abelian group or field G. For any such G, the finite formal linear combinations $c_n = \sum g_i T_i$ of singular n-simplices in X with $g_i \in G$ form an abelian group $C_n(X; G)$, which is a vector space whenever G is a field. The boundary homomorphism $\partial_n : C_n(X; G) \to C_{n-1}(X; G)$ is defined by $\partial_n(c_n) = \sum g_i \partial T_i$, and is a linear map when G is a field. Again $\partial_n \partial_{n+1} = 0$, so that $\{C_n(X; G), \partial_n\}$ is a chain complex; the corresponding homology groups $H_n(X; G)$ are called the *singular homology groups of X with coefficients in G*. If G is a field, then $H_n(X; G)$ (being the quotient of a vector space by a linear subspace) is a vector space. We remark that in this terminology, $H_n(X)$ is $H_n(X; Z)$, and in fact, the chain complex $\{C_n(X; G), \partial_n\}$ is precisely $\{C_n(X) \otimes G, \partial_n \otimes 1\}$, where $\otimes$ is the tensor product.

If (X, A) is a pair of spaces, the relative singular homology groups $H_n(X, A; G)$ are defined analogously, as the homology groups of the chain complex $C_*(X, A) \otimes G$.

We remark that singular homology with coefficients in a field will be used in this chapter as a basic tool in Lefschetz theory.

We conclude this section by showing that singular homology has compact supports. This property, expressed in terms of direct systems of abelian groups (see Appendix), is given in the following:

(1.6) THEOREM. *Let (X, A) be a pair of spaces, and let $\{(K_\alpha, L_\alpha)\}_{\alpha\in\mathcal{D}}$ be the family of all compact pairs $(K_\alpha, L_\alpha) \subset (X, A)$ ordered by inclusion. Then the family $\{H_*(K_\alpha, L_\alpha)\}$ of groups, together with the homomorphisms induced by inclusions, forms a direct system, and*

$$H_*(X, A) \cong \varinjlim H_*(K_\alpha, L_\alpha).$$

PROOF. We first consider the absolute case, when we are given a space X together with the collection $\{K_\alpha\}_{\alpha\in\mathcal{D}}$ of all compact subsets of X. We are going to show that $H_*(X) \cong \varinjlim H_*(K_\alpha)$. For convenience we assume that there is a one-to-one correspondence $\alpha \leftrightarrow K_\alpha$ between the indices in $\mathcal{D}$ and the compact subsets of X; we write $\alpha \preceq \beta$ whenever $K_\alpha \subset K_\beta$, which makes $\mathcal{D}$ a directed set. We form the direct system of chain complexes $\{C_*(K_\alpha), i_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{D}}$, where $i_{\alpha\beta} : C_*(K_\alpha) \to C_*(K_\beta)$, $\alpha \preceq \beta$, is the inclusion. Let $C_*(X)$ be the singular complex of X, and $f_\alpha : C_*(K_\alpha) \to C_*(X)$ the inclusion. We note that the family $\{f_\alpha\}$ is compatible and that both conditions (i) and (ii) of Theorem (D.3)(II) in the Appendix are satisfied (because the support of any chain in $C_*(X)$ is compact). Thus, by that theorem, $C_*(X) \cong \varinjlim\{C_*(K_\alpha), i_{\alpha\beta}\}$. Now the asserted isomorphism follows from the fact that the homology functor commutes with direct limits (see Appendix, Theorem (D.7)).

The proof of the theorem in the relative case is analogous, and the details are left to the reader. □

2. Invariance of Homology under Barycentric Subdivision

In the singular homology of X mod A, chains on A are identified with 0. This suggests that removal of a set $E \subset A$ should not affect homology, i.e.,

$$H_*(X - E, A - E) \cong H_*(X, A).$$

This fundamental property will be established in Section 3 under the condition $\overline{E} \subset \operatorname{Int} A$; this is to guarantee that certain simplices can be dropped. The technique consists in cutting down the size of the support of singular simplices so that any element $z \in H_n(X, A)$ is represented by a relative cycle z, all of whose singular simplices have their support either in A or in the complement of E. The part on A can then be identified to zero, so z is represented by a relative cycle of $X - E$ mod $A - E$. Cutting down the size of the singular simplices is done by means of the barycentric subdivision β of the model Δ^n; again it is a question of identifying the members of $\beta(\mathrm{id}_n)$

with singular simplices $T : \Delta^n \to \Delta^n$, and the process has affinities with barycentric subdivision of polyhedra.

In this section we establish the invariance of singular homology under barycentric subdivision. This implies that to compute singular homology we can confine ourselves to singular simplices with arbitrarily small supports.

Presentability of the functor C_n

In presenting the technique of barycentric subdivision, it will be convenient to use the notion of a presentable functor. Let $\mathbf{T}$ be any category and $\mathbf{Ab}$ be the category of abelian groups. A functor $F : \mathbf{T} \to \mathbf{Ab}$ is called *presentable* if there exists an object M in $\mathbf{T}$ and an element $s \in F(M)$ such that for each $X \in \mathbf{T}$, the set $\{F(u)s \mid u \in \mathbf{T}(M, X)\}$ is a basis for $F(X)$. In particular, then, whenever F is a presentable functor, each $F(X)$ must be a free abelian group; and the presentability condition states that there is a canonical way to specify a basis in each $F(X)$; M is called the *model object*, and the couple (M, s) is called a *presentation* of F.

(2.1) PROPOSITION. *The functor $C_n : \mathbf{Top} \to \mathbf{Ab}$ is presentable and the pair $(\Delta^n, \mathrm{id}_n)$ consisting of Δ^n and $\mathrm{id}_n \in C_n(\Delta^n)$ is a presentation of C_n.*

PROOF. Indeed, given any space X, we find for each $T : \Delta^n \to X$ that $C_n(T)(\mathrm{id}_n) = [T : \Delta^n \to X] \in C_n(X)$. Consequently, the set $\{C_n(T)(\mathrm{id}_n) \mid T \in \mathbf{Top}(\Delta^n, X)\}$ is precisely a basis for $C_n(X)$. This means exactly that C_n is presentable and $(\Delta^n, \mathrm{id}_n)$ is a presentation of C_n. $\square$

Two important categorical properties of the functor C_n are given in

(2.2) PROPOSITION. *Let $G : \mathbf{Top} \to \mathbf{Ab}$ be any functor. Then:*

(i) *for each $g \in G(\Delta^n)$ there is a unique natural transformation $\alpha : C_n \to G$ determined by setting $\alpha^{\Delta^n}(\mathrm{id}_n) = g$ and "transporting" g by $\alpha^X C_n(T)\,\mathrm{id}_n = G(T)g$,*

(ii) *given any two natural transformations $\alpha, \beta : C_n \to G$, the condition $\alpha^{\Delta^n}(\mathrm{id}_n) = \beta^{\Delta^n}(\mathrm{id}_n)$ implies that $\alpha = \beta$.*

Thus, to construct a natural transformation $\alpha : C_n \to G$ we need only define a single element $\alpha^{\Delta^n}(\mathrm{id}_n) \in G(\Delta^n)$; on the other hand, to verify that two natural transformations $\alpha, \beta : C_n \to G$ are identical, it is sufficient to show that on the model Δ^n, we have $\alpha(\mathrm{id}_n) = \beta(\mathrm{id}_n)$.

The proof of (2.2), based on the presentability of C_n, is left to the reader.

Natural chain maps

Let $\alpha_n : C_n \to C_n$ be a natural transformation, one for each $n \geq 0$. We say that $\alpha = \{\alpha_n\}$ is a *natural chain map* provided $\partial\alpha_n = \alpha_{n-1}\partial$ for each

$n > 0$; $\partial : C_n \to C_{n-1}$ is regarded here as a natural transformation of functors.

(2.3) LEMMA. *Let $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ be two natural chain maps. If $\alpha(\mathrm{id}_0) = \beta(\mathrm{id}_0)$, then there exists a natural chain homotopy $D = \{D_n\}$ such that*

$$(*) \qquad \partial D_n + D_{n-1}\partial = \alpha_n - \beta_n.$$

PROOF. The method of proof goes by the name of "acyclic models" and is due to Eilenberg-MacLane. Since $\alpha(\mathrm{id}_0) - \beta(\mathrm{id}_0) = 0$, it follows that $(*)$ holds for $n = 0$ with $D_0 = 0$. Assuming inductively that $D_k : C_k \to C_{k+1}$ has already been found for all $k < n$ with

$$(**) \qquad \partial D_k + D_{k-1}\partial = \alpha_k - \beta_k$$

we establish by induction the existence of $D_n : C_n \to C_{n+1}$ for which the formula $(*)$ holds. Consider the map $\gamma_n : C_n \to C_n$ given by

$$\gamma_n = \alpha_n - \beta_n - D_{n-1}\partial$$

(which exists by the inductive hypothesis) and denote by c_n the chain in $C_n(\Delta^n)$ given by $c_n = \gamma_n^{\Delta^n}(\mathrm{id}_n)$, where $\gamma_n^{\Delta^n}$ is the Δ^n component of γ_n. Using the fact that α and β are chain maps and the inductive hypothesis (under which $(**)$ is true for $k = n-1$), we get

$$\begin{aligned}\partial\gamma_n &= \partial\alpha_n - \partial\beta_n - \partial D_{n-1}\partial \\ &= \partial\alpha_n - \partial\beta_n - [\alpha_{n-1}\partial - \beta_{n-1}\partial - D_{n-2}\partial\partial] = 0.\end{aligned}$$

From this we conclude that the chain c_n is a cycle in $C_n(\Delta^n)$, and because Δ^n is acyclic (as a convex set), it is a boundary, i.e., $c_n = \partial b_{n+1}$ for some b_{n+1} in $C_{n+1}(\Delta^n)$. Letting now $D_n(\mathrm{id}_n) = b_{n+1} \in C_{n+1}(\Delta^n)$, we observe, by presentability of C_n, that $D_n : C_n \to C_{n+1}$ is defined (by transportation) and it is a natural map. Then, from the formula

$$\partial D_n(\mathrm{id}_n) + D_{n-1}\partial(\mathrm{id}_n) = \partial b_{n+1} + D_{n-1}\partial(\mathrm{id}_n) = \alpha_n(\mathrm{id}_n) - \beta_n(\mathrm{id}_n),$$

in view of naturality of the maps involved and again by presentability of C_n, we conclude that our assertion $(*)$ is true for $k = n$. □

Barycentric subdivision

We shall now construct for each $n \geq 0$ a natural transformation $\beta_n : C_n \to C_n$, which for each space X replaces a chain in $C_n(X)$ by a chain also in $C_n(X)$ with "smaller" simplices. Geometrically speaking, barycentric subdivision will be constructed as the cone of the subdivided boundary. So we begin by recalling the cone construction.

Let Δ^q be the standard simplex in R^{q+1}, and $\{A_n(\Delta^q)\}$ the subcomplex of $\{C_n(\Delta^q)\}$ generated by the affine singular simplices in Δ^q. Given a fixed point b in Δ^q, the *cone operator* $\varkappa$ assigns to a generator $\sigma = (p_0 p_1 \dots p_n)$ in $A_n(\Delta^q)$ the affine simplex of one higher dimension,

$$\varkappa(\sigma) = b \times \sigma = (b p_0 p_1 \dots p_n),$$

and so determines (by linearity) a homomorphism $\varkappa : A_n(\Delta^q) \to A_{n+1}(\Delta^q)$. A straightforward verification gives

$$(*) \qquad \partial(b \times c) = c - b \times \partial c \quad \text{for any chain } c \in A_n(\Delta^q).$$

We are now ready to define the subdivision operator β and prove its properties.

(2.4) DEFINITION. We define $\beta = \{\beta_n\} : \{C_n\} \to \{C_n\}$ inductively:

(i) $\beta_0 = \mathrm{id}$;

(ii) if $\beta_{n-1} : C_{n-1} \to C_{n-1}$ is defined, then $\beta_{n-1}^{\Delta^n} : C_{n-1}(\Delta^n) \to C_{n-1}(\Delta^n)$ is also defined, and we let

$$\beta_n^{\Delta^n}(\mathrm{id}_n) = b_n \times \beta_{n-1}(\partial\, \mathrm{id}_n),$$

where b_n is the barycenter of Δ^n. Because C_n is presentable, the above formula defines a natural transformation $\beta_n : C_n \to C_n$. For any integer $r \geq 1$ we define the r-fold iterate β^r by letting $\beta^1 = \beta$ and $\beta^r = \beta \circ \beta^{r-1}$ for $r \geq 2$.

(2.5) THEOREM. *The operations β and β^r have the following properties:*

(i) *β^r is a natural chain map for any $r \in N$.*

(ii) *For each space X, β^r is naturally chain homotopic to the identity map of the singular chain complex into itself.*

(iii) *For any space X, every cycle $z \in Z(X)$ is homologous to the cycle $\beta^r(z)$.*

(iv) *For a given $n \in N$ and any $\delta > 0$ there exists an integer $s \in N$ (depending on n and δ) such that the support of any affine simplex s_i appearing in the chain $\beta_n^s(\mathrm{id}_n)$ has diameter* $\operatorname{diam}(|s_i|) \leq \delta$.

PROOF. (i) Because every functor C_n is presentable, to show that $\beta = \{\beta_n\}$ is a chain map we simply need to verify that in the diagram

$$\begin{tikzcd}
C_n(\Delta^n) \arrow[r, "\partial_n"] \arrow[d, "\beta_n"'] & C_{n-1}(\Delta^n) \arrow[d, "\beta_{n-1}"] \\
C_n(\Delta^n) \arrow[r, dashed, "\partial_n"] & C_{n-1}(\Delta^n)
\end{tikzcd}$$

$\mathrm{id}_n \in C_n(\Delta^n)$ transports around commutatively. This being true for $n = 1$, we use induction and (*) to find

$$\begin{aligned}\partial_n\beta_n(\mathrm{id}_n) &= \partial_n[b_n \times \beta_{n-1}\partial_n(\mathrm{id}_n)]\\ &= \beta_{n-1}\partial_n(\mathrm{id}_n) - b_n \times \partial_{n-1}\beta_{n-1}\partial_n(\mathrm{id}_n)\\ &= \beta_{n-1}\partial_n(\mathrm{id}_n) - b_n \times \beta_{n-1}\partial_{n-1}\partial_n(\mathrm{id}_n)\\ &= \beta_{n-1}\partial_n(\mathrm{id}_n),\end{aligned}$$

and the proof is complete. Since β is a chain map, so is β^r for any $r \geq 1$.

(ii) Because $\beta^r = \{\beta^r_n\}$ and the identity $\mathrm{id} = \{\mathrm{id}_n\}$ are both natural chain maps of $\{C_n\}$ into itself and they agree in dimension 0, our assertion follows from Lemma (2.3).

(iii) Let X be any space, $n \geq 0$ a fixed integer, and D^r a natural chain homotopy joining the chain maps β^r and id:

$$\beta^r, \mathrm{id} : \{C_n(X)\} \to \{C_n(X)\}.$$

Given any cycle $z \in Z_n(X)$ we have $z - \beta^r(z) = D^r\partial z + \partial D^r z = \partial D^r z$; thus z and $\beta^r(z)$ belong to the same coset of $B_n(X)$, and our assertion is proved.

(iv) This follows clearly from (8.2.3). □

Small simplices

(2.6) DEFINITION. Let X be a space and $\mathbf{V} = \{V_\lambda \mid \lambda \in \Lambda\}$ a family of subsets of X whose interiors $\{\mathring{V}_\lambda \mid \lambda \in \Lambda\}$ cover X. A singular simplex $T : \Delta^n \to X$ will be said to *belong to* $\mathbf{V}$ (or to be $\mathbf{V}$-*small*) provided its support $|T|$ lies in at least one of the sets V_λ.

By confining our attention to singular simplices that are $\mathbf{V}$-small, we can introduce several notions which are strictly analogous to those of Section 1.

We let $C^{\mathbf{V}}_*(X)$ be the subcomplex of $C_*(X)$ generated by the $\mathbf{V}$-small singular simplices. If $A \subset X$, we define $C^{\mathbf{V}}_*(A)$ to be the subcomplex of $C^{\mathbf{V}}_*(X)$ generated by the singular simplices that belong to $\mathbf{V} \cap A = \{V_\lambda \cap A\}_{\lambda\in\Lambda}$, and $C^{\mathbf{V}}_*(X, A)$ to be the quotient complex $C^{\mathbf{V}}_*(X)/C^{\mathbf{V}}_*(A)$.

Thus associated with $\mathbf{V}$ are three chain complexes

$$C^{\mathbf{V}}_*(X), \quad C^{\mathbf{V}}_*(A), \quad C^{\mathbf{V}}_*(X, A),$$

and the corresponding graded homology groups

$$H^{\mathbf{V}}_*(X) = \{H^{\mathbf{V}}_n(X)\}, \quad H^{\mathbf{V}}_*(A) = \{H^{\mathbf{V}}_n(A)\}, \quad H^{\mathbf{V}}_*(X, A) = \{H^{\mathbf{V}}_n(X, A)\}.$$

(2.7) THEOREM. *Let (X, A) be a pair and $\mathbf{V} = \{V_\lambda\}$ a family of subsets of X whose interiors cover X. Then the injection $j : C^{\mathbf{V}}_*(X, A) \to C_*(X, A)$ induces an isomorphism $j_* : H^{\mathbf{V}}_*(X, A) \to H_*(X, A)$ of homology groups.*

PROOF. We first write the following commutative diagram of chain complexes with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & C_*^{\mathbf{V}}(A) \arrow[r] \arrow[d, "i^A"] & C_*^{\mathbf{V}}(X) \arrow[r] \arrow[d, "i^X"] & C_*^{\mathbf{V}}(X,A) \arrow[r] \arrow[d, "j"] & 0 \\ 0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X,A) \arrow[r] & 0 \end{tikzcd}$$

(where i^A, i^X are the injections), and then the corresponding commutative ladder of homology groups:

$$\begin{tikzcd} \cdots \arrow[r] & H_q^{\mathbf{V}}(A) \arrow[r] \arrow[d, "i_*^A"] & H_q^{\mathbf{V}}(X) \arrow[r] \arrow[d, "i_*^X"] & H_q^{\mathbf{V}}(X,A) \arrow[r, "\partial"] \arrow[d, "j_*"] & H_{q-1}^{\mathbf{V}}(A) \arrow[r] \arrow[d, "i_*^A"] & H_{q-1}^{\mathbf{V}}(X) \arrow[r] \arrow[d, "i_*^X"] & \cdots \\ \cdots \arrow[r] & H_q(A) \arrow[r] & H_q(X) \arrow[r] & H_q(X,A) \arrow[r, "\partial"] & H_{q-1}(A) \arrow[r] & H_{q-1}(X) \arrow[r] & \cdots \end{tikzcd}$$

We now observe that to establish our assertion it is clearly enough to consider the absolute case. For once we have proved that i_*^A and i_*^X are isomorphisms, the invertibility of j_* will follow, with the aid of the 5-lemma, from the fact that the above commutative diagram of homology groups has exact rows. Thus we must show that

$$i_*^X : H_*^{\mathbf{V}}(X) \cong H_*(X)$$

or, in equivalent geometric terms, that each homology class in $H_*(X)$ is representable by a cycle having all singular simplices in $\mathbf{V}$.

We first prove that

(∗) if c is a chain in $C_n(X)$, then there is an $r \in N$ such that all singular simplices of the chain $\beta^r(c)$ belong to $\mathbf{V}$.

Clearly, it is sufficient to establish (∗) for singular simplices.

Let $T : \Delta^n \to X$ be a singular simplex in X. The family $\{T^{-1}(\mathring{V}_\lambda)\}$ is an open covering of the compact set Δ^n, hence has a Lebesgue number $\delta > 0$: any set $A \subset \Delta^n$ with $\operatorname{diam} A < \delta$ lies in one of the sets $T^{-1}(\mathring{V}_\lambda)$. In particular, if r is sufficiently large, then every affine simplex s_i appearing in the chain $\beta^r(\mathrm{id}_n)$ has its support $|s_i|$ in one of the sets $T^{-1}(\mathring{V}_\lambda)$. It follows that the support of any singular simplex appearing in $\beta^r T$ (being the image of a geometric simplex with diameter $\leq \delta$) will lie in a set of $\{V_\lambda\}$.

Now letting z be the given singular n-cycle, we can pick r so large that $\beta^r z$ has all simplices lying in $\mathbf{V}$; in view of Theorem (2.5), this will be the required representative. Thus we have proved that i_*^X is surjective; the proof that it is injective is left to the reader. □

3. Excision

As the main consequence of (2.7) we come to the following basic

(3.1) THEOREM (Excision theorem). *Let (X, A) be any pair of spaces, and let $E \subset X$ be any subset such that $\overline{E} \subset \operatorname{Int} A$ (i.e., E is "far from the boundary"). Then the inclusion map $i : (X - E, A - E) \hookrightarrow (X, A)$ induces an isomorphism in homology:*

$$i_* : H_*(X - E, A - E) \cong H_*(X, A).$$

PROOF. Consider the open covering $\mathbf{V} = \{\operatorname{Int} A, X - \overline{E}\}$ of X.

(a) i_* *is epic.* Let c be any cycle mod A; then $c = \sum n_i T_i$, and by Theorem (2.7), we can assume that each T_i is $\mathbf{V}$-small. In particular, the support of each singular simplex is then either entirely in $\operatorname{Int} A$, or entirely in $X - \overline{E}$, or possibly in both. We break up c into two parts: $c = c_{X-E} + r$ by collecting all the simplices of c whose supports lie entirely in $X - E$, and letting r be the remaining part of c. We now claim that c_{X-E} is a cycle mod $A - E$, and then $|c_{X-E} - c| = |r| \subset A$, so c and c_{X-E} are homologous mod $A - E$. Indeed, since $X - \overline{E} \subset X - E$, the supports of the singular simplices of r cannot be entirely in $X - \overline{E}$, and therefore $|r| \subset \operatorname{Int} A \subset A$. Focusing on c_{X-E}, we have $\partial c_{X-E} = \partial c - \partial r$. Now, since $|c_{X-E}| \subset X - E$, we know that $|\partial c_{X-E}| \subset X - E$; but this must be equal to $|\partial c - \partial r|$, which is contained in A. Thus, $|\partial c_{X-E}| \subset A \cap (X - E) = A - E$, and therefore c_{X-E} is a cycle mod $A - E$. Because c_{X-E} and c are homologous mod A, it follows that $i_*([c_{X-E}]) = [c] \in H_*(X, A)$, and thus i_* is epic.

(b) i_* *is monic.* Let $z_{X-E} \in Z_*(X - E, A - E)$ be a cycle homologous to zero in $Z_*(X, A)$; this says that $z_{X-E} = \partial q_X + \varrho_A$, where we can assume that all chains involved are $\mathbf{V}$-small (by taking β^r on both sides if necessary). As before, we break up q_X into two pieces $q_X = q_{X-E} + r_A$, getting $z_{X-E} - \partial q_{X-E} = \partial r_A + \varrho_A$; the left side is in $X - E$, the right in A, so both are in $A - E$. Thus, z_{X-E} is, in fact, bounding on $X - E$ mod $A - E$, i.e., i_* is monic. □

(3.2) DEFINITION. Let (A, B) be a pair of subsets of a space X with $X = A \cup B$. We say that the pair (A, B) is *excisive* if the inclusion

$$(A, A \cap B) \hookrightarrow (A \cup B, B) = (X, B)$$

induces an isomorphism in homology.

With this terminology, the excision theorem (3.1) can be put in another equivalent form, which is sometimes formally easier to use and involves sets whose interiors cover X (it resembles the usual group-theoretic isomorphism theorem).

Below we write $\mathring{A}$ for $\operatorname{Int} A$ and $\complement A$ for $X - A$.

(3.3) THEOREM. *If A and B are any two sets with $X = A \cup B = \mathring{A} \cup \mathring{B}$, then the pair (A, B) is excisive.*

PROOF. Consider (X, B). The hypothesis gives $X - \mathring{A} \subset \mathring{B}$. Now, $\mathring{A} = \complement\overline{\complement A}$, so $\overline{\complement A} = \complement\mathring{A} \subset \mathring{B}$. Thus, by (3.1), we can excise $\complement A$, and it suffices to note that $(X - \complement A, B - \complement A) = (A, A \cap B)$. □

Conversely, (3.3) implies (3.1): suppose we have (X, A) with $\overline{U} \subset \mathring{A}$. Then $X = \mathring{A} \cup (X - U)^\circ$. Indeed, $(X - U)^\circ = \complement[\overline{\complement\complement U}] = \complement\overline{U}$, so $X - \mathring{A} \subset X - \overline{U} = (X - U)^\circ$, proving the claim.

Thus, by (3.3),

$$H(X, A) = H(\complement U \cup A, A) \cong H(\complement U, (\complement U) \cap A) = H(X - U, A - U).$$

This says that whenever $X = \mathring{A} \cup \mathring{B}$, then for considering (X, B), we can confine ourselves to A and remove all of B not in A.

EXAMPLES. (i) Let $A \subset S^n$ be closed nonempty and assume $* \in S^n - A$. Then $H_n(S^n - A, *) = 0$. Indeed, consider the exact sequence of the triple $\{*\} \subset S^n - A \subset S^n$:

$$\cdots \to H_{n+1}(S^n, S^n - A) \xrightarrow{\partial} H_n(S^n - A, *) \to H_n(S^n, *) \to H_n(S^n, S^n - A)$$
$$\to H_{n-1}(S^n - A, *) \to H_{n-1}(S^n, *) \to \cdots.$$

Choosing any $p \in A$, we find a factorization $S^n - A \subset S^n - \{p\} \subset S^n$ of $i : S^n - A \hookrightarrow S^n$; but $S^n - \{p\}$ is contractible, so i_* is the zero homomorphism, showing $H_n(S^n - A, *) = \partial(H_{n+1}(S^n, S^n - A))$. But $H_{n+1}(S^n, S^n - A) = 0$ because $\dim S^n = n$, and the assertion follows.

(ii) Let $A \subset S^n$ be closed nonempty and $* \in S^n - A$. Then

$$H_n(S^n, S^n - A) \cong H_{n-1}(S^n - A, *) \oplus H_n(S^n, *);$$

therefore:

(a) $H_n(S^n, S^n - A)$ always contains an infinite cyclic group.

(b) If $S^n - A$ is contractible over itself (e.g. $A =$ point), then we have $H_n(S^n, S^n - A) \cong H_n(S^n, *)$.

Indeed, from the exact sequence in (i), and because we have $H_n(S^n - A, *) = H_{n-1}(S^n, *) = 0$, we obtain the short exact sequence

$$0 \to H_n(S^n, *) \xrightarrow{i_*} H_n(S^n, S^n - A) \to H_{n-1}(S^n - A, *) \to 0.$$

We note that the diagram

$$\begin{tikzcd} H_n(S^n, *) \arrow[r, "i_*"] \arrow[dr, "k_*"', "\cong"] & H_n(S^n, S^n - A) \arrow[d, "j_*"] \\ & H_n(S^n, S^n - \{p\}) \end{tikzcd}$$

commutes; this implies that the above exact sequence splits, and our assertion follows.

(iii) Let $V \subset S^n$ be open, and $A \subset V$ nonempty and compact. Then inclusion induces an isomorphism

$$e : H_n(V, V - A) \cong H_n(S^n, S^n - A).$$

For write $(S^n, S^n - A) = (V \cup (S^n - A), S^n - A)$; by excision, this has the same homology as $(V, V \cap (S^n - A)) = (V, V - A)$.

4. Axiomatization

In order to get quickly to the essentials of the algebraic techniques we shall now present the axiomatic approach.

There are many homology theories; the following description of what is to be meant by a homology theory has been given by Eilenberg–Steenrod.

Each homology theory is defined on some category $\mathbf{K}$ whose objects are pairs (X, A) of topological spaces such that $A \subset X$, and whose morphisms are continuous maps of such pairs. The category $\mathbf{K}$ must contain the space $\emptyset$, and we denote a pair $(X, \emptyset)$ simply by X.

(4.1) DEFINITION. A *homology theory on* $\mathbf{K}$ is a sequence $h_* = \{h_n\}$, $n = 0, 1, \ldots$, of functors from $\mathbf{K}$ to the category of abelian groups, together with a family of natural transformations

$$\partial_n : h_n(X, A) \to h_{n-1}(A),$$

one for each (X, A, n), such that the following axioms hold:

(1) (*Homotopy*) If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then $h_n(f_0) = h_n(f_1)$ for all n.

(2) (*Exactness*) For each $(X, A) \in \mathbf{K}$, the sequence

$$\cdots \to h_n(A) \to h_n(X) \to h_n(X, A) \xrightarrow{\partial_n} h_{n-1}(A) \to h_{n-1}(X) \to \cdots \to 0$$

is exact, where all unmarked maps are induced by inclusions.

(3) (*Strong excision*) The inclusion $e : (A, A \cap B) \to (A \cup B, B)$ induces an isomorphism

$$h_n(e) : h_n(A, A \cap B) \cong h_n(A \cup B, B)$$

for all n.

For a homology theory h_*, the graded group $\{h_n(*)\}$, where $*$ is a one-point space, is called the *group of coefficients* of the theory. By an *ordinary homology theory* h_* we shall mean one with $h_n(*) = 0$ unless $n = 0$; if

moreover $h_0(*) = G$, then h_* is called an *ordinary homology theory with coefficients in G*.

The construction given in §8 shows that there exists an ordinary homology theory on the category of finite polyhedra. For the singular homology groups, because of (1.4), the exactness axiom is satisfied. The strong excision axiom is not valid for all A, B, but in view of (3.2), it does hold for all excisive pairs (A, B); in particular (cf. (3.3)), the axiom is valid if A and B are open subsets of some space X. It is somewhat more delicate to verify, and will not be proved here, that if A, B are closed in $A \cup B$ and $A \cap B$ is a strong neighborhood deformation retract in A, then the pair (A, B) is excisive. The homotopy axiom can be verified in a manner essentially similar to that in (8.7.4) (for another proof see (D.3) in "Miscellaneous results and examples"). Thus, singular homology is an ordinary homology theory on suitable pairs of topological spaces.

Some consequences of the axioms

We now derive some properties that follow immediately from the axioms, and are therefore common to all homology theories.

Recall that two pairs (X, A), (Y, B) are *homotopy equivalent* if there exist maps $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ such that $gf \simeq 1_{(X,A)}$ and $fg \simeq 1_{(Y,B)}$.

(4.2) PROPOSITION. *If two pairs (X, A) and (Y, B) are homotopy equivalent (in particular, homeomorphic), then $h_n(X, A) \cong h_n(Y, B)$ for all n.*

PROOF. If $f : (X, A) \to (Y, B)$ is a homotopy equivalence with homotopy inverse g, then $g \circ f \simeq \text{id}$, so that $g_* \circ f_* = \text{id}$. Similarly, $f_* \circ g_* = \text{id}$. Thus $f_* : h_n(X, A) \to h_n(Y, B)$ is an isomorphism with inverse g_*. □

(4.3) PROPOSITION. *If $A \subset X$ is a deformation retract of X, then*

$$h_n(X, A) = 0 \quad \textit{for all } n \in Z.$$

PROOF. Because the sequence

$$h_n(A) \cong h_n(X) \to h_n(X, A) \to h_{n-1}(A) \cong h_{n-1}(X)$$

is exact and X is homotopy equivalent to A, our assertion follows clearly from (4.2). □

(4.4) PROPOSITION. $h_n(X, X) = 0$ *for every* X.

(4.5) PROPOSITION. *If A is a retract of X, then*

$$h_n(X) \cong h_n(A) \oplus h_n(X, A).$$

PROOF. Let $r : X \to A$ be the retraction. Then $r \circ i = \mathrm{id}$, so that $r_* \circ i_* = \mathrm{id}$; from this we see that $i_* : h_n(A) \to h_n(X)$ is monic and $r_* : h_n(X) \to h_n(A)$ is epic. Moreover, since i_* is monic, the exact sequence

$$0 \to h_n(A) \to h_n(X) \to h_n(X, A) \to 0$$

splits as $h_n(X) \cong \mathrm{Im}\, i_* \oplus \mathrm{Ker}\, r_* \cong h_n(A) \oplus \mathrm{Ker}\, r_*$. This shows that $h_n(X, A) \cong h_n(X)/h_n(A) = \mathrm{Ker}\, r_*$, completing the proof. □

(4.6) PROPOSITION. *If X is the disjoint union $X = X_1 \cup \cdots \cup X_k$, then $h_n(X) \cong \bigoplus_{i=1}^k h_n(X_i)$.*

PROOF. We prove this for $n = k = 2$, the general case following by induction. Assume that $X = X_1 \cup X_2$; then X_1 is a retract of X, so

$$h_2(X) = h_2(X_1) \oplus h_2(X_1 \cup X_2, X_1),$$

and by excision, the second term is isomorphic to $h_2(X_2, \emptyset) \equiv h_2(X_2)$. □

Reduced groups

Let X be nonempty and $r : X \to *$ the unique map to a one-point space $* \in X$. Our first task is to find a relation between $h_n(X)$ and $h_n(X, *)$.

(4.7) PROPOSITION. *We have $h_n(X, *) = \mathrm{Ker}[r_* : h_n(X) \to h_n(*)]$, with r_* epic. In fact,*

$$h_n(X) = h_n(X, *) \oplus h_n(*).$$

PROOF. Since $*$ is a retract of X, our assertion follows at once from (4.5).□

The groups $h_n(X, *)$ are easier to work with, and are called the *reduced homology groups* of X. We have

(4.8) THEOREM (Reduced exact homology sequence). *For any pair (X, A) and $* \in A$, the sequence*

$$\cdots \to h_n(A, *) \to h_n(X, *) \to h_n(X, A) \to h_{n-1}(A, *) \to \cdots \to 0$$

is exact.

PROOF. We consider

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & h_n(A) & \xrightarrow{i} & h_n(X) & \xrightarrow{j} & h_n(X, A) & \xrightarrow{\partial} & h_{n-1}(A) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & \\
\cdots \longrightarrow & h_n(*) & \xrightarrow{\lambda} & h_n(*) & \xrightarrow{\mu} & h_n(*, *) & \longrightarrow & h_{n-1}(*) & \longrightarrow \cdots
\end{array}$$

Since the squares commute,

$$\mathrm{Ker}\, \alpha \xrightarrow{i} \mathrm{Ker}\, \beta \xrightarrow{j} \mathrm{Ker}\, \gamma \xrightarrow{\partial} \mathrm{Ker}\, \delta;$$

and since the top sequence is exact, we clearly have Im $\subset$ Ker at each stage of the last sequence. The task is to show the converse, that Ker $\subset$ Im for each pair.

(i) Let $x \in \operatorname{Ker}\beta$, $j(x) = 0$. Then there is an $a \in h_n(A)$ with $i(a) = x$ because of exactness. We have to show $a \in \operatorname{Ker}\alpha$. So, consider $\alpha(a)$; then $\lambda\alpha(a) = \beta i(a) = \beta x = 0$, so $\alpha(a) \in \operatorname{Ker}\lambda = 0$, and therefore $a \in \operatorname{Ker}\alpha$.

(ii) Let $\xi \in h_n(X, A) = \operatorname{Ker}\gamma$, $\partial\xi = 0$. Then there exists $x \in h_n(X)$ with $jx = \xi$. The problem is to pick such an x in $\operatorname{Ker}\beta$. Let us see how far off we are with x. We find $\mu\beta(x) = \gamma j(x) = 0$, so $\beta x \in \operatorname{Ker}\mu = \operatorname{Im}\lambda$, showing $\beta x = \lambda(a')$, and since α is surjective, $\beta x = \lambda\alpha(a)$. Now consider $x - ia$. We have $j(x - ia) = jx = \xi$ because of exactness; and we have $\beta(x - ia) = \beta x - \beta ia = \beta x - \lambda\alpha(a) = 0$. Thus, the sequence is exact at $h_n(X, A)$.

(iii) Let $\zeta \in \operatorname{Ker}\delta$, $i(\zeta) = 0$. Then there is a ξ with $\partial\xi = \zeta$, and clearly $\xi \in \operatorname{Ker}\gamma$. □

We now consider the reduced groups for the special case where the space is contractible (i.e., $\mathrm{id} : X \to X$ is homotopic, in X, to a constant map $F : X \to *$).

(4.9) PROPOSITION. *Let X be contractible. Then $h_n(X, *) = 0$.*

PROOF. We have $X \xrightarrow{F} * \xrightarrow{i} X$ and clearly $Fi = \mathrm{id}_*$. But also $iF \simeq \mathrm{id}_X$ because of the contractibility, and consequently, $*$ being a deformation retract of X, our assertion follows from (4.3). □

(4.10) PROPOSITION. *Let $A \subset X$ be a subspace contractible in itself. Then $h_n(X, A) \cong h_n(X, *)$.*

PROOF. $0 = h_n(A, *) \to h_n(X, *) \to h_n(X, A) \to h_{n-1}(A, *) = 0$, and the result follows. □

The final global mechanism that we seek is the Mayer–Vietoris theorem.

(4.11) THEOREM (Mayer–Vietoris). *Let A, B be any two spaces, and $* \in A \cap B$. Then there is an exact sequence*

$$\cdots \to h_n(A \cap B, *) \xrightarrow{(i_1, -i_2)} h_n(A, *) \oplus h_n(B, *) \xrightarrow{j_1 + j_2} h_n(A \cup B, *) \xrightarrow{\partial} h_{n-1}(A \cap B, *) \to \cdots,$$

where i_1, i_2, j_1, j_2 are induced by inclusions, and ∂ is defined in the proof.

PROOF. The excision $\gamma : (A, A \cap B) \to (A \cup B, B)$ induces a morphism of

the corresponding exact sequences:

$$\begin{tikzcd}
\to h_{n+1}(A, A\cap B) \arrow[r] \arrow[d, "\gamma"', "\cong"] & h_n(A\cap B,*) \arrow[r, "i_1"] \arrow[d, "i_2"] & h_n(A,*) \arrow[r, "k_1"] \arrow[d, "j_1"] & h_n(A, A\cap B) \arrow[r, "\delta"] \arrow[d, "\gamma"', "\cong"] & h_{n-1}(A\cap B,*) \to \arrow[d] \\
\to h_{n+1}(A\cup B, B) \arrow[r] & h_n(B,*) \arrow[r, "j_2"] & h_n(A\cup B,*) \arrow[r, "k_2"] & h_n(A\cup B, B) \arrow[r] & h_{n-1}(B,*) \to
\end{tikzcd}$$

in which every fourth vertical arrow γ is an isomorphism. It is a general result in the theory of exact sequences, easily established by diagram chasing (cf. the Whitehead–Barratt lemma in the Appendix), that in such a case the sequence

$$\cdots \to h_n(A\cap B,*) \xrightarrow{(i_1,-i_2)} h_n(A,*)\oplus h_n(B,*) \xrightarrow{j_1+j_2} h_n(A\cup B,*)$$
$$\xrightarrow{\partial} h_{n-1}(A\cap B,*) \to \cdots \to 0$$

with $\partial = \delta\gamma^{-1}k_2$ is exact. □

Homology groups of spheres

We now give some applications and determine the homology of some simple spaces:

(a) $h_q(B^n,*) = 0$ in any homology theory.

(b) $h_q(S^n,*) = \begin{cases} h_{q-n}(*), & q \geq n, \\ 0, & q < n. \end{cases}$

We start with $S^n = S^n_+ \cup S^n_-$, where S^n_+. S^n_- are the "northern" and "southern" closed hemispheres of S^n. Then, by the Mayer–Vietoris theorem,

$$\cdots \to h_q(S^n_+,*)\oplus h_q(S^n_-,*) \to h_q(S^n_+\cup S^n_-,*)$$
$$\to h_{q-1}(S^n_+\cap S^n_-,*) \to h_{q-1}(S^n_+,*)\oplus h_{q-1}(S^n_-,*) \to \cdots$$

is exact. But S^n_+ and S^n_- are contractible, and therefore their homology groups are zero; so

$$h_q(S^n,*) \cong h_{q-1}(S^{n-1},*).$$

We repeat this again to find

$$h_q(S^n,*) \cong \begin{cases} h_{q-1}(S^{n-1},*) \cong \cdots \cong h_0(S^{n-q},*), & q < n, \\ h_q(S^0,*), & q = n, \\ h_{q-n}(S^0,*), & q > n. \end{cases}$$

Now, $h_0(S^{n-q},*) = 0$, for we have

$$h_0(S^{n-q}_+,*)\oplus h_0(S^{n-q}_-,*) \xrightarrow{\alpha} h_0(S^{n-q}_+\cup S^{n-q}_-,*) \to 0;$$

since the direct sum is zero and α is onto, $h_0(S^{n-q},*) = 0$.

Next, $h_{q-n}(S^0,*) = h_{q-n}(p\cup *,*) \cong h_{q-n}(p)$, and this proves the assertion.

Thus, in any homology theory, $h_n(S^n, *) \cong h_0(*)$, the coefficients, and nonvanishing $h_q(S^n, *)$ can occur only for $q \geq n$; for all $q < n$, we have $h_q(S^n, *) = 0$.

5. Comparison of Homologies. Künneth Theorem

In this section we derive a simple principle for the comparison of homology theories when they are both applicable to some space X; being primarily interested in polyhedra (where the simplicial homology is effectively computable) we will require X to have a system of subsets, called a structure, that behave as the subcomplexes of a complex. More precisely:

(5.1) DEFINITION. A *structure* $\mathcal{S}$ on a topological space X is a lattice of subspaces (join is union, meet is intersection) containing $\emptyset$, X and satisfying the descending chain condition: each strictly decreasing sequence $A_1 \supset A_2 \supset \cdots$ is finite. An element $S \in \mathcal{S}$ is called *indecomposable* if $S \neq A \cup B$, where $A, B \in \mathcal{S}$ and $A \neq S$, $B \neq S$.

For example, if X is a finite polyhedron, then the set $\mathcal{S}$ of all subcomplexes of X is a structure, and the indecomposables are the closed simplices.

Further examples of structures are provided by the general

(5.2) LEMMA. *Let $\mathcal{S}$ be a structure on X, and let $p : Z \to X$ be any surjection. Then $p^{-1}(\mathcal{S}) = \{p^{-1}(A) \mid A \in \mathcal{S}\}$ is a structure on Z having $\{p^{-1}(S) \mid S$ indecomposable$\}$ as its indecomposables.*

The simple proof of this lemma is left to the reader.

Each structure $\mathcal{S}$ on a space X determines the *structure category* $\mathbf{K}(\mathcal{S})$ in which (a) the objects are all pairs (E, A) with $E, A \in \mathcal{S}$ and $E \supset A$, and (b) the set of morphisms $(E, A) \to (F, B)$ consists simply of the inclusion map if both $E \subset F$ and $A \subset B$, and is empty otherwise.

The comparison theorem relates homology theories on a structure category $\mathbf{K}(\mathcal{S})$; observe that due to the simple nature of the morphisms, the homotopy axiom (4.1)(1) for a homology theory on $\mathbf{K}(\mathcal{S})$ is superfluous, and a natural transformation of functors is required to commute with the images of at most one morphism.

(5.3) THEOREM (Dugundji comparison principle). *Let $\mathbf{K}(\mathcal{S})$ be a structure category on a space X, and h, $\widehat{h}$ two homology theories on $\mathbf{K}(\mathcal{S})$. Assume that there is a natural transformation $t : h \to \widehat{h}$ such that $t(S) : h_*(S) \to \widehat{h}_*(S)$ is an isomorphism for each indecomposable $S \in \mathcal{S}$. Then $t(E, A) : h_*(E, A) \cong \widehat{h}_*(E, A)$ for every $(E, A) \in \mathbf{K}(\mathcal{S})$, and in particular, $t(X) : h_*(X) \cong \widehat{h}_*(X)$.*

PROOF. We argue by contradiction, supposing that $t(E)$ is not an isomorphism for some $E \in \mathcal{S}$. Then there exists an $A \neq \emptyset$ such that $t(A)$ is not an isomorphism, but $t(B)$ is an isomorphism for every proper subset $B \subset A$. Indeed, start with E; if there is some proper subset $E_1 \subset E$ with $t(E_1)$ not an isomorphism, replace E with E_1 and repeat the process with E_1. This gives a strictly descending chain $E \supset E_1 \supset E_2 \supset \cdots$, which must be finite; the final term A is a set having the specified properties; and $A \neq \emptyset$ because of (4.1)(2).

By hypothesis, A is not indecomposable, so $A = B \cup C$, where B, C are proper subsets of A; then $B \cap C \in \mathcal{S}$ is also a proper subset of A, so we conclude that $t(B)$, $t(C)$, and $t(B \cap C)$ are all isomorphisms.

Using now the exact sequences for the homologies of $(B, B \cap C)$ we have the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & h_n(B\cap C) \arrow[r] \arrow[d, "t(B\cap C)"] & h_n(B) \arrow[r] \arrow[d, "t(B)"] & h_n(B, B\cap C) \arrow[r] \arrow[d, "t(B, B\cap C)"] & h_{n-1}(B\cap C) \arrow[r] \arrow[d, "t(B\cap C)"] & h_{n-1}(B) \arrow[r] \arrow[d, "t(B)"] & \cdots \\
\cdots \arrow[r] & \widehat{h}_n(B\cap C) \arrow[r] & \widehat{h}_n(B) \arrow[r] & \widehat{h}_n(B, B\cap C) \arrow[r] & \widehat{h}_{n-1}(B\cap C) \arrow[r] & \widehat{h}_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Since $t(B)$ and $t(B \cap C)$ are all isomorphisms, the 5-lemma shows that all $t(B, B \cap C)$ are isomorphisms. By strong excision and naturality, we find from

$$\begin{tikzcd}
h_*(B, B\cap C) \arrow[r, "t(B, B\cap C)", "\cong"'] \arrow[d, "e"', "\cong"] & \widehat{h}_*(B, B\cap C) \arrow[d, "\widehat{e}", "\cong"'] \\
h_*(B\cup C, C) \arrow[r, "t(B\cup C, C)", "\cong"'] & \widehat{h}_*(B\cup C, C)
\end{tikzcd}$$

that $t(B \cup C, C)$ is an isomorphism. The 5-lemma applied to the exact sequences for $(B \cup C, C)$ now shows that all $t(B \cup C)$ are isomorphisms. But this is a contradiction, because $B \cup C = A$.

Thus, every $t(A)$ is an isomorphism, and in particular, $t(X) : h_*(X) \cong \widehat{h}_*(X)$. Moreover, using the 5-lemma once again for the exact sequences of any pair (E, A) shows all $t(E, A)$ to be isomorphisms, and the proof is complete. □

As an immediate application, we give another proof of the fact that the simplicial homology of a polyhedron depends only on the topology of the carrier, and not on the particular triangulation used; thus any convenient triangulation can serve for calculating the homology of a triangulable space:

(5.4) THEOREM (Topological invariance of simplicial homology). *Let P be a triangulable space and $h_*(P)$ its singular homology. Let $H_*(P)$ denote the simplicial homology of P, using some triangulation. Then*

$H_*(K) \cong h_*(K)$ for every subcomplex $K \subset P$ Because singular homology is topologically invariant, simplicial homology is also topologically invariant, and in particular independent of the triangulation used in its computation.

PROOF. Let $\mathcal{S}$ be the structure on P formed by the subcomplexes in the triangulation being used. By (8.9.3), H has the strong excision property. and so also does h, since subpolyhedra are strong neighborhood deformation retracts; thus both singular and simplicial homologies are homology theories on $\mathbf{K}(\mathcal{S})$. On $\mathbf{K}(\mathcal{S})$, there is a natural transformation $t : H \to h$ induced by the chain transformation from simplicial to singular chains which sends each generator $\sigma = |p_0, \ldots, p_n|$ to the singular simplex $T_\sigma : \Delta^n \to P$ given by $T_\sigma(t_0, \ldots, t_n) = t_0 p_0 + \cdots + t_n p_n$. The indecomposables of $\mathcal{S}$ are the closed simplices σ, and we know that $t_0(\sigma) : H_0(\sigma) \cong Z \cong h_0(\sigma)$ and $H_n(\sigma) = 0 = h_n(\sigma)$ for $n > 0$. Thus $t(\sigma)$ is an isomorphism on each indecomposable of $\mathcal{S}$, so by Theorem (5.3), $t(E, A) : H_*(E, A) \cong h_*(E, A)$ for every pair $(E, A) \in \mathbf{K}(\mathcal{S})$. □

We remark that in the same way as in (5.4) one can show that the homology of ordered chains is isomorphic to the homology of oriented chains on P.

Künneth theorem and universal coefficients theorem [1]

As another application of the comparison principle (5.3) we shall give a proof of the Künneth theorem concerned with the determination of the homology of a Cartesian product $K \times L$ of two complexes in terms of the homology of each factor. Although $K \times L$ is a complex, we will use singular homology throughout: by (5.4), this theory is isomorphic to the simplicial homology, and its use avoids working with an explicit triangulation of $K \times L$.

Each singular simplex $T : \Delta^n \to K \times L$ is completely specified by its projections $p_K T, p_L T$ into the two factors; moreover, the face operators δ_i also satisfy $p_K \delta_i T = \delta_i p_K T$ and $p_L \delta_i T = \delta_i p_L T$. Now, for each $n \geq 0$, consider the free abelian group $\mathbf{C}_n = C_n(K) \times C_n(L)$ generated by the $S \times T$, where S, T are singular simplices in K and L respectively; with the boundary operator $\widehat{\partial} : \mathbf{C}_n \to \mathbf{C}_{n-1}$ given by

$$\widehat{\partial}(S \times T) = (\delta_0 S) \times (\delta_0 T) - (\delta_1 S) \times (\delta_1 T) + \cdots \pm (\delta_n S) \times (\delta_n T)$$

we obtain a chain complex $\mathbf{C}_* = \{C_n(K) \times C_n(L), \widehat{\partial}\}$. Since the correspondence $T \mapsto p_K T \times p_L T$ yields an isomorphism $C_n(K \times L) \cong C_n(K) \times C_n(L)$, calculation of the homology $H_*(K \times L)$ is reduced to that of $\mathbf{C}_*$.

[1] The reader may omit this subsection on a first reading.

We now consider the two chain complexes $(C_n(K), \partial_K)$ and $(C_n(L), \partial_L)$ separately. Their tensor product $\mathbf{D}_*$ has the grading

$$\mathbf{D}_n = \bigoplus_{i=0}^{n} C_i(K) \otimes C_{n-i}(L)$$

and the boundary operator $\partial : \mathbf{D}_n \to \mathbf{D}_{n-1}$ given by

$$\partial(S_i \otimes T_{n-i}) = \partial_K S_i \otimes T_{n-i} + (-1)^i S_i \otimes \partial_L T_{n-i}.$$

The homology of the chain complex $\mathbf{D}_*$ is denoted by $H_*(K \otimes L)$.

A chain transformation $\tau : \mathbf{C}_* \to \mathbf{D}_*$ can be constructed by setting, for each $n \geq 0$ and each generator $S \times T \in C_n(K) \times C_n(L)$,

$$\begin{aligned} \tau(S \times T) = {} & \delta_1 \circ \cdots \circ \delta_n S \otimes T + \delta_2 \circ \cdots \circ \delta_n S \otimes \delta_0 T \\ & + \delta_3 \circ \cdots \circ \delta_n S \otimes \delta_0^2 T + \cdots + S \otimes \delta_0^n T. \end{aligned}$$

Expressed directly in terms of maps, $\delta_{k+1} \circ \cdots \circ \delta_n S \otimes \delta_0^k T$ is the tensor product of the singular simplex $A(t_0, \ldots, t_k) = S(t_0, \ldots, t_k, 0, \ldots, 0)$, called the front k-face of S, with $B(t_0, \ldots, t_{n-k}) = T(0, \ldots, 0, t_0, \ldots, t_{n-k})$, the back $(n-k)$-face of T. This transformation is natural, being defined in terms of the face operators, and a straightforward computation shows that $\partial\tau = \tau\hat{\partial}$, so that τ is indeed a chain transformation.

Consider finally another chain complex $\mathbf{E}_*$, where

$$\mathbf{E}_n = \bigoplus_{i=0}^{n} C_i(K) \times H_{n-i}(L)$$

and $\partial : \mathbf{E}_n \to \mathbf{E}_{n-1}$ is $\partial \otimes 1$. Observe that since $\partial \otimes 1 : C_i(K) \otimes H_{n-i}(L) \to C_{i-1}(K) \otimes H_{n-i}(L)$ for each n and i, it follows from the definition of homology over an arbitrary abelian group that the nth homology of $\mathbf{E}_*$ is the direct sum

$$H_n(\mathbf{E}_*) = \bigoplus_{i=0}^{n} H_i(K, H_{n-i}(L)).$$

To construct a chain transformation $\lambda : \mathbf{D}_* \to \mathbf{E}_*$ we begin with the observation that because $B_{n-1}(L) \subset C_{n-1}(L)$ is free abelian, there exists a homomorphism α making the diagram

$$\begin{tikzcd} & B_{n-1} \arrow[dl, dashed, "\alpha"'] \arrow[d, "1"] & \\ C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

commutative. Since $\partial\alpha = 1$, this implies $C_n(L) = \operatorname{Im}\alpha \oplus \operatorname{Ker}\partial = \operatorname{Im}\alpha \oplus Z_n(L)$, showing that the group of cycles is a direct summand of the group

of chains. Consequently, there is a homomorphism $\omega_n : C_n(L) \to H_n(L)$ obtained by projection into Z_n and then into the homology group. We now define $\lambda : \mathbf{D}_* \to \mathbf{E}_*$ by setting, for each $n \geq 0$,

$$\lambda\Big(\sum_{i=0}^{n} S_i \otimes T_{n-i}\Big) = \sum_{i=0}^{n} S_i \otimes \omega_{n-i} T_{n-i}.$$

This is a chain transformation: for $\lambda\partial(S_i \otimes T_{n-i}) = \lambda[\partial S_i \otimes T_{n-i} \pm S_i \otimes \partial T_{n-i}]$, and because ω annihilates boundaries, we get $\lambda\partial = (\partial \otimes 1)\lambda$ as required.

We can now establish a version of the Künneth theorem:

(5.5) THEOREM. *Let K and L be finite complexes. Then*

$$(\lambda\tau)_* : H_n(K \times L) \cong \bigoplus_{i=0}^{n} H_i(K; H_{n-i}(L))$$

for each $n \geq 0$.

PROOF. Let $\mathcal{S}$ be the structure in $K \times L$ determined by the sets $\{K_\mu \times L\}$, where K_μ runs over the subcomplexes of K. Define

$$h_n(E \times L, A \times L) = H_n(E \times L, A \times L),$$

$$\widehat{h}_n(E \times L, A \times L) = \sum_{i=0}^{n} H_i(E, A; H_{n-i}(L)).$$

Clearly, h is a homology theory on $\mathbf{K}(\mathcal{S})$, and so also is $\widehat{h}$, since it is a direct sum of homology theories on $\mathbf{K}(\mathcal{S})$. The morphisms in $\mathbf{K}(\mathcal{S})$ being inclusions $j \times 1$, it follows that $(\lambda\tau)_* : h \to \widehat{h}$ is a natural transformation of homology theories. We examine their behavior on the indecomposables of $\mathcal{S}$, which are the sets of the form $\sigma^n \times L$. We have $h_s(\sigma^n \times L) = H_s(\sigma^n \times L) \cong H_s(L)$ because for any vertex $p_0 \in \sigma^n$ the subspace $p_0 \times L \cong L$ is a strong deformation retract of $\sigma^n \times L$; and $\widehat{h}_s(\sigma^n \times L) = \bigoplus_{i=0}^{s} H_i(\sigma^n; H_{s-i}(L)) = H_0(\sigma^n; H_s(L)) = H_s(L)$. It is easy to see that $(\lambda\tau)_*$ establishes this isomorphism, so by (5.3), the proof is complete. □

To get a more explicit form for the homology of $K \times L$, we establish a result in homological algebra:

(5.6) THEOREM (Universal coefficients theorem). *If C is a free chain complex and G is an abelian group, then*

$$H_n(C; G) = H_n(C) \otimes G \oplus \operatorname{Tor}(H_{n-1}(C), G).$$

In particular, if Q is torsion-free or a field of characteristic zero, then

$$H_n(C; Q) = H_n(C) \otimes Q.$$

Proof. We factor the boundary homomorphism $\partial : C_n \to C_{n-1}$ as $C_n \xrightarrow{\partial_0} B_{n-1} \xrightarrow{j} C_{n-1}$, where j is inclusion, and consider the commutative diagram

$$\begin{tikzcd}
 & B_n \arrow[d, "k"'] & C_{n+1} \arrow[l, "\bar\partial_0"] \arrow[d, "\bar\partial"] & & & \\
0 \arrow[r] & Z_n \arrow[r, "i"] & C_n \arrow[rr, "\partial_0"] \arrow[d, "\partial"'] & & B_{n-1} \arrow[r] \arrow[d, "1"] & 0 \\
 & & C_{n-1} & & B_{n-1} \arrow[ll, "j"] \arrow[llu, dashed, "\alpha"'] \arrow[dl, "\bar k"] & \\
 & & & Z_{n-1} \arrow[ul, "\bar i"] & &
\end{tikzcd}$$

of boundary homomorphisms and inclusions. Because B_{n-1} is free, there is, as we have seen before, a homomorphism α with $\partial_0\alpha = 1$; since $\partial\alpha = (j\partial_0)\alpha = j$, the diagram with α is also commutative. Now take the tensor product of this diagram with G; the diagram is still commutative. Moreover, since the long row is exact and B_{n-1} is free, the tensored sequence remains exact.

From $(\partial_0 \otimes 1) \circ (\alpha \otimes 1) = 1$, we see that $\alpha \otimes 1$ is monic and

$$C_n \otimes G = \operatorname{Ker}(\partial_0 \otimes 1) \oplus \operatorname{Im}(\alpha \otimes 1).$$

From $j\partial_0 = \partial$ we find $\operatorname{Ker}(\partial_0 \otimes 1) \subset \operatorname{Ker}(\partial \otimes 1) = Z_n(C \otimes G)$, so that

$$Z_n(C \otimes G) = \operatorname{Ker}(\partial_0 \otimes 1) \oplus [\operatorname{Im}(\alpha \otimes 1) \cap \operatorname{Ker}(\partial \otimes 1)].$$

Because $\partial_0\widehat{\partial} = 0$, we have $B_n(C \otimes G) = \operatorname{Im}(\widehat{\partial} \otimes 1) \subset \operatorname{Ker}(\partial_0)$ so

$$H_n(C \otimes G) = \frac{Z_n(C \otimes G)}{B_n(C \otimes G)} = \frac{\operatorname{Ker}(\partial_0 \otimes 1)}{\operatorname{Im}(\widehat{\partial} \otimes 1)} \oplus [\operatorname{Im}(\alpha \otimes 1) \cap \operatorname{Ker}(\partial \otimes 1)].$$

It remains to simplify the expressions for the direct summands. For the first factor: by exactness, we have $\operatorname{Ker}(\partial_0 \otimes 1) = \operatorname{Im}(i \otimes 1)$; moreover, $\operatorname{Im}(\widehat{\partial} \otimes 1) = \operatorname{Im}[(i \otimes 1) \circ (k \otimes 1) \circ (\widehat{\partial}_0 \otimes 1)]$, which, because $\partial_0 \otimes 1$ is epic, gives $\operatorname{Im}(\widehat{\partial} \otimes 1) = (i \otimes 1)[\operatorname{Im}(k \otimes 1)]$. Observing that $i \otimes 1$ is monic, we find

$$\frac{\operatorname{Ker}(\partial_0 \otimes 1)}{\operatorname{Im}(\widehat{\partial} \otimes 1)} = \frac{\operatorname{Im}(i \otimes 1)}{(i \otimes 1)[\operatorname{Im}(k \otimes 1)]} \cong \frac{Z_n \otimes G}{\operatorname{Im}(k \otimes 1)}.$$

Now, $0 \to B_n \xrightarrow{k} Z_n \to H_n \to 0$ is exact, so $B_n \otimes G \xrightarrow{k \otimes 1} Z_n \otimes G \to H_n \otimes G \to 0$ is also exact, and therefore the first direct factor is $H_n(C) \otimes G$.

For the second factor, because $\alpha \otimes 1$ and $\widehat{\imath} \otimes 1$ are monic, we have

$$\begin{aligned}\operatorname{Im}(\alpha \otimes 1) \cap \operatorname{Ker}(\partial \otimes 1) &= (\alpha \otimes 1)(\operatorname{Ker}[(\partial \otimes 1) \circ (\alpha \otimes 1)]) \\ &\cong \operatorname{Ker}(j \otimes 1) = \operatorname{Ker}[(\widehat{\imath} \otimes 1) \circ (\widehat{k} \otimes 1)] \\ &\cong \operatorname{Ker}(\widehat{k} \otimes 1).\end{aligned}$$

Now, as Z_{n-1} is free, the exact sequence $0 \to B_{n-1} \xrightarrow{\hat{k}} Z_{n-1} \to H_{n-1} \to 0$, when tensored with G, yields the exact sequence

$$0 \to \operatorname{Tor}(H_{n-1}, G) \to B_{n-1} \otimes G \xrightarrow{\hat{k} \otimes 1} Z_{n-1} \otimes G \to H_{n-1} \otimes G \to 0,$$

so that the second summand is $\cong \operatorname{Tor}(H_{n-1}, G)$, and the proof is complete.

The second part is immediate, since $\operatorname{Tor}(A, Q) = 0$ whenever Q is torsion-free, for any abelian A. □

(5.7) COROLLARY (Künneth formula). *Let K, L be finite complexes. Then*

$$H_n(K \times L) = \bigoplus_{i=0}^{n} H_i(K) \otimes H_{n-i}(L) \oplus \bigoplus_{i=1}^{n} \operatorname{Tor}(H_{i-1}(K), H_{n-i}(L)).$$

PROOF. Immediate, using (5.6) to simplify the formula in (5.5). □

EXAMPLE. By the Künneth formula, the homology of $S^p \times S^q$ for $p \neq q$ is

$$H_n(S^p \times S^q) = \begin{cases} Z & \text{if } n = 0,\ p,\ q \text{ or } p+q, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the homology of the torus $S^1 \times S^1$ is

$$H_n(S^1 \times S^1) = \begin{cases} Z, & n = 0, \\ Z \oplus Z, & n = 1, \\ Z, & n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

6. Homology and Topological Degree

In this section we illustrate the use of singular homology by presenting a homological approach to topological degree.

We begin by a preliminary discussion which will lead us to an appropriate definition.

One of the properties of the degree we have found is that it depends only on the boundary values. So the question is: if we know the boundary values, can we calculate the degree directly?

Let us start with the simple case of a map $f : \overline{U} \to R^n$, where $\overline{U}$ is some convex body, and with $f^{-1}(0) = K \subset U$, i.e. $K \cap \partial U = \emptyset$. We have $\partial U = \Sigma^{n-1}$, an $(n-1)$-sphere, and we can consider the map

$$\pi f : \Sigma^{n-1} \to S^{n-1}, \quad \text{where } \pi(x) = x / \|x\| \text{ for } x \in R^n - \{0\}.$$

Such maps have a well-defined degree, $\deg \pi f$, and in order to define this, we need a few preliminary remarks about orientation in $\boldsymbol{R}^n$

Recall that two orderings of the vertices of a simplex are called equivalent if they differ by an even permutation. There are exactly two equivalence classes, and each class is called an orientation of that simplex. The order in which the vertices are written represents the selected orientation.

If $\sigma = (p_0, p_1, \ldots, p_n)$ is contained in $\boldsymbol{R}^n$, then the orientation of σ induced by the orientation of $\boldsymbol{R}^n$ represented by $(e_1, \ldots, e_n)$ is that class of orderings for which the unique affine map sending $(p_0, p_1, \ldots, p_n)$ to $(0, e_1, \ldots, e_n)$ has positive determinant. This orientation of σ depends, in fact, only on the orientation of $\boldsymbol{R}^n$, and not on the representation of that orientation.

An orientation for $\boldsymbol{S}^{n-1}$ is a definite generator $b \in H_{n-1}(\boldsymbol{S}^{n-1})$; since $H_{n-1}(\boldsymbol{S}^{n-1}) = \boldsymbol{Z}$, a sphere has exactly two orientations, b and $-b$.

An orientation of $\boldsymbol{R}^n$ induces an orientation of each $\Sigma^{n-1} \subset \boldsymbol{R}^n$ as follows:

(a) Regard Σ^{n-1} as the triangulated boundary of an n-simplex σ.
(b) Take $\sigma = (p_0, p_1, \ldots, p_n)$ with the orientation induced by $\boldsymbol{R}^n$
(c) The homology class $b \in H_{n-1}(\Sigma^{n-1})$ containing $\partial(p_0 p_1 \ldots p_n)$ is the induced orientation.

Let $f : \Sigma^{n-1} \to \boldsymbol{S}^{n-1}$ be a map of oriented spheres, as considered above. The degree of f is the unique number $\deg f \in \boldsymbol{Z}$ with $f_*(b') = (\deg f)b$. Obviously, the degree is not uniquely determined until the orientations have been specified; changing both orientations does not change the degree, but changing only one of them does. However, if Σ^{n-1} and $\boldsymbol{S}^{n-1}$ are always both taken with the orientation induced by some orientation of $\boldsymbol{R}^n$, then the degree is uniquely determined, independently of the orientation chosen in $\boldsymbol{R}^n$, since the opposite orientation changes the sign of orientations of both spheres.

Returning now to our map $\pi f : \Sigma^{n-1} \to \boldsymbol{S}^{n-1}$, we find that it has a definite degree, and it depends only on the boundary values of f.

It is instructive to calculate first the degree $\deg L$ of a nonsingular linear map $L : (p_0, \ldots, p_n) \to (L(p_0), \ldots, L(p_n))$. For this, we first need orientations of $(p_0, \ldots, p_n)$ and $(L(p_0), \ldots, L(p_n))$.

Assume $(p_0, \ldots, p_n)$ has the required orientation; then $(L(p_0), \ldots, L(p_n))$ must have the induced orientation, so that we obtain $\pm(L(p_0), \ldots, L(p_n))$ depending on whether the simplex has the same orientation as $(p_0, \ldots, p_n)$ or not. Thus, $(\pi f)_*$ sends the generator $\partial(p_0 \ldots p_n)$ to $\partial(L(p_0) \ldots L(p_n))$, so we conclude that

$$\deg L = \operatorname{sign} \det L = [\![p_0, \ldots, p_n]\!][\![L(p_0), \ldots, L(p_n)]\!].$$

Therefore, as far as linear transformations are concerned, we find that

$\deg L = d(L, B(0,1))$ as considered in §10, and at least for such linear maps, we have a determination of $d(L, B(0,1))$ simply and directly from the boundary values.

We now analyze the process more carefully, trying to see what can be done if we are given $f : \overline{U} \to \boldsymbol{R}^n$, where $\overline{U}$ is not necessarily a convex body. We assume that $\overline{U}$ is compact and $0 \notin f(\partial U)$; the set $f^{-1}(0) \cap \overline{U} = K$ is therefore a compact subset of $\operatorname{Int} U$.

We started off with $\overline{V} - K \xrightarrow{f} \boldsymbol{R}^n - \{0\} \xrightarrow{\pi} S^{n-1}$, where $V \supset K$ is a ball, giving $(\pi f)_* : H_{n-1}(\overline{V} - K) \to H_{n-1}(S^{n-1})$ in homology. Then we took the boundary $\dot{V} \subset \overline{V} - K$ to get

$$H_{n-1}(\dot{V}) \xrightarrow{i} H_{n-1}(\overline{V} - K) \xrightarrow{(\pi f)_*} H_{n-1}(S^{n-1}),$$

and set $\deg f$ to be the coefficient of the generator of $H_{n-1}(S^{n-1})$ in $(\pi f)_* i(b)$, where $b \in H_{n-1}(\dot{V})$.

In terms of the original diagram, this can be interpreted as saying that we have selected a distinguished element $y = i(b)$ of $H_{n-1}(\overline{V} - K)$ and let $(\pi f)_*(y)$ determine the degree of f

This distinguished element y came about easily because $\dot{V}$ is an $(n-1)$-sphere, and so has a unique basic cycle. We cannot find y so easily if $\dot{V}$ is not a sphere, so we seek another way to get it.

For this purpose, we consider the commutative diagram

$$\begin{tikzcd}
H_n(\overline{V}, \dot{V}) \arrow[r, "\partial", "\cong"'] \arrow[d, "j"'] & H_{n-1}(\dot{V}) \arrow[d, "i"] & \\
H_n(\overline{V}, \overline{V} - K) \arrow[r, "\partial"] & H_{n-1}(\overline{V} - K) \arrow[r, "(\pi f)_*"] & H_{n-1}(S^{n-1})
\end{tikzcd}$$

This is somewhat simpler, because the generator of $H_n(\overline{V}, \dot{V})$ is simply σ^n, the top line is an isomorphism, so that $i(b) = \partial j(\sigma^n)$. Thus, the degree of f is determined by $(\pi f)_* \partial j(\sigma^n)$. The distinguished element in $H_{n-1}(\overline{V} - K)$ is $\partial j(\sigma^n)$, the image of the distinguished element $j(\sigma^n)$ in $H_n(\overline{V}, \overline{V} - K)$. Thus, our problem will be settled if we can find a distinguished element in $H_n(\overline{V}, \overline{V} - K)$. So we consider a further extension of the diagram:

$$\begin{tikzcd}
 & H_n(S^n, S^n - V) \arrow[dd, "i"] & H_n(\overline{V}, \dot{V}) \arrow[l, "e'"', "\cong"] \arrow[dd, "j"] \\
H_n(S^n) \arrow[ur, "l"] \arrow[dr, "k"'] & & \\
 & H_n(S^n, S^n - K) \arrow[r, "e"] & H_n(\overline{V}, \overline{V} - K)
\end{tikzcd}$$

Now, l is an isomorphism, because $S^n - V$ is contractible over itself, and e' is

an isomorphism because it is an excision, as is also e. By commutativity, we find $j(\sigma^n) = ek(b')$, so that the degree of f is determined by $(\pi f)_* \partial ek(b')$, and this form does not explicitly use the fact that $\overline{V}$ is a ball.

Indeed, if $\overline{U}$ is the closure of any open set and $K \subset U$ is compact, then

$$H_n(\overline{U}, \overline{U} - K) \cong H_n(S^n, S^n - K).$$

For since $K \subset U$, we have $\overline{\complement U} = \complement U \subset \complement K = \operatorname{Int} \complement K$, so we can excise $\complement U$ to obtain the above isomorphism in homology.

We are now prepared to define the degree using singular homology.

(6.1) DEFINITION. Let $f : \overline{U} \to \boldsymbol{R}^n$ be continuous with $f^{-1}(0) = K \subset U$. The *homological degree* of f is the integer $\deg_U(f, K)$ given by the image of the generator $b \in H_n(S^n)$ under the composition

$$\begin{aligned} &H_n(S^n) \to H_n(S^n, S^n - K) \to H_n(U, U - K) \\ &\quad \to H_{n-1}(U - K) \to H_{n-1}(\boldsymbol{R}^n - \{0\}) \to H_{n-1}(S^{n-1}) = \boldsymbol{Z}. \end{aligned}$$

Thus $\deg_U(f, K)$ is given by $(\pi f)_* \partial ck(b) = \deg_U(f, K) \cdot \beta$, where β is a generator of $H_{n-1}(S^{n-1})$.

We note that the definition is formally applicable whenever the set $f^{-1}(0)$ is replaced by a larger compact set, and when U is cut down to any nbd of that set.

We now state an important localization property of the degree.

(6.2) THEOREM. *If $K \subset B \subset V \subset U$, i.e., if K is enlarged and U made smaller, then*

$$\deg_U(f, K) = \deg_V(f, B).$$

PROOF. Since $V \subset U$ and $V - B \subset V - K \subset U - K$, we have a commutative diagram

$$\begin{array}{ccccc}
H_n(S^n, S^n - B) & \longrightarrow & H_n(V, V - B) & \longrightarrow & H_{n-1}(V - B) \\
\uparrow & & & & \downarrow \\
H_n(S^n) & & & & H_{n-1}(\boldsymbol{R}^n - \{0\}) \\
\downarrow & & & & \uparrow \\
H_n(S^n, S^n - K) & \longrightarrow & H_n(U, U - K) & \longrightarrow & H_{n-1}(U - K)
\end{array}$$

so the images of b across the top and across the bottom must be the same, showing that $\deg_U(f, K)$ is independent of the choice of U and K. □

Theorem (6.2) is a handy computational tool, showing that if two maps agree on any nbd of $f^{-1}(0)$, then their degrees are the same. This tool will now permit us to prove the homotopy invariance of the degree (enlarging the compact sets is required here).

(6.3) THEOREM. *Let $f, g : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ be homotopic. Then* $\deg_U(f) = \deg_U(g)$.

PROOF. Letting $h_t : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ be a homotopy joining f and g, set $C = \{x \in U \mid h(x,t) = 0 \text{ for some } t\} = p_U h^{-1}(0)$. This set is compact, being the projection of a compact set. Moreover, $f^{-1}(0) \subset C$ and $g^{-1}(0) \subset C$, so we can use U and C to compute the degree of both f and g.

Now, $h_t : U - C \to \boldsymbol{R}^n - \{0\}$ shows $f|(U-C) \simeq g|(U-C)$, and therefore they induce the same homomorphism $H_{n-1}(U-C) \to H_{n-1}(\boldsymbol{R}^n - \{0\})$. This completes the proof. □

Next, we show that the degree has the additivity property. This requires cutting down open sets.

(6.4) THEOREM. *Let U_1 and U_2 be disjoint open sets in U, and $K = f^{-1}(0) = K_1 \cup K_2$ with $K_i \subset U_i$. Then*

$$\deg_U(f) = \deg_{U_1}(f|U_1) + \deg_{U_2}(f|U_2).$$

PROOF. Because we can cut down open sets containing K, we need only consider f on $U_1 \cup U_2$. Now,

$$\begin{aligned} H_{n-1}(U_1 \cup U_2 - (K_1 \cup K_2)) &= H_{n-1}((U_1 - K_1) \cup (U_2 - K_2)) \\ &= H_{n-1}(U_1 - K_1) \oplus H_{n-1}(U_2 - K_2), \end{aligned}$$

since these are disjoint. So we have to show that the split parts of the distinguished element in $H_{n-1}(U_1 \cup U_2 - (K_1 \cup K_2))$ are each the distinguished elements. To see this, it suffices to consider the diagram

$$\begin{tikzcd} H_n(S^n, S^n - K) \arrow[r, "\cong"] & H_n(U, U - K) \\ H_n(S^n) \arrow[u] \arrow[d] & \bigoplus_j H_n(U_j, U_j - K_j) \arrow[u, "\cong"'] \\ H_n(S^n) \oplus H_n(S^n) \arrow[r] & \bigoplus_j H_n(S^n, S^n - K_j) \arrow[u] \end{tikzcd}$$

□

The next result follows at once from the definitions involved.

(6.5) THEOREM. *If $0 \in U$ and the map $f : \overline{U} \to \boldsymbol{R}^n$ is the inclusion, then* $\deg_U(f) = 1$. □

The three theorems above indicate that if we assign to each $f : (\overline{U}, \partial U) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ the integer $\deg_U(f)$, then the function $f \mapsto \deg_U(f)$ is precisely the same as the function $f \mapsto d(f, U)$ that we obtained in an elementary fashion in Chapter IV. In particular, we infer immediately that:

(1) if $f : U \to \boldsymbol{R}^n$ has isolated zeros, then the additivity reduces the calculation of the degree to calculating it at each of those isolated zeros;
(2) since we can use balls, the degree is particularly easy to calculate, as just the degree of a map of spheres.

As a consequence, we obtain at once

(6.6) THEOREM (Poincaré–Hopf). *Let $f : \overline{U} \to \boldsymbol{R}^n$ have isolated zeros $a_1, \ldots, a_n$ in U, where $\overline{U}$ is a convex body. Then*

$$\sum_{i=1}^{n} d(f, a_i) = \deg(\pi f : \partial U \to S^{n-1}) = d(f, U). \qquad \square$$

Thus, for convex bodies we can get the degree as the sum of the individual degrees, or directly from the boundary values (cf. (E.2)(d) in §13).

7. Miscellaneous Results and Examples

A. Categories and functors

Throughout this section **C** stands for an arbitrary category and $\mathcal{D}$ for a directed set.

(A.1) Let $\mathcal{D} = \{\alpha, \beta, \ldots\}$. Show how to represent $\mathcal{D}$ as a category.

[Regard elements $\alpha, \beta, \ldots$ as objects with one morphism $\alpha \to \beta$ if $\alpha \preceq \beta$, and none otherwise.]

(A.2) Identify all functors from a directed set $\mathcal{D}$ to a category **C**.

[Any such functor $F : \mathcal{D} \to \mathbf{C}$ (called a *direct system* in **C**) consists of (i) objects $F(\alpha) = F_\alpha$ in **C**, one for each $\alpha \in \mathcal{D}$, (ii) morphisms $F[(\alpha \to \beta)] = f_{\alpha\beta} : F_\alpha \to F_\beta$ in **C**, one for each $\alpha \preceq \beta$ in $\mathcal{D}$, satisfying:
(i) $f_{\alpha\alpha} = \mathrm{id}$,
(ii) $f_{\alpha\gamma} = f_{\beta\gamma} f_{\alpha\beta}$ for $\alpha \preceq \beta \preceq \gamma$.]

(A.3) Let $\mathbf{C}^{\mathcal{D}}$ be the category (called the *category of direct systems* in **C**) whose objects are functors from $\mathcal{D}$ to **C** and whose morphisms are natural transformations $\tau : F \to G$ between such functors. Identify morphisms in the category $\mathbf{C}^{\mathcal{D}}$

(A.4) Let F $\mathcal{D} \to \mathbf{C}$ be a direct system in a category **C**. Let $\{v_\alpha : F_\alpha \to A\}$ be a family of morphisms in **C** indexed by $\mathcal{D}$. Call such a family *compatible* if $v_\alpha = v_\beta f_{\alpha\beta}$ for any $\alpha \preceq \beta$ in $\mathcal{D}$. A *direct limit* $\varinjlim F$ of F is a pair $(F^\infty, \{u_\alpha\})$ consisting of an object $F^\infty \in \mathbf{C}$ together with a compatible family $\{u_\alpha : F_\alpha \to F^\infty\}$ of morphisms in **C** (called a *universal family*) such that for any other compatible family $\{v_\alpha$ $F_\alpha \to A\}$ of morphisms in **C** there is a unique morphism $v^\infty : F^\infty \to A$ satisfying $v^\infty u_\alpha = v_\alpha$ for each $\alpha \in \mathcal{D}$. Show:
(a) If a direct limit of F exists, then it is unique up to isomorphism.
(b) A direct limit operation exists in **Ab** and is a functor from $\mathbf{Ab}^{\mathcal{D}}$ to **Ab**.

(A.5) (*Adjoint functors*) Given categories **C** and **D** consider two functors $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$. We say that F is a *left adjoint* of G if there is a natural equivalence

$$\mathbf{D}(F(\cdot), \cdot) \simeq \mathbf{C}(\cdot, G(\cdot))$$

of functors from $\mathbf{C}^* \times \mathbf{D}$ to **Ens**, i.e., there is a bijective map

$$\eta = \eta_{C,D} : \mathbf{D}(F(C), D) \to \mathbf{C}(C, G(D))$$

which is natural in C and D (recall that $\mathbf{C}^*$ is the opposite category of $\mathbf{C}$).

Let F **Ens** $\to$ **Ab** be the *free group functor*, which associates with every set X the free abelian group $F(X)$ generated by X; and let $G : \mathbf{Ab} \to \mathbf{Ens}$ be the *forgetful functor* which associates with every abelian group A its underlying set $G(A)$. Show: F is a left adjoint of G.

[Given $X \in \mathbf{Ens}$ and $A \in \mathbf{Ab}$, define a transformation

$$\eta = \eta_{X,A} : [F(X), A]_{\mathbf{Ab}} \to [X, G(A)]_{\mathbf{Ens}}$$

by assigning to a homomorphism $\phi : F(X) \to A$ its restriction $\phi|X : X \to G(A)$. This transformation is natural in both X and A, and the defining property of free abelian groups implies that $\eta_{X,A}$ is bijective.]

(A.6) If $\mathcal{D}$ is a directed set, the *constant functor* $\mathscr{K} : \mathbf{C} \to \mathbf{C}^{\mathcal{D}}$ is defined as follows: $\mathscr{K}$ associates with every $A \in \mathbf{C}$ the functor $\mathscr{K}_A : \mathcal{D} \to \mathbf{C}$ that takes each $\alpha \in \mathcal{D}$ to A and each relation $\alpha \preceq \beta$ to 1_A, and with every $f \in \mathbf{C}(A, B)$ the obvious natural transformation $\mathscr{K}_A \to \mathscr{K}_B$. Prove: The direct limit functor $\varinjlim : \mathbf{C}^{\mathcal{D}} \to \mathbf{C}$ is a left adjoint of the constant functor $\mathscr{K} : \mathbf{C} \to \mathbf{C}^{\mathcal{D}}$

[To show that the maps

$$\eta = \eta_{F,A} : [F^{\infty}, A]_{\mathbf{C}} = [\varinjlim F, A]_{\mathbf{C}} \cong \mathrm{Nat}[F, \mathscr{K}_A]$$

are natural and bijective, observe that each natural transformation $v : F \to \mathscr{K}_A$ is just a compatible family of morphisms $\{v_\alpha : F_\alpha \to A\}$.]

(A.7) Given categories $\mathbf{C}$ and $\mathbf{D}$ let $F : \mathbf{C} \to \mathbf{D}$ be a left adjoint of $G : \mathbf{D} \to \mathbf{C}$. Prove: F preserves direct limits, i.e., if $H \in \mathbf{C}^{\mathcal{D}}$ is any functor such that $\varinjlim H$ exists in $\mathbf{C}$, then $\varinjlim FH = F(\varinjlim H)$.

[By definition, for all D in $\mathbf{D}$, we have

$$[F(\varinjlim H), D]_{\mathbf{D}} \cong [\varinjlim H, G(D)]_{\mathbf{C}}.$$

Observe that a morphism $\varinjlim H \to G(D)$ corresponds to a compatible family of morphisms $\{H(\alpha) \to G(D)\}$, which, by naturality of adjointness, corresponds to a compatible family $\{FH(\alpha) \to D\}$. From this, using the universal property defining $\varinjlim FH$, deduce that $\varinjlim FH = F(\varinjlim H)$.]

B. Functor-chains

(B.1) Let $\mathbf{T}$ be a category. A *functor-chain* $C_* = \{C_n, \partial_n\}$ on $\mathbf{T}$ is a sequence of covariant functors $C_n : \mathbf{T} \to \mathbf{Ab}$ for $n \geq 0$ together with natural transformations $\partial_n : C_n \to C_{n-1}$ for $n \geq 1$ such that $\partial_{n-1}\partial_n = 0$. For each $X \in \mathbf{T}$, we set

$$H_n(X) = H_n(C_*(X)) = \mathrm{Ker}\,\partial_n^X / \mathrm{Im}\,\partial_{n+1}^X, \qquad n \geq 1,$$
$$H_0(X) = C_0(X)/\mathrm{Im}\,\partial_1^X,$$

and call $H_n(X)$ the nth *homology group* of $C_*(X)$. Prove: For each $n \geq 0$, the map $X \mapsto H_n(X)$ is the object function of a covariant functor $H_n : \mathbf{T} \to \mathbf{Ab}$.

(B.2) Define a natural transformation $\partial_0 : C_0 \to H_0$ to be, for each $X \in \mathbf{T}$, the projection $\partial_0^X : C_0(X) \to C_0(X)/\operatorname{Im}\partial_1^X$. Show:

$$\cdots \to C_n \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} H_0 \to 0$$

is also a functor-chain (called the *augmented functor-chain*) $\widehat{C}_* : \mathbf{T} \to \mathbf{Ab}$.

(B.3) A functor-chain is said to be *n-acyclic* at X if $H_n(X) = 0$. Show: For any functor-chain $C_* : \mathbf{T} \to \mathbf{Ab}$, the augmented functor-chain $\widehat{C}_* : \mathbf{T} \to \mathbf{Ab}$ is 0-acyclic for all X.

(B.4) By a *natural transformation* $T_* : C_* \to G_*$ of two functor-chains from $\mathbf{T}$ to $\mathbf{Ab}$ is meant a sequence of natural transformations

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_0 \\
& \downarrow{\scriptstyle T_n} & & \downarrow{\scriptstyle T_{n-1}} & & \downarrow{\scriptstyle T_0} \\
\cdots \longrightarrow & G_n & \xrightarrow{d_n} & G_{n-1} & \longrightarrow \cdots \longrightarrow & G_0
\end{array}$$

where each square commutes. Show:

(a) If $S_*, T_* : C_* \to G_*$ are two natural transformations of functor-chains, then so are $S_* \pm T_*$ (the meaning is clear, since values can be added in $\mathbf{Ab}$).

(b) $R_* \circ (S_* - T_*) = R_* \circ S_* - R_* \circ T_*$.

(B.5) Two natural transformations $S_*, T_* : C_* \to G_*$ of functor-chains are called *chain homotopic*, written $S_* \simeq T_*$, if there exists a sequence $D_n : C_n \to G_{n+1}$ of natural transformations such that $S_n - T_n = d_{n+1}D_n + D_{n-1}\partial_n$. Show: The chain homotopy is an equivalence relation in the set of natural functor-chain transformations from C_* to G_*.

(B.6) Let $S_*, T_* : C_* \to G_*$ be such that $S_* \simeq T_*$. Show: If $R_* : G_* \to G_*^1$ and $P_* : C_*^1 \to C_*$ are any two natural transformations, then $S_*P_* \simeq T_*P_*$ and $R_*S_* \simeq R_*T_*$.

(B.7) Let $S_*, T_* : C_* \to G_*$. Show:

(a) For each $n \geq 0$ and $X \in \mathbf{T}$ we have the induced homomorphism

$$H_n(S_*)(X) : H_n(C_*(X)) \to H_n(G_*(X)).$$

(b) If $S_* \simeq T_*$, then $H_n(S_*)(X) = H_n(T_*)(X)$ for all n and X.

C. The acyclic model theorem

(C.1) Let $G_* : \mathbf{T} \to \mathbf{Ab}$ be a functor-chain and $C : \mathbf{T} \to \mathbf{Ab}$ be any covariant functor. Assume that:

(1) C is presentable with model (M, s),

(2) G_* is q-acyclic at M, i.e., the qth homology group of $G_*(M)$ is zero.

Prove: If $S : C \to G_q$ is a natural transformation with $d_q \circ S = 0$, then there exists a natural transformation $\widehat{S} : C \to G_{q+1}$ such that $S = d_{q+1} \circ \widehat{S}$.

[C being presentable with model (M, s), consider $d_q(M)S(M)s$. Show that there is $g \in G_{q+1}(M)$ with $d_{q+1}(M)g = S(M)s$ and let $\widehat{S}(M)s = g$.]

(C.2) (*Acyclic model theorem*) Let $C_*, G_* : \mathbf{T} \to \mathbf{Ab}$ be two functor-chains such that:

(1) for each $n \geq 0$, C_n is presentable with model (M_n, s_n),

(2) for each $n \geq 1$, G_* is n-acyclic at M_n.

Prove:

(a) Every natural transformation $S : H_0(C_*) \to H_0(G_*)$ extends to a natural $S_* : C_* \to G_*$.

(b) Let $S, T : H_0(C_*) \to H_0(G_*)$ be two natural transformations such that $S(M_0)[s_0] = T(M_0)[s_0]$, where $[s_0] \in H_0(C_*(M_0))$ is the class of $s_0 \in C_0(M_0)$. Then any extensions S_*, T_* are chain homotopic. In particular, any two extensions of a fixed S are chain homotopic.

[Consider the augmented functor-chains

$$\begin{tikzcd} \cdots \arrow[r] & C_1 \arrow[r, "\partial_1"] & C_0 \arrow[r, "\partial_0"] \arrow[d, dashed, "S_0"] & H_0(C_*) \arrow[r] \arrow[d, "S"] & 0 \\ \cdots \arrow[r] & G_1 \arrow[r, "d_1"] & G_0 \arrow[r, "d_0"] & H_0(G_*) \arrow[r] & 0 \end{tikzcd}$$

and construct first S_0 using the fact that $\widehat{G}_*$ is 0-acyclic for all X (see (B.3)). Then the construction proceeds by induction. The proof of (b) is similar.]

D. Homotopy invariance of singular homology

(D.1) Let $C_* : \mathbf{Top} \to \mathbf{Ab}$ be the singular complex functor-chain and $G_* : \mathbf{Top} \to \mathbf{Ab}$ any functor-chain. Show: To construct a natural transformation $T_* : C_* \to G_*$, we need only specify $T_n(\Delta^n)s_n \in G_n(\Delta^n)$ for each n in such a way that the following diagram is commutative at s_n for each n:

$$\begin{tikzcd} C_n(\Delta^n) \arrow[r, "\partial_n"] \arrow[d, "T_n"'] & C_{n-1}(\Delta^n) \arrow[d, "T_{n-1}"] \\ G_n(\Delta^n) \arrow[r, "d_n"] & G_{n-1}(\Delta^n) \end{tikzcd}$$

(D.2) Let C_* be the singular complex functor-chain and G_* another functor-chain such that for each $n \geq 1$, G_* is n-acyclic at Δ^n. Show:

(a) To construct $T_* : C_* \to G_*$, we need only specify $T : H_0(C_*) \to H_0(G_*)$.

(b) Any two $S, T : H_0(C_*) \to H_0(G_*)$ yielding the same value at $[s_0]$ produce transformations $S_*, T_* : C_* \to G_*$ such that $S_* \simeq T_*$.

(D.3) (*Homotopy invariance*) Let $C_* : \mathbf{Top} \to \mathbf{Ab}$ be the singular complex functor-chain. Define another functor-chain $G_* \cdot \mathbf{Top} \to \mathbf{Ab}$ by

$$G_n(X) = C_n(X \times I), \qquad G_n(f) = C_n(f \times \mathrm{id})$$

and two natural transformations $h_{0*}, h_{1*} : C_* \to G_*$ corresponding to the natural homomorphisms $h_0(X), h_1(X) : C_n(X) \to C_{n+1}(X \times I)$ induced by the maps $h_0 : x \mapsto (x, 0)$ and $h_1 : x \mapsto (x, 1)$ of X into $X \times I$. Show: $h_{0*} \simeq h_{1*}$.

(D.4) Let $f, g : X \to Y$ be homotopic maps. Show that the induced maps f_*, g_* of singular chain complexes are chain homotopic.

[Given a homotopy $H : X \times I \to Y$, observe that $f = H \circ h_0$ and $g = H \circ h_1$, and apply (D.3).]

E. Exact sequence of a triple

In this subsection the homology groups are taken with rational coefficients and are therefore considered as vector spaces over Q. Recall that given a pair (X, A), the *Betti numbers* $b_n(X, A)$ are defined by

$$b_n(X, A) = \dim H_n(X, A), \qquad n = 0, 1, 2, \ldots,$$

and the *Euler characteristic* is given by

$$\chi(X,A)=\sum(-1)^i\dim H_i(X,A).$$

A pair (X,A) is said to be of *finite type* if the graded vector space $\{H_n(X,A)\}$ is of finite type. The *Poincaré polynomial* $P(t,X,A)$ of such a pair is defined by $P(t,X,A)=\sum b_n(X,A)t^n$

Let (X,Y,Z) be a *triple* of topological spaces, i.e., $Z\subset Y\subset X$, and let

$$\cdots\to H_{n+1}(X,Y)\xrightarrow{\partial_{n+1}}H_n(Y,Z)\xrightarrow{i_*}H_n(X,Z)\xrightarrow{j_*}H_n(X,Y)\xrightarrow{\partial_n}H_{n-1}(Y,Z)\to\cdots$$

be the corresponding long exact sequence of vector spaces. Let $\varrho_n(X,Y,Z)=\dim\operatorname{Im}\partial_n$.

(E.1) Let (X,Y,Z) be a triple. Establish the following relations between the vector spaces appearing in the exact sequence of (X,Y,Z):

$$\begin{aligned}b_n(Y,Z)&=\varrho_{n+1}(X,Y,Z)+\dim\operatorname{Im}i_*,\\ b_n(X,Z)&=\dim\operatorname{Im}i_*+\dim\operatorname{Im}j_*,\\ b_n(X,Y)&=\dim\operatorname{Im}j_*+\varrho_n(X,Y,Z).\end{aligned}$$

[Use the observation that if $A\xrightarrow{\alpha}B\xrightarrow{\beta}C$ is an exact sequence of vector spaces, then $\dim B=\dim\operatorname{Im}\alpha+\dim\operatorname{Im}\beta$.]

(E.2) Let $Z\subset Y\subset X$ be a triple such that Z is a deformation retract of Y Show: $H_n(X,Y)=H_n(X,Z)$ for all $n\geq 0$.

(E.3) Let $X\subset A\subset X'$ and $Y\subset B\subset Y'$ be two triples such that $X'\subset Y$ Assume that X (resp. Y) is a deformation retract of X' (resp. Y'). Show: $b_n(Y,X)\leq b_n(B,A)$ for all $n\geq 0$.

(E.4) Let $Z\subset Y\subset X$ be a triple such that the pairs (X,Y) and (Y,Z) are of finite type. Show:

(a) The pair (X,Z) is of finite type.
(b) $b_n(X,Y)+b_n(Y,Z)=b_n(X,Z)+\varrho_n(X,Y,Z)+\varrho_{n+1}(X,Y,Z)$.
(c) $b_n(X,Z)\leq b_n(X,Y)+b_n(Y,Z)$ for all $n\geq 0$.
(d) $\chi(X,Z)=\chi(X,Y)+\chi(Y,Z)$.
(e) $P(t,X,Y)+P(t,Y,Z)=P(t,X,Z)+(1+t)Q(t,X,Y,Z)$, where $Q(t,X,Y,Z)=\sum\varrho_{n+1}(X,Y,Z)t^n$

[For (e), use (E.1) and the fact that $\varrho_0(X,Y,Z)=0$.]

(E.5) Let $X_0\subset X_1\subset\cdots\subset X_s$ be a finite increasing sequence of spaces such that each pair (X_j,X_{j-1}) $(j=1,\dots,s)$ is of finite type. Show:

(a) $b_i(X_s,X_0)\leq\sum_j b_i(X_j,X_{j-1})$ for each i.
(b) $\chi(X_s,X_0)=\sum_j\chi(X_j,X_{j-1})$.
(c) $\sum_j P(t,X_j,X_{j-1})=P(t,X_s,X_0)+(1+t)Q(t)$, where $Q(t)$ is a polynomial with nonnegative coefficients.

F. Abstract critical point theory

Let X be a space and $J:X\to\boldsymbol{R}$ a real-valued function. For any $\alpha\in\boldsymbol{R}$, we let

$$(J\leq\alpha)=\{x\in X\mid J(x)\leq\alpha\},\qquad (J<\alpha)=\{x\in X\mid J(x)<\alpha\}.$$

Below, the homology groups H_q are taken with rational coefficients and regarded as vector spaces over $\boldsymbol{Q}$. We say that $J:X\to\boldsymbol{R}$ is an *h-function* if:

(h.1) for any $q \in N$ and $x \in H_q(X)$, $x \neq 0$, there exists $c \in R$ such that

$$x \in \operatorname{Im}[H_q(J \leq c) \to H_q(X)] \quad \text{but} \quad x \notin \operatorname{Im}[H_q(J < c) \to H_q(X)],$$

(h.2) for any $q \in N$, $\alpha \in R$ and $x \in H_q(J < \alpha)$, $x \neq 0$, there exists $c < \alpha$ such that

$$x \in \operatorname{Im}[H_q(J \leq c) \to H_q(J < \alpha)] \quad \text{but} \quad x \notin \operatorname{Im}[H_q(J < c) \to H_q(J < \alpha)],$$

(h.3) for any $q \in N$, any $\alpha \in R$ and any $x \in \operatorname{Ker}[H_q(J \leq \alpha) \to H_q(X)]$, $x \neq 0$, there exists $c' > \alpha$ such that

$$x \in \operatorname{Ker}[H_q(J \leq \alpha) \to H_q(J \leq c')] \quad \text{but} \quad x \notin \operatorname{Ker}[H_q(J \leq \alpha) \to H_q(J < c')].$$

(F.1) (*Critical levels*) A level $\alpha \in R$ is an *ordinary level* of J if $H_n(J \leq \alpha, J < \alpha) = 0$ for all $n \in N$. A level c is *critical* if it is not ordinary; it is *q-critical* for some $q \in N$ provided $H_q(J \leq c, J < c) \neq 0$.

Assuming that J is an h-function, show:

(a) If $H_q(X) \neq 0$, then J admits at least one q-critical level.

(b) If $H_q(J < \alpha) \neq 0$ for some $\alpha \in R$, then J admits at least one q-critical level c with $c < \alpha$.

(c) If $\operatorname{Ker}[H_q(J \leq \alpha) \to H_q(X)] \neq 0$ for some $\alpha \in R$, then J has at least one $(q+1)$-critical level c' with $\alpha < c'$.

(F.2) (*Type numbers of critical levels*) If $c \in R$ is a q-critical level of J, then we let $m_q(c) = \dim H_q(J \leq c, J < c)$ and define the *q-type number* of J by $M_q = \sum_c m_q(c)$.

Let $q \in N$ and assume that for each $\alpha \in R$ the h-function J has only a finite number of q-critical levels $c < \alpha$. Show: $b_q(X) \leq M_q$.

(F.3) (*Morse inequalities*) Assume that for any $j = 0, 1, \ldots, q+1$ the h-function J admits only a finite number of j-critical levels. Show: If the type numbers $M_0, M_1, \ldots, M_{q+1}$ are finite, then so are the Betti numbers $b_0(X), b_1(X), \ldots, b_q(X)$, and

$$\sum_{j=0}^{q+1}(-1)^{q-j}M_j \leq \sum_{j=0}^{q}(-1)^{q-j}b_j(X) \leq \sum_{j=0}^{q}(-1)^{q-j}M_j.$$

(F.4) (*Critical points*) Let $J : X \to R$ be continuous and $A \subset X$. A deformation $d : A \times I \to X$ of A into X is a *J-deformation* provided $J(d(a,t)) \leq J(a)$ for all $(a,t) \in A \times I$. A subset $Z \subset X$ is said to be the *set of critical points* of J if:

(d.1) Z is discrete in X,

(d.2) for any $\alpha \in R$, there exists a J-deformation $d : (J \leq \alpha) \times I \to X$ satisfying

$$d(z,t) = z \qquad \text{for } z \in Z \cap (J \leq \alpha),$$
$$J(d(x,1)) < J(x) \qquad \text{for } x \in (J \leq \alpha) \cap (X - Z),$$

(d.3) for each $\alpha \in R$ there exists a $\beta > \alpha$ such that $(J \leq \alpha)$ is a deformation retract of $(J < \beta)$.

We say that $J : X \to R$ is a *d-function* provided it is continuous and bounded from below, the sets $(J \leq \alpha)$ are compact for all $\alpha \in R$, and J admits a nonempty set of critical points.

Assuming that J is a d-function, show:

(a) J is an h-function.

(b) For any $z \in X$, let $X_z = (J < J(z))$ and $m_q(z) = \dim H_q(X_z \cup \{z\}, X_z)$. Then:

(i) If $m_q(z) \neq 0$, then z is a critical point of J.

(ii) $m_q(c) = \sum\{m_q(z) ; J(z) = c\}$

[For (a), use the fact that singular homology has compact carriers: for any $x \in H_q(X, A)$, there is a compact pair $(X', A') \subset (X, A)$ such that $x \in \mathrm{Im}[H_q(X', A') \to H_q(X, A)]$.]

(The above results are due to Eilenberg [1950].)

G. Selected problems

(G.1) Let X be a space and $A \subset X$ a closed strong deformation retract of some open nbd U of A (over U). Show:

(a) $H_n(X, A) \cong H_n(X/A, *)$.

(b) The above isomorphism is induced by the identification $p : (X, A) \to (X/A, *)$.

[For (a), first from $A \subset U \subset X$ and $H_n(A) \cong H_n(U)$, using exactness and excision, deduce that $H_n(X, A) \cong H_n(X, U) \cong H_n(X - A, U - A)$. Then, using (B.6) of §11, observe that $p_* : H_n(X - A, U - A) \cong H_n(X/A - *, U/A - *)$, and thus

$$H_n(X, A) \cong H_n(X/A - *, U/A - *) \cong H_n(X/A, U/A) \cong H_n(X/A, *).$$

For (b), consider the diagram

$$\begin{tikzcd}
(X - A, U - A) \arrow[r, "i_1"] \arrow[d, "p"] & (X, U) & (X, A) \arrow[l, "j_1"'] \\
(X/A - *, U/A - *) \arrow[r, "i_1'"] & (X/A, U/A) & (X/A, *).] \arrow[l, "j_1'"']
\end{tikzcd}$$

(G.2) (*Relative homeomorphisms*) A map $f : (X, A) \to (Y, B)$ is a *relative homeomorphism* if f maps $X - A$ homeomorphically onto $Y - B$.

Assume that (X, A) is a compact pair such that A is a strong deformation retract of some closed nbd N of A and let $f : (X, A) \to (Y, B)$ be a relative homeomorphism, where B is closed in Y Show: $f_* : H_n(X, A) \cong H_n(Y, B)$ for all n.

[Let $d : N \times I \to N$ be a strong deformation retraction of N onto A. Letting $\widehat{N} = f(N) \cup B$, prove that $\widehat{d} : \widehat{N} \times I \to \widehat{N}$ given by

$$\begin{aligned} \widehat{d}(b, t) &= b && \text{for } b \in B,\ t \in I, \\ \widehat{d}(y, t) &= fd(f^{-1}(y), t) && \text{for } y \in f(N),\ t \in I, \end{aligned}$$

is a strong deformation retraction of $\widehat{N}$ onto B. Conclude the proof by showing the commutativity of the diagram

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\cong"] \arrow[d, "f_*"] & H_n(X, N) & H_n(X - A, N - A) \arrow[l, "\cong"'] \arrow[d, "\cong"] \\
H_n(Y, B) \arrow[r, "\cong"] & H_n(Y, \widehat{N}) & H_n(Y - B, \widehat{N} - B).] \arrow[l, "\cong"']
\end{tikzcd}$$

(G.3) (*Mayer–Vietoris homomorphism*) (a) Consider the commutative diagram of abelian groups and homomorphisms

$$\begin{tikzcd}
 & H_1 \arrow[d, "i_1"'] \arrow[dr, "k_2"] & \\
G_1 \arrow[r, "j_1"] \arrow[dr, "k_1"'] & G \arrow[r, "j_2"] \arrow[d, "i_2"] & G_2 \\
 & H_2 &
\end{tikzcd}$$

in which both horizontal and vertical lines are exact and k_1 and k_2 are isomorphisms. Show: i_1 and j_1 are monic and $G = i_1(H_1) + j_1(G_1)$.

(b) Let (A, B) be an excisive pair and consider the Mayer–Vietoris homomorphisms

$$\Delta_B : H_n(A \cup B) \to H_n(A \cup B, B) \cong H_n(A, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B),$$
$$\Delta_A : H_n(A \cup B) \to H_n(A \cup B, A) \cong H_n(B, A \cap B) \xrightarrow{\partial} H_{n-1}(A \cap B),$$

and the following commutative (hexagonal) diagram:

$$\begin{tikzcd}
 & & H_n(A \cup B) \arrow[dll] \arrow[drr] \arrow[dd] & & \\
H_n(A \cup B, B) & & & & H_n(A \cup B, A) \\
 & & H_n(A \cup B, A \cap B) \arrow[ull] \arrow[urr] \arrow[dd] & & \\
H_n(A, A \cap B) \arrow[uu, "e"', "\cong"] \arrow[urr] \arrow[drr] & & & & H_n(B, A \cap B) \arrow[uu, "\cong", "e"'] \arrow[ull] \arrow[dll] \\
 & & H_{n-1}(A \cap B) & &
\end{tikzcd}$$

in which the diagonals are exact. Show: The homomorphisms from top to bottom along the edges differ in sign, i.e., $\Delta_B(\alpha) = -\Delta_A(\alpha)$ for $\alpha \in H_n(A \cup B)$.

[First, using (a), observe that

$$H_n(A \cup B, A \cap B) \cong H_n(A, A \cap B) \oplus H_n(B, A \cap B).$$

Then, taking $\alpha \in H_n(A \cup B)$, write the direct sum decomposition of its projection into $H_n(A \cup B, A \cap B)$; push that down into $H_{n-1}(A \cap B)$ to get zero; then commutativity gives the images along both sides.]

8. Notes and Comments

Singular homology

Singular homology in its present form is due to S. Eilenberg. The reader interested in studying the theory in full detail may want to consult the books of Dold [1980] and Vick [1973]. The axiomatic treatment of the ordinary homology (and cohomology) theories is due to Eilenberg and Steenrod and appears in their book [1952]. General homology and cohomology theories were introduced by G.W. Whitehead [1962]; for details and other references, the reader may consult Switzer's book [1975]. The reader interested in historical details is referred to the book of Dieudonné [1989].

The comparison of homology principle given in Theorem (5.3) is due to Dugundji [1966].

Cohomology theories

Cohomology can be axiomatized in the same way as homology. Let $\mathbf{K}$ be a category whose objects are pairs (X, A) of topological spaces and whose morphisms are continuous maps of such pairs

A *cohomology theory on* $\mathbf{K}$ is a sequence $h^* = \{h^n \mid n = 0, \pm 1, \pm 2, \ldots\}$ of contravariant functors $h^n : \mathbf{K} \to \mathbf{Ab}$ and a family of natural transformations

$$\partial^n : h^n(A) \to h^{n+1}(X, A)$$

such that the following axioms hold:

(1) (*Homotopy*) If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then $h^n(f_0) = h^n(f_1)$ for all n.

(2) (*Exactness*) For each $(X, A) \in \mathbf{K}$, the sequence

$$\cdots \to h^n(X, A) \to h^n(X) \to h^n(A) \xrightarrow{\partial^n} h^{n+1}(X, A) \to h^{n+1}(X) \to \cdots$$

is exact, where all unmarked maps are induced by inclusions.

(3) (*Strong excision*) The inclusion $e : (A, A \cap B) \to (A \cup B, B)$ induces an isomorphism

$$h^n(e) : h^n(A \cup B, B) \cong h^n(A, A \cap B)$$

for all n.

For a cohomology theory h^*, the graded group $\{h^n(*)\}$, where $*$ is a one-point space, is called the *group of coefficients* of the theory. By an *ordinary cohomology theory* h^* is meant one with $h^n(*) = 0$ unless $n = 0$; if moreover $h^0(*) = G$, then h^* is called an *ordinary cohomology theory with coefficients in* G.

As in the case of ordinary homology, any two ordinary cohomology theories with the same coefficient group coincide on the category of polyhedra.

Both the Čech cohomology theory and singular cohomology theory satisfy the above axioms and are in fact ordinary cohomology theories on suitable pairs of topological spaces. These two cohomology theories differ for some pairs (X, A).

For notational convenience we use the symbol H^* to denote an ordinary cohomology theory.

Singular cohomology

Let X be a space, $C_*(X) = \{C_n(X), \partial_n\}$ the singular chain complex of X, and G a fixed abelian group. Given an abelian group A, write $\mathrm{Hom}(A, G)$ for the set of homomorphisms from A to G and note that the assignment $A \mapsto \mathrm{Hom}(A, G)$ defines a cofunctor $\lambda_G : \mathbf{Ab} \to \mathbf{Ab}$. Just as in the construction of singular homology $H_*(X; G)$ with coefficients in G, instead of using the tensor product $\otimes G$, we can apply the cofunctor λ_G to the singular chain complex $C_*(X)$ to obtain a cochain complex $\mathrm{Hom}(C_*(X), G) = \{C^n(X), \delta^n\}$, where $C^n(X) = \mathrm{Hom}(C_n(X), G)$ and $\delta^n f \in \mathrm{Hom}(C_{n+1}(X), G)$ is defined by $(\delta^n f)(c) = f(\partial_{n+1} c)$, $c \in C_{n+1}(X)$. The cohomology groups of this cochain

complex are by definition the *singular cohomology groups* of X with coefficients in G and are denoted by $H^n(X;G)$. In a similar way, one defines the singular cohomology groups $H^n(X,A;G)$ for a pair (X,A); it is in fact a cofunctor from the category of pairs of spaces to the category of abelian groups. Singular cohomology is an example of an ordinary cohomology theory on suitable pairs of topological spaces.

Although the behavior of cohomology groups resembles that of homology groups, cohomology in general has a considerable advantage over homology in that it is possible to define a product of elements of $H^*(X,A) = \bigoplus H^q(X,A)$, and thus to convert $H^*(X,A)$ into a ring; in the context of Lefschetz theory, this ring structure frequently permits one to obtain more refined fixed point results than could be obtained without it.

Čech homology and cohomology

We outline the Čech method of defining the homology and cohomology groups of a space Y with coefficients in a group G. Let $\alpha = \{U(i)\}_{i\in I}$ be an indexed open cover of Y. For each $(q+1)$-tuple $s = (s_0, s_1, \dots, s_q)$ of elements of the index set I we let $|\alpha(s)| = \bigcap\{U(s_i) \mid i = 0, 1, \dots, q\}$.

Let $N(\alpha)$ be the nerve of the cover α, i.e., the realization of an abstract simplicial complex with q-dimensional simplices $N_q(\alpha) = \{s \in I^{q+1} \mid |\alpha(s)| \neq 0\}$. The q-dimensional cochain group $C^q(\alpha)$ is $\{f \mid f : N_q(\alpha) \to G\}$, and the usual coboundary operator from $C^q(\alpha)$ to $C^{q+1}(\alpha)$ defines the cohomology groups $H^q(\alpha)$ of the cover.

The set $\mathcal{D}(Y)$ of all open covers of Y is directed by refinement: a cover $\beta = \{V(j)\}_{j\in J}$ in $\mathcal{D}(Y)$ refines α if $V(j) \subset U(\varrho_j)$ for some suitably chosen function $\varrho : J \to I$. We call ϱ a *refining function*. Being defined on vertices, ϱ induces a refining map from $N_q(\beta)$ to $N_q(\alpha)$, which in turn induces refining homomorphisms $r_{\beta\alpha} : H_q(\beta) \to H_q(\alpha)$ and $r^{\alpha\beta} : H^q(\alpha) \to H^q(\beta)$. These homomorphisms do not depend on the choice of the particular refining function ϱ. Using the directed system $\mathcal{D}(Y)$ we define the *Čech homology* and *cohomology groups* of Y with coefficients in G by

$$H_q(Y;G) = \varprojlim_{\alpha\in\mathcal{D}(Y)} \{H_q(\alpha), r_{\beta\alpha}\}, \qquad H^q(Y;G) = \varinjlim_{\alpha\in\mathcal{D}(Y)} \{H^q(\alpha), r^{\alpha\beta}\}.$$

A detailed exposition of the Čech theory can be found in Eilenberg–Steenrod's book [1952].

Vietoris homology and cohomology groups

Let Y be a space and $\mathcal{D}(Y)$ the directed set of all open covers of Y, partially ordered by refinement. With each cover $\alpha \in \mathcal{D}(Y)$, we associate an abstract simplicial complex $\mathcal{V}(\alpha)$ (called the *Vietoris complex*) defined as follows: the vertices of $\mathcal{V}(\alpha)$ are the points of Y, and $q+1$ points of Y form a q-simplex of $\mathcal{V}(\alpha)$ if they are contained in a common open

set of α. The q-dimensional chain group $C_q(\mathcal{V}(\alpha))$ is the free abelian group generated by the ordered q-simplices of $\mathcal{V}(\alpha)$; the boundary operator $\partial_q : C_q(\mathcal{V}(\alpha)) \to C_{q-1}(\mathcal{V}(\alpha))$ is defined by

$$\partial((x_0,\dots,x_q)) = \sum_{i=0}^{q}(-1)^i(x_0,\dots,\widehat{x}_i,\dots,x_q),$$

where $(x_0,\dots,\widehat{x}_i,\dots,x_q)$ is the q-tuple obtained by deleting x_i. Then $H_q(\mathcal{V}(\alpha))$ is defined as the homology of the chain complex (C_q,∂_q).

When a cover $\beta \in \mathcal{D}(Y)$ refines another cover $\alpha \in \mathcal{D}(Y)$ (we write $\alpha \preceq \beta$), then $\mathcal{V}(\beta) \subset \mathcal{V}(\alpha)$, so that the inclusion $i_{\alpha\beta} : \mathcal{V}(\beta) \to \mathcal{V}(\alpha)$ induces a natural homomorphism $(i_{\alpha\beta})_* : H_q(\mathcal{V}(\beta)) \to H_q(\mathcal{V}(\alpha))$. The q-dimensional *Vietoris homology group* is defined as the inverse limit

$$H_q(Y) = \varprojlim_{\alpha\in\mathcal{D}(Y)} \{H_q(\mathcal{V}(\alpha)),(i_{\alpha\beta})_*\}.$$

For compact metric spaces, this group was introduced by Vietoris [1927].

To define the Vietoris cohomology of a space Y with coefficients in an abelian group G, let $\alpha \in \mathcal{D}(Y)$. We define $C^q(\mathcal{V}(\alpha)) = \mathrm{Hom}(C_q(\mathcal{V}(\alpha)),G)$ and let the coboundary map δ^q $C^q(\mathcal{V}(\alpha)) \to C^{q+1}(\mathcal{V}(\alpha))$ be defined by $(\delta^q f)(c) = f(\partial_{q+1}c)$, where $f \in C^q(\mathcal{V}(\alpha))$ and $c \in C_{q+1}(\mathcal{V}(\alpha))$. The cohomology groups of this cochain complex are denoted by $H^q(\mathcal{V}(\alpha);G)$. If the cover $\beta \in \mathcal{D}(Y)$ refines $\alpha \in \mathcal{D}(Y)$, then the inclusion $C_q(\mathcal{V}(\beta)) \hookrightarrow C_q(\mathcal{V}(\alpha))$ induces the dual homomorphism $C^q(\mathcal{V}(\alpha)) \to C^q(\mathcal{V}(\beta))$, and therefore the homomorphism $i^*_{\alpha\beta} : H^q(\mathcal{V}(\alpha);G) \to H^q(\mathcal{V}(\beta);G)$. The q-dimensional *Vietoris cohomology group* of Y with coefficients in G is defined by

$$H^q(Y;G)) = \varinjlim_{\alpha\in\mathcal{D}(Y)} \{H^q(\mathcal{V}(\alpha);G),i^*_{\alpha\beta}\}.$$

A special feature of the Vietoris cohomology is that it coincides with the Alexander–Kolmogoroff cohomology (called also by some authors the Alexander–Spanier cohomology); for details see the books of Spanier [1966] and Hilton–Wylie [1960].

THEOREM (Hurewicz–Dugundji–Dowker [1948]). *If a space Y is paracompact, then the Vietoris and Čech cohomology groups of Y are isomorphic.*

PROOF ([1]) Let U be an open nbd of the diagonal $\Delta = \{(y,y) \mid y \in Y \subset Y \times Y\}$. For each $x \in Y$, let $U_x = \{y \in Y \mid (x,y) \in U\}$ be the x-slice of U and denote by U^* the cover $\{U_x \mid x \in Y\}$; we note that the space Y itself is the indexing set for the cover U^* Using paracompactness of Y, one can prove that every open cover of Y has a refinement which is of the form U^*; in other words, the set of covers of the form U^* is cofinal in the directed set $\mathcal{D}(Y)$ of all open covers of Y The class of open nbd's of Δ is directed by inclusion, and we see that if $U \subset V$ are such nbd's, then the cover U^* is a refinement of V^*; moreover, there is a natural choice of the refining function which carries the index set Y of U^* into the index set Y of V^*, namely, the identity. The set $N_q(U^*)$ of q-simplices of the cover U^* is in fact a subset of $N_q(V^*)$, and the induced refining map $C^q(V^*) \to C^q(U^*)$ is the restriction, that is, the image of $f \in C^q(V^*)$ is $f|N_q(U^*)$. It follows that the Čech cohomology group $H^q(Y)$ is isomorphic to the direct limit, under the homomorphisms induced by restriction, of $H^q(U^*)$ for all neighborhoods U of the diagonal in $Y \times Y$

([1]) This proof is due to J.L. Kelley

§15. Lefschetz Theory for Maps of ANRs

In this paragraph we return to the consideration of homological methods that we first encountered in §9 devoted to the Lefschetz–Hopf theorem. Now that we have singular homology available as a tool, we shall be able to extend the Lefschetz–Hopf theorem to various classes of maps of ANRs.

We first establish a Lefschetz-type theorem for compact maps of ANRs: this involves a suitable notion of the generalized Lefschetz number. The relevant preliminaries are presented separately before we proceed to the proof. In Sections 5–7 we introduce and study four classes of noncompact maps arising in asymptotic fixed point theory. For maps from each of these classes, an asymptotic Lefschetz-type theorem is established.

In the last section we introduce the periodicity index of a map and using this invariant we establish a version of the Lefschetz theorem that conveys information about the existence not only of fixed points but also of periodic points.

1. The Leray Trace

One of the main tools of Lefschetz theory is an algebraic refinement of the factorization technique, based on the notion of the Leray trace and the generalized Lefschetz number.

Let $f : Y \to Y$ be a map and suppose f factors through a space P:

$$\begin{tikzcd} & Y \arrow[dl, "\alpha"'] \arrow[d, "f"] \\ P \arrow[r, "\beta"] & Y \end{tikzcd}$$

Then, if $\alpha\beta : P \to P$ has a fixed point, so also does f; this information is conveyed by the commutative diagram

$$\begin{tikzcd} P \arrow[r, "\beta"] \arrow[d, "\alpha\beta"'] & Y \arrow[dl, "\alpha"'] \arrow[d, "f"] \\ P \arrow[r, "\beta"] & Y \end{tikzcd}$$

and the fixed point question for f is now thrown into the same question for an associated map of another space.

Just as polytopes—specifically, maps into nerves—provide a bridge between algebraic and topological methods, so also the factorization technique serves as a bridge in fixed point theory. For consider the case where P is a finite polytope; then, for the above factorization, we know that $\alpha\beta$ will have a fixed point whenever its Lefschetz number $\lambda(\alpha\beta)$ is nonzero.

On the other hand, if we take singular homology over a field, then $H_*(Y) = \{H_q(Y)\}$ need not be a graded vector space of finite type. So our first task is to define the generalized Lefschetz number $\Lambda(f)$ of f in such a way that whenever the factorization exists, we have $\Lambda(f) = \lambda(\alpha\beta)$. This requires a suitable definition of trace for appropriate endomorphisms of arbitrary vector spaces.

We begin by recalling some properties of the ordinary trace that will be needed. In what follows all the vector spaces are taken over a fixed field K. For an endomorphism $\varphi : E \to E$ of a finite-dimensional vector space E, we let $\operatorname{tr}\varphi$ denote the ordinary trace of φ.

The following are the two basic properties of the trace function tr:

(a) (*Commutativity*) Let

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "\varphi"'] & E'' \arrow[d, "\psi"] \arrow[dl, "g"'] \\ E' \arrow[r, "f"'] & E'' \end{tikzcd}$$

be a commutative diagram of linear maps of finite-dimensional vector spaces. Then $\operatorname{tr}\varphi = \operatorname{tr}\psi$; in other words, $\operatorname{tr}(fg) = \operatorname{tr}(gf)$.

(b) (*Additivity*) Let

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r, "f"] \arrow[d, "\varphi'"'] & E \arrow[r, "g"] \arrow[d, "\varphi"'] & E'' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\ 0 \arrow[r] & E' \arrow[r, "f"] & E \arrow[r, "g"] & E'' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of linear maps of finite-dimensional vector spaces with both rows exact. Then $\operatorname{tr}\varphi = \operatorname{tr}\varphi' + \operatorname{tr}\varphi''$.

(1.1) DEFINITION. Let E be an arbitrary vector space. An endomorphism $\varphi : E \to E$ is called *nilpotent* if $\varphi^n = 0$ for some n, where $\varphi^n : E \to E$ is the nth iterate of φ; more generally, φ is called *weakly nilpotent* provided for every $x \in E$ there is an integer n_x (depending on x) such that $\varphi^{n_x}(x) = 0$.

(1.2) PROPOSITION. *If* $\dim E < \infty$ *and the endomorphism* $\varphi : E \to E$ *is nilpotent, then* $\operatorname{tr}\varphi = 0$.

PROOF. By assumption, $\operatorname{Ker}\varphi^n = E$ for some n. Write the commutative diagram

$$\begin{tikzcd}
\operatorname{Ker}\varphi \arrow[r, hook] \arrow[d, "\varphi_0"'] & \operatorname{Ker}\varphi^2 \arrow[r, hook] \arrow[d] \arrow[dl] & \cdots \arrow[r, hook] & \operatorname{Ker}\varphi^{n-1} \arrow[r, hook] \arrow[d] & \operatorname{Ker}\varphi^n = E \arrow[d, "\varphi"] \arrow[dl] \\
\operatorname{Ker}\varphi \arrow[r, hook] & \operatorname{Ker}\varphi^2 \arrow[r, hook] & \cdots \arrow[r, hook] & \operatorname{Ker}\varphi^{n-1} \arrow[r, hook] & \operatorname{Ker}\varphi^n = E
\end{tikzcd}$$

in which all vertical arrows are contractions of the endomorphism φ. Since $\varphi_0 = 0$, by applying commutativity of the trace repeatedly, we see that the trace of every vertical arrow is zero. In particular, $\operatorname{tr}\varphi = 0$. □

The category of endomorphisms and the functor "~"

We introduce some terminology to simplify the exposition of basic facts on the Leray trace.

Denote by $\mathbf{V}$ the category of vector spaces and by $\mathbf{End(V)} = \mathbf{E}$ the category whose objects are pairs (E, φ) consisting of a vector space E and an endomorphism $\varphi : E \to E$, and whose morphisms from (E', φ') and (E'', φ'') are linear maps $f : E' \to E''$ such that $\varphi'' \circ f = f \circ \varphi'$. Let $\mathbf{E}_{\text{mono}}$ be the full subcategory of $\mathbf{E}$ whose objects are pairs (E, φ) with φ injective. The category $\mathbf{E}$ (respectively $\mathbf{E}_{\text{mono}}$) is called the *category of endomorphisms* (respectively the *category of monomorphisms*).

Let $\varphi : E \to E$ be an endomorphism of a vector space E. The kernels $\operatorname{Ker}\varphi \subset \operatorname{Ker}\varphi^2 \subset \cdots$ form an increasing sequence of φ-invariant linear subspaces of E, and hence their union

$$N_\varphi = \bigcup_{n \geq 1} \operatorname{Ker}\varphi^n = \{x \in E \mid \varphi^n(x) = 0 \text{ for some } n\}$$

is a φ-invariant linear subspace of E. Consequently, the formula

$$\widetilde{\varphi}(x + N_\varphi) = \varphi(x) + N_\varphi, \qquad x \in E,$$

defines an endomorphism $\widetilde{\varphi} : \widetilde{E} \to \widetilde{E}$ on the factor space $\widetilde{E} = E/N_\varphi$.

(1.3) PROPOSITION. *For every $\varphi : E \to E$, the map $\widetilde{\varphi} : \widetilde{E} \to \widetilde{E}$ is injective.*

PROOF. Clearly, it is sufficient to prove that $\varphi^{-1}(N_\varphi) = N_\varphi$. If $x \in \varphi^{-1}(N_\varphi)$, then $\varphi(x) \in N_\varphi$, i.e., for some n, we have $\varphi^n(\varphi(x)) = 0 = \varphi^{n+1}(x)$; hence $x \in N_\varphi$. Conversely, if $x \in N_\varphi$, then $\varphi^n(x) = 0$ for some n; thus $\varphi^n(\varphi(x)) = 0$, and hence $\varphi(x) \in N_\varphi$, i.e., $x \in \varphi^{-1}(N_\varphi)$. □

Thus, to each object (E, φ) in $\mathbf{E}$ we have assigned an object $(\widetilde{E}, \widetilde{\varphi})$ in $\mathbf{E}_{\text{mono}}$. We note that if $f : (E', \varphi') \to (E'', \varphi'')$ is a morphism in $\mathbf{E}$, then $f : E' \to E''$ maps $N_{\varphi'}$ into $N_{\varphi''}$, and therefore determines a linear factor map $\widetilde{f} : \widetilde{E}' \to \widetilde{E}''$ by $x + N_{\varphi'} \mapsto f(x) + N_{\varphi''}$. It is easily seen that $\widetilde{f} \circ \widetilde{\varphi}' = \widetilde{\varphi}'' \circ \widetilde{f}$, and thus $\widetilde{f}$ is a morphism from $(\widetilde{E}', \widetilde{\varphi}')$ to $(\widetilde{E}'', \widetilde{\varphi}'')$ in $\mathbf{E}$.

(1.4) PROPOSITION. *The assignments* $(E,\varphi) \mapsto \widetilde{(E,\varphi)} = (\widetilde{E},\widetilde{\varphi})$ *and* $f \mapsto \widetilde{f}$ *define a functor* "$\widetilde{\ }$" *from* $\mathbf{E}$ *to the subcategory* $\mathbf{E}_{\mathrm{mono}}$.

PROOF. An easy verification that $\widetilde{\ }$ is a functor is left to the reader; the fact that $\widetilde{\ }$ sends arbitrary endomorphisms to monomorphisms was already established in (1.3). □

Using (1.4), the reader can easily verify the following

(1.5) PROPOSITION. *Assume that the diagram*

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r, "f"] \arrow[d, "\varphi'"'] & E \arrow[r, "g"] \arrow[d, "\varphi"'] & E'' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\ 0 \arrow[r] & E' \arrow[r, "f"] & E \arrow[r, "g"] & E'' \arrow[r] & 0 \end{tikzcd}$$

has exact rows and commutes. If $\dim \widetilde{E}' < \infty$, *then the corresponding diagram*

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{E}' \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{\varphi}'"'] & \widetilde{E} \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{\varphi}"'] & \widetilde{E}'' \arrow[r] \arrow[d, "\widetilde{\varphi}''"'] & 0 \\ 0 \arrow[r] & \widetilde{E}' \arrow[r, "\widetilde{f}"] & \widetilde{E} \arrow[r, "\widetilde{g}"] & \widetilde{E}'' \arrow[r] & 0 \end{tikzcd}$$

has the same property. □

The Leray trace

(1.6) DEFINITION. Let E be an arbitrary vector space. An endomorphism $\varphi : E \to E$ is called *admissible* whenever $\dim \widetilde{E} < \infty$. For such a φ, the endomorphism $\widetilde{\varphi} : \widetilde{E} \to \widetilde{E}$ is an isomorphism, and we define the *Leray trace* $\operatorname{Tr}\varphi$ by putting $\operatorname{Tr}\varphi = \operatorname{tr}\widetilde{\varphi}$.

We first show that the Leray trace *is* a generalization of the ordinary trace function.

(1.7) PROPOSITION. *Let* $\varphi : E \to E$ *be an endomorphism. If* $\dim E < \infty$, *then* $\operatorname{Tr}\varphi = \operatorname{tr}\varphi$.

PROOF. Write a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & N_\varphi \arrow[r] \arrow[d, "\varphi'"'] & E \arrow[r] \arrow[d, "\varphi"'] & \widetilde{E} \arrow[r] \arrow[d, "\widetilde{\varphi}"'] & 0 \\ 0 \arrow[r] & N_\varphi \arrow[r] & E \arrow[r] & \widetilde{E} \arrow[r] & 0 \end{tikzcd}$$

By additivity of the trace, we have $\operatorname{tr}\varphi = \operatorname{tr}\varphi' + \operatorname{tr}\widetilde{\varphi}$. Since $\dim E < \infty$, we have $N_\varphi = \operatorname{Ker}\varphi^n$ for some n, i.e., $\varphi^n(N_\varphi) = \{0\}$. From this we infer that $\operatorname{tr}\varphi' = 0$ (because φ' is nilpotent), and hence $\operatorname{tr}\varphi = \operatorname{tr}\widetilde{\varphi} = \operatorname{Tr}\varphi$. □

Two basic properties of the Leray trace are given in the following

(1.8) THEOREM.
(A) (Commutativity) *Let*

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "\varphi"'] & E'' \arrow[d, "\psi"] \arrow[dl, "g"'] \\ E' \arrow[r, "f"] & E'' \end{tikzcd}$$

be a commutative diagram in the category of arbitrary vector spaces. If one of φ, ψ is an admissible endomorphism, then so also is the other, and $\operatorname{Tr}\varphi = \operatorname{Tr}\psi$.

(B) (Additivity) *Let*

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r, "f"] \arrow[d, "\varphi'"'] & E \arrow[r, "g"] \arrow[d, "\varphi"'] & E'' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\ 0 \arrow[r] & E' \arrow[r, "f"] & E \arrow[r, "g"] & E'' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram in the category of vector spaces with both rows exact and such that one of the following conditions holds:
(i) *φ is admissible,*
(ii) *φ' and φ'' are admissible.*
Then all three endomorphisms are admissible, and

$$\operatorname{Tr}\varphi = \operatorname{Tr}\varphi' + \operatorname{Tr}\varphi''.$$

PROOF. In view of (1.4), property (A) follows from the commutativity of tr; the proof of additivity of Tr, based on (1.5), is left to the reader. □

We now give some examples of admissible endomorphisms:

(1.9) PROPOSITION. *Every weakly nilpotent endomorphism $\varphi : E \to E$ is admissible and* $\operatorname{Tr}\varphi = 0$.

PROOF. If $\varphi : E \to E$ is weakly nilpotent, then $E = N_\varphi$, $\widetilde{E} = 0$, and the conclusion follows. □

(1.10) PROPOSITION. *Let $\varphi : E \to E$ be an endomorphism, $E_0 \subset E$ an invariant subspace, and assume that for each $x \in E$ there is a natural n_x such that $\varphi^{n_x}(x) \in E_0$. Let $\varphi_0 : E_0 \to E_0$ be the contraction of φ.*

If one of φ_0, φ is an admissible endomorphism, then so also is the other, and $\operatorname{Tr}\varphi = \operatorname{Tr}\varphi_0$.

PROOF. Since $E_0 \subset E$ is invariant, we may write the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & E & \longrightarrow & E/E_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_1} & & \\ 0 & \longrightarrow & E_0 & \longrightarrow & E & \longrightarrow & E/E_0 & \longrightarrow & 0 \end{array}$$

Let $x + E_0$ be an arbitrary element of E/E_0. By assumption there is an integer n such that $\varphi^n(x) \in E_0$, and hence $\varphi_1^n(x + E_0) = \varphi^n(x) + E_0 = E_0$. Consequently, by (1.9), φ_1 is admissible and $\operatorname{Tr}\varphi_1 = 0$. This, in view of the additivity of the Leray trace, implies both of our assertions (i) and (ii), because $\operatorname{Tr}\varphi = \operatorname{Tr}\varphi_0 + \operatorname{Tr}\varphi_1$. □

(1.11) COROLLARY. *Any of the following conditions implies that an endomorphism $\varphi : E \to E$ is admissible:*
- (i) $\dim \varphi(E) < \infty$,
- (ii) $\dim \varphi^n(E) < \infty$ *for some n,*
- (iii) *there is an invariant finite-dimensional subspace $E_0 \subset E$ such that for each $x \in E$ there is an n with $\varphi^n(x) \in E_0$.* □

2. Generalized Lefschetz Number

We recall that a graded vector space $\{E_q\}$ is of *finite type* if (i) $\dim E_q < \infty$ for all $q \in N$ and (ii) $E_q = 0$ for all sufficiently large q. We now consider a class of graded endomorphisms of degree zero for which a generalized Lefschetz number can be defined.

(2.1) DEFINITION. Let $\varphi = \{\varphi_q\}$ be an endomorphism of a graded vector space $E = \{E_q\}$, and let $\tilde{\varphi} = \{\tilde{\varphi}_q\}$ be the induced endomorphism on the graded vector space $\tilde{E} = \{\tilde{E}_q\}$. We say that φ is a *Leray endomorphism* if $\tilde{E}$ is of finite type. For such a φ we define the *generalized Lefschetz number* $\Lambda(\varphi)$ by

$$\Lambda(\varphi) = \sum_q (-1)^q \operatorname{Tr}\varphi_q = \sum_q (-1)^q \operatorname{tr}\tilde{\varphi}_q,$$

and the *Euler number* $\chi(\varphi)$ by

$$\chi(\varphi) = \chi(\tilde{E}) = \sum_q (-1)^q \dim \tilde{E}_q.$$

If $E = \{E_q\}$ is of finite type then clearly $\lambda(\varphi) = \Lambda(\varphi)$.

The following propositions express two basic properties of the generalized Lefschetz number:

(2.2) THEOREM.
(A) (Commutativity) *Let*

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "\varphi"'] & E'' \arrow[d, "\psi"] \arrow[dl, "g"'] \\ E' \arrow[r, "f"] & E'' \end{tikzcd}$$

be a commutative diagram in the category of graded vector spaces. If one of φ, ψ is a Leray endomorphism, then so also is the other, and $\Lambda(\varphi)=\Lambda(\psi)$.

(B) (Additivity) *Let*

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r, "f"] \arrow[d, "\varphi'"'] & E \arrow[r, "g"] \arrow[d, "\varphi"'] & E'' \arrow[r] \arrow[d, "\varphi''"'] & 0 \\ 0 \arrow[r] & E' \arrow[r, "f"] & E \arrow[r, "g"] & E'' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram in the category of graded vector spaces with both rows exact and such that one of the following conditions holds:

(i) *φ is a Leray endomorphism,*
(ii) *φ' and φ'' are Leray endomorphisms.*

Then φ', φ, and φ'' are Leray endomorphisms, and

$$\Lambda(\varphi)=\Lambda(\varphi')+\Lambda(\varphi'').$$

PROOF. Clearly, this is an obvious consequence of the definitions and of the additivity and commutativity of the Leray trace. □

(2.3) THEOREM. *Let $a=\{a_q\}$, $b=\{b_q\}$, $c=\{c_q\}$ be endomorphisms of the graded vector spaces $A=\{A_q\}$, $B=\{B_q\}$, and $C=\{C_q\}$, respectively, and assume that the following diagram in the category of vector spaces is commutative and has exact rows:*

$$\begin{tikzcd} \arrow[r] & C_{q+1} \arrow[r, "\partial_{q+1}"] \arrow[d, "c_{q+1}"] & A_q \arrow[r, "i_q"] \arrow[d, "a_q"] & B_q \arrow[r, "j_q"] \arrow[d, "b_q"] & C_q \arrow[r, "\partial_q"] \arrow[d, "c_q"] & A_{q-1} \arrow[r] \arrow[d, "a_{q-1}"] & \cdot \\ \arrow[r] & C_{q+1} \arrow[r, "\partial_{q+1}"] & A_q \arrow[r, "i_q"] & B_q \arrow[r, "j_q"] & C_q \arrow[r, "\partial_q"] & A_{q-1} \arrow[r] & \cdots \end{tikzcd}$$

If any two of a, b, c are Leray endomorphisms, then so also is the third, and $\Lambda(c)=\Lambda(b)-\Lambda(a)$.

PROOF. Set

$$\begin{aligned} A'_q &= \operatorname{Im}\partial_{q+1} = \operatorname{Ker} i_q, & A' &= \{A'_q\}, \\ B'_q &= \operatorname{Im} i_q = \operatorname{Ker} j_q, & B' &= \{B'_q\}, \\ C'_q &= \operatorname{Im} j_q = \operatorname{Ker}\partial_q, & C' &= \{C'_q\}. \end{aligned}$$

By assumption, we have the following commutative diagrams with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A'_q \arrow[r] \arrow[d, "a'_q"] & A_q \arrow[r] \arrow[d, "a_q"] & B'_q \arrow[r] \arrow[d, "b'_q"] & 0 \\ 0 \arrow[r] & A'_q \arrow[r] & A_q \arrow[r] & B'_q \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & B'_q \arrow[r] \arrow[d, "b'_q"] & B_q \arrow[r] \arrow[d, "b_q"] & C'_q \arrow[r] \arrow[d, "c'_q"] & 0 \\ 0 \arrow[r] & B'_q \arrow[r] & B_q \arrow[r] & C'_q \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & C'_q \arrow[r] \arrow[d, "c'_q"] & C_q \arrow[r] \arrow[d, "c_q"] & A'_{q-1} \arrow[r] \arrow[d, "a'_{q-1}"] & 0 \\ 0 \arrow[r] & C'_q \arrow[r] & C_q \arrow[r] & A'_{q-1} \arrow[r] & 0 \end{tikzcd}$$

In the above diagrams a'_q, b'_q, c'_q are the evident contractions of a_q, b_q, c_q, respectively, and we let $a' = \{a'_q\}$, $b' = \{b'_q\}$, $c' = \{c'_q\}$.

By examining the above diagrams and taking into account Theorem (2.2), we easily verify (i). To prove (ii), we note that from the first and second diagrams we get, using additivity,

$$\Lambda(a) = \Lambda(a') + \Lambda(b'), \qquad \Lambda(b) = \Lambda(b') + \Lambda(c'),$$

and from the third (because of the shift in dimension), $\Lambda(c) = \Lambda(c') - \Lambda(a')$. Consequently, $\Lambda(c) = \Lambda(b) - \Lambda(a)$. □

3. Lefschetz Maps and Lefschetz Spaces

Consider the category $\mathbf{Top}^2$ of pairs of topological spaces and continuous maps, and let H_* be the singular homology functor from $\mathbf{Top}^2$ to the category of graded vector spaces over K. Thus, for a pair (X, A) of spaces, $H_*(X, A) = \{H_q(X, A)\}$ is a graded vector space, $H_q(X, A)$ being the q-dimensional relative homology group with coefficients in K. For a map $f : (X, A) \to (Y, B)$, $H_*(f)$ is the induced linear map $f_* = \{f_{*q}\}$, where $f_{*q} : H_q(X, A) \to H_q(Y, B)$.

A map $f : X \to X$ is called *homologically trivial* provided that the induced homomorphisms $f_{*q} : H_q(X) \to H_q(X)$ are trivial for all $q \geq 1$ and $H_0(X) \cong K$, the scalar field. A space X is said to be *acyclic* if (i) $H_0(X) \cong K$ and (ii) $H_q(X) = 0$ for all $q \geq 1$.

Given a map $f : (X, A) \to (X, A)$ we denote by $f_X : X \to X$ and $f_A : A \to A$ the evident maps defined by f.

(3.1) DEFINITION. A map $f : (X, A) \to (X, A)$ is called a *Lefschetz map* if $f_* : H_*(X, A) \to H_*(X, A)$ is a Leray endomorphism. For such an f we define the *generalized Lefschetz number* $\Lambda(f)$ by $\Lambda(f) = \Lambda(f_*)$ and the *Euler number* by $\chi(f) = \chi(f_*)$.

Clearly, the property of f to be a Lefschetz map depends only on the homotopy class of f; and if $\Lambda(f)$ is defined and $f \simeq g$ then $\Lambda(f) = \Lambda(g)$.

Some simple examples of Lefschetz maps are given in

(3.2) LEMMA. *Let X be a pathwise connected space and $f : X \to X$ a map such that for some $n \geq 1$ one of the following conditions is satisfied:*
(i) *$f^n : X \to X$ is homologically trivial,*
(ii) *$f^n : X \to X$ is nullhomotopic,*
(iii) *$f^n(X)$ is contained in an acyclic subset A of X,*
(iv) *X is acyclic.*
Then f is a Lefschetz map and $\Lambda(f) = 1$.

PROOF. (i) Let $q \geq 1$; since $(f^n)_{*q} = 0$, it follows that f_{*q} is nilpotent, and hence $\operatorname{Tr} f_{*q} = 0$ by (1.9). From this, because X is connected, we get $\Lambda(f) = 1$.

(ii) Because f^n is homotopic to a constant map, $f^n_{*q} = 0$ for each $q \geq 1$, and therefore $\Lambda(f) = 1$ as in (i).

(iii) This case reduces to (i) by observing that f^n factors as

$$X \to A \overset{j}{\hookrightarrow} X,$$

and since $j_* = 0$, we get $f^n_{*q} = 0$, again giving $\Lambda(f) = 1$; (iv) is obvious. □

We are now ready to state the basic properties of Lefschetz maps and generalized Lefschetz numbers:

(3.3) THEOREM.
(i) (Commutativity) *Let*

$$\begin{tikzcd}
(X,A) \arrow[r, "f"] \arrow[d, "gf"'] & (Y,B) \arrow[d, "fg"] \arrow[dl, "g"'] \\
(X,A) \arrow[r, dashed, "f"] & (Y,B)
\end{tikzcd}$$

be a commutative diagram of pairs of spaces and maps. If one of the composites $g \circ f$, $f \circ g$ is a Lefschetz map, then so also is the other, and $\Lambda(g \circ f) = \Lambda(f \circ g)$.

(ii) (Additivity) *Let $f : (X, A) \to (X, A)$ be a map. If any two of f, f_X, f_A are Lefschetz maps, then so also is the third, and $\Lambda(f) = \Lambda(f_X) - \Lambda(f_A)$.*

PROOF. To prove (i), we apply the functor H_* to the above diagram and obtain the commutative diagram of graded vector spaces

$$\begin{tikzcd} H_*(X,A) \arrow[r, "f_*"] \arrow[d, "(g\circ f)_*"'] & H_*(Y,B) \arrow[d, "(f\circ g)_*"] \arrow[dl, "g_*"'] \\ H_*(X,A) \arrow[r, "f_*"'] & H_*(Y,B) \end{tikzcd}$$

Now our assertion follows clearly from the definitions involved and Theorem (2.2).

To establish (ii), write the commutative diagram with exact rows

$$\begin{tikzcd} \cdots \arrow[r] & H_q(A) \arrow[r, "i_q"] \arrow[d, "(f_A)_{*q}"] & H_q(X) \arrow[r, "j_q"] \arrow[d, "(f_X)_{*q}"] & H_q(X,A) \arrow[r, "\partial_q"] \arrow[d, "f_{*q}"] & H_{q-1}(A) \arrow[r] \arrow[d, "(f_A)_{*q-1}"] & \cdots \\ \cdots \arrow[r] & H_q(A) \arrow[r, "i_q"] & H_q(X) \arrow[r, "j_q"] & H_q(X,A) \arrow[r, "\partial_q"] & H_{q-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

(= the endomorphism of the homology sequence of the pair (X, A) induced by f). Now our assertion follows from the definitions involved and (2.3). □

(3.4) DEFINITION. A map $f : X \to X$ is called *strongly Lefschetz* provided (i) $\Lambda(f)$ is defined and (ii) $\Lambda(f) \neq 0$ implies that f has a fixed point. A space X is said to be a *Lefschetz space* if any compact map $f : X \to X$ is strongly Lefschetz.

With this terminology, we have two simple general results that are frequently used:

(3.5) LEMMA. *Let $\alpha : K \to X$ and $\beta : X \to K$. Then $\beta\alpha : K \to K$ is strongly Lefschetz if and only if so is $\alpha\beta : X \to X$. In other words, given the commutative diagrams*

$$\begin{tikzcd} & K \arrow[dl, "\alpha"'] \arrow[d, "f"] \\ X \arrow[r, "\beta"'] & K \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & X \arrow[dl, "\beta"'] \arrow[d, "F"] \\ K \arrow[r, "\alpha"'] & X \end{tikzcd}$$

we have: f is strongly Lefschetz $\Leftrightarrow$ F is strongly Lefschetz.

PROOF. Our assertion follows at once from (3.3) and (0.4.1). □

(3.6) LEMMA. *Let K be a compact space and assume that $f : K \to K$ factors through a Lefschetz space X, i.e., there are maps α and β making the diagram*

$$\begin{tikzcd} & K \arrow[dl, "\alpha"'] \arrow[d, "f"] \\ X \arrow[r, "\beta"'] & K \end{tikzcd}$$

commutative. Then f is strongly Lefschetz.

PROOF. This is an obvious consequence of (3.5). □

We now describe two simple ways of forming new Lefschetz spaces from old ones.

(3.7) THEOREM. *If X is a Lefschetz space, then so is every retract of X.*

PROOF. Let $r : X \to A$ be the retraction and $i : A \to X$ the inclusion. Consider a compact $f : A \to A$; then $ifr : X \to X$ is also compact. Letting $\alpha = i : A \to X$ and $\beta = fr : X \to A$, we have $\alpha\beta = ifr$ and $\beta\alpha = f$. Because $\alpha\beta$ is, by assumption, strongly Lefschetz, (3.5) implies that so is $\beta\alpha = f$; since f was arbitrary, our assertion follows. □

(3.8) THEOREM. *Let K be a compact space such that every $f : K \to K$ factors through a Lefschetz space. Then K is a Lefschetz space.*

PROOF. This is an obvious consequence of (3.6). □

4. Lefschetz Theorem for Compact Maps of ANRs

The Lefschetz theorem for compact maps of ANRs, which will be established in this section, includes several known fixed point results, both in topology and in nonlinear analysis.

First, by combining the Lefschetz–Hopf theorem for polyhedra with the approximation theorem (12.3.1) we obtain the following preliminary result.

(4.1) THEOREM. *Any open subset of a normed linear space is a Lefschetz space.*

PROOF. Let U be open in a normed linear space E, and let $f : U \to U$ be compact. We first show that f is a Lefschetz map. Applying (12.3.1) to the map f, choose a positive $\varepsilon < \operatorname{dist}(\overline{f(U)}, \partial U)$ and an ε-approximation $f_\varepsilon : U \to U$ of f such that $f_\varepsilon(U) \subset K_\varepsilon \subset U$, where K_ε is a finite polytope and f_ε is homotopic to f.

To prove that f is Lefschetz, consider the commutative diagram

$$\begin{tikzcd} K_\varepsilon \arrow[r, hook, "i"] \arrow[d, "f'_\varepsilon"'] & U \arrow[d, "f_\varepsilon"] \arrow[dl] \\ K_\varepsilon \arrow[r, hook, "i"] & U \end{tikzcd}$$

with the obvious contractions of f_ε. Since $H_*(K_\varepsilon)$ is of finite type, f'_ε is Lefschetz, and hence by (3.3), so also is f_ε, and $\lambda(f'_\varepsilon) = \Lambda(f_\varepsilon)$. Because $f \simeq f_\varepsilon$, we infer that f is Lefschetz and

$$\Lambda(f) = \Lambda(f_\varepsilon) = \lambda(f'_\varepsilon).$$

Assume now that $\Lambda(f) \neq 0$. Then $\lambda(f'_\varepsilon) \neq 0$, and applying the Lefschetz–Hopf Theorem (9.2.4) to $f'_\varepsilon : K_\varepsilon \to K_\varepsilon$, we get a point $x_0 \in U$ such that $x_0 = f'_\varepsilon(x_0) = f_\varepsilon(x_0)$. Because $\|f(x_0) - x_0\| = \|f(x_0) - f_\varepsilon(x_0)\| < \varepsilon$, x_0 is an ε-fixed point for f. Since f is compact and ε was arbitrary, this completes the proof. □

(4.2) THEOREM.

(a) *Let K be a compact metric space and suppose that $f : K \to K$ can be factored as $K \xrightarrow{\alpha} X \xrightarrow{\beta} K$, where X is an ANR. Then f is strongly Lefschetz.*

(a)* *Let X be an ANR and suppose that $F : X \to X$ can be factored as $X \xrightarrow{\beta} K \xrightarrow{\alpha} X$, where K is a compact metric space. Then F is strongly Lefschetz.*

PROOF. In view of (3.5), it is clearly enough to establish (a). To this end consider the diagram

$$\begin{tikzcd} E & U \arrow[l, hook'] \arrow[dd, dashed, "\varphi"'] & & \widehat{K} \arrow[ll, hook', "j"'] \arrow[dl, shift left, "s^{-1}"] \\ & & K \arrow[ur, shift left, "s"] \arrow[dl, "\alpha"'] \arrow[d, "f"] & \\ & X \arrow[r, "\beta"] & K & \end{tikzcd}$$

where X is an ANR, $E = l^2$ is the Hilbert space, and $s : K \to \widehat{K}$ is a homeomorphism of K onto $\widehat{K} \subset E$ with inverse $s^{-1} : \widehat{K} \to K$. Since X is an ANR, there is an extension $\varphi : U \to X$ of the map $\alpha s^{-1} : \widehat{K} \to X$ over an open nbd U of $\widehat{K}$ in E, i.e., $\varphi j = \alpha s^{-1}$, where $j : \widehat{K} \to U$ is the inclusion. Now consider the composite

$$K \xrightarrow{js} U \xrightarrow{\beta\varphi} K.$$

We have $(\beta\varphi)(js) = \beta(\varphi j)s = \beta(\alpha s^{-1})s = \beta\alpha = f$ Thus, in view of (4.1), f factors through a Lefschetz space, and hence by (3.6), we infer that f is a strongly Lefschetz map. □

As a special case of (4.2)(a*) we obtain the first main result of this paragraph:

(4.3) THEOREM. *Any ANR is a Lefschetz space.* □

(4.4) COROLLARY (Lefschetz). *Any compact ANR is a Lefschetz space.* □

(4.5) COROLLARY (Browder–Eells). *Any Banach manifold and more generally any Fréchet manifold is a Lefschetz space.* □

As a direct consequence of (4.3) and (3.2) we have

(4.6) THEOREM. *Let X be a connected ANR and, in particular, a connected metric space in one of the following classes:*
(i) *finite-dimensional manifolds,*
(ii) *Banach or Fréchet manifolds,*
(iii) *finite unions of closed convex sets in metrizable locally convex spaces.*

Assume furthermore that $f : X \to X$ is a compact map such that for some $n \geq 1$ one of the following conditions is satisfied:
(a) *$f^n : X \to X$ is homologically trivial,*
(b) *$f^n : X \to X$ is nullhomotopic,*
(c) *$f^n(X)$ is contained in an acyclic subset A of X.*

Then $\Lambda(f) = 1$, and f has a fixed point. □

5. Asymptotic Fixed Point Theorems for ANRs

In Sections 5–7 we develop some fixed point theorems for ANRs in which the existence of a fixed point of a map f is established from assumptions on the iterates f^n of f. In the next section we shall define and study the following four classes of locally compact maps arising in asymptotic fixed point theory:

$\mathscr{K}_{\text{ev}}$ = the class of *eventually compact* maps,

$\mathscr{K}_{\text{as}}$ = the class of *asymptotically compact* maps,

$\mathscr{K}_{\text{at}}$ = the class of maps *of compact attraction*,

$\mathscr{K}_{\text{ca}}$ = the class of *compactly absorbing* maps.

For convenience, any of these classes is called *basic*, and a map belonging to a basic class is simply called a *basic map*. With this terminology, our goal is to establish the following general Lefschetz-type theorem: If X is an ANR and $f : X \to X$ is a basic map, then f is a strongly Lefschetz map.

We begin by introducing some terminology and notation.

Given a map $f : X \to X$ and a subset $K \subset X$, we let

$$C_f = \bigcap_{n\geq 1} f^n(X) \quad \text{and} \quad O_K = \bigcup_{n\geq 1} f^n(K);$$

the set C_f is called the *core* of the map f, and O_K is said to be the *orbit* of K under f. If $K = \{x\}$ consists of a single point $x \in X$, then $O_x = O_K$ is the orbit of the point x. A set $A \subset X$ is *invariant* if $f(A) \subset A$. Clearly, the core C_f and all the orbits O_K, $K \subset X$, are invariant subsets for f.

(5.1) DEFINITION. Let $f : X \to X$ be a map, and let A and K be two subsets of X. We say that:

(i) A *absorbs* K if $f^n(K) \subset A$ for all sufficiently large n,

(ii) A *attracts* a point $x \in X$ if $\overline{O}_x \cap A \neq \emptyset$,

(iii) A is an *attractor* for f if it attracts all the points in X.

(iv) A is a *stable attractor* for f if:

(a) A is an invariant compact attractor for f,

(b) A admits arbitrarily small open nbds $U \supset A$ such that $f(U) \subset U$ and $f|U : U \to U$ is a compact map.

(5.2) PROPOSITION. *Let $f : X \to X$ be a map. If the orbit O_x of a point $x \in X$ is relatively compact, then the core C_f is nonempty and attracts the point x.*

PROOF. Set $K = \overline{O}_x$; since by assumption, K is compact, it follows that $f(K) \subset K$. This, in turn, implies that $K \supset f(K) \supset \cdots \supset f^n(K) \supset \cdots$ is a descending sequence of compact sets, so $\bigcap_{n\geq 1} f^n(K)$ is nonempty and contained in both $K = \overline{O}_x$ and C_f. □

(5.3) PROPOSITION. *Let $f : X \to X$ be a map and U an open invariant set for f that absorbs every point in X. Then U absorbs every compact set in X.*

PROOF. By assumption, for each $x \in X$ there is an integer $n(x)$ such that $f^{n(x)}(x) \in U$. By continuity, there is an open neighbourhood U_x of x in X such that $f^{n(x)}(y) \in U$ for all $y \in U_x$. Let K be a compact subset of X. Then there is a finite covering $\{U_{x_1}, \ldots, U_{x_l}\}$ of K. For $n \geq N = \max\{n(x_1), \ldots, n(x_l)\}$ and $x \in K$ we have $f^n(x) \in U$. □

6. Basic Classes of Locally Compact Maps

We are now ready to describe the basic classes of maps $\mathscr{K}_{\mathrm{ev}}$, $\mathscr{K}_{\mathrm{as}}$, $\mathscr{K}_{\mathrm{at}}$ and $\mathscr{K}_{\mathrm{ca}}$.

A map $f : X \to Y$ between topological spaces is called *locally compact* provided each $x \in X$ has a nbd U_x such that the restriction $f|U_x : U_x \to Y$ is compact.

(6.1) DEFINITION. Let X be a regular space and $f : X \to X$ a locally compact map.

(i) f is called *eventually compact*, written $f \in \mathscr{K}_{\text{ev}}$, provided the iterate $f^n : X \to X$ of f is compact for some $n \geq 2$.

(ii) f is called *asymptotically compact*, written $f \in \mathscr{K}_{\text{as}}$, provided the core C_f is relatively compact and all the orbits O_x are relatively compact.

(iii) f is said to be of *compact attraction*, written $f \in \mathscr{K}_{\text{at}}$, if f has a compact attractor.

(iv) f is called *compactly absorbing*, written $f \in \mathscr{K}_{\text{ca}}$, if there is an open set $U \subset X$ satisfying: (a) $f(U) \subset U$, (b) U absorbs compact sets in X, (c) $f_U : U \to U$ is a compact map.

We shall now establish several connections between the basic classes of maps and the class $\mathscr{K}$. These results, though more general than required in the context of ANRs, are stated explicitly for future use.

To relate compact and eventually compact maps we need the following

(6.2) LEMMA. *Let X be a regular space and $f : X \to X$ a locally compact map such that $f^2 : X \to X$ is compact. Then there is an open set $U \subset X$ satisfying*:

(i) $f(U) \subset U$,

(ii) $f_U : U \to U$ *is a compact map*,

(iii) $\overline{f^2(X)} \subset U$,

(iv) $\operatorname{Fix}(f) \subset U$

PROOF. We shall use the following elementary observation: if $K \subset X$ is compact and U is a nbd of K, then by the regularity of X there is a nbd W of K such that $W \subset \overline{W} \subset U$.

Consider the compact set $K = \overline{f^2(X)}$ and note that $f(K) \subset K$, because $f(f^2(X)) \subset f^2(X)$ and by continuity $f(K) = f(\overline{f^2(X)}) \subset \overline{f^2(X)} = K$. Since f is locally compact, there is a nbd V_0 of K such that $\overline{f(V_0)} = L$ is compact. By the above observation we can find an open set W_0 such that

$$K \subset W_0 \subset \overline{W_0} \subset V_0.$$

Letting $V_1 = f^{-1}(W_0)$, we observe that since $K \subset f^{-1}(K)$ and $f^2(L) \subset W_0$, we have

$$(*) \qquad \overline{f(V_1)} \subset V_0, \qquad K \cup f(L) \subset V_1.$$

We now choose an open set W_1 such that

$$K \cup f(L) \subset W_1 \subset \overline{W_1} \subset V_1,$$

and letting $V_2 = f^{-1}(W_1)$ we obtain, by a similar argument,

$$(**) \qquad \overline{f(V_2)} \subset V_1, \qquad K \cup L \subset V_2.$$

We define $U = V_0 \cap V_1 \cap V_2$ and show that properties (i)–(iv) are satisfied: indeed, using (*) and (**) we get

$$\begin{aligned} f(U) &\subset f(V_0) \cap f(V_1) \cap f(V_2) \subset L \cap \overline{f(V_1)} \cap \overline{f(V_2)} \\ &\subset V_2 \cap V_0 \cap V_1 = U, \end{aligned}$$

and thus (i) and (ii) are established. From $f^2(X) \subset K \subset U$, it follows that (iii) and (iv) are also satisfied. □

Repeated application of this lemma leads to a result relating compact and eventually compact maps:

(6.3) THEOREM. *Let X be a regular space and $f : X \to X$ an eventually compact map. Then there is an open set $U \subset X$ such that*
 (i) $\overline{f(U)} \subset U$,
 (ii) $\overline{f^n(X)} \subset U$ *for some* $n \geq 2$,
 (iii) $f_U : U \to U$ *is a compact map,*
 (iv) $\mathrm{Fix}(f) \subset U$ □

We relate maps of compact attraction and eventually compact maps in

(6.4) THEOREM. *Let X be a regular space and $f : X \to X$ a map of compact attraction. Then there is an open set $U \subset X$ satisfying:*
 (i) $f(U) \subset U$,
 (ii) *U absorbs compact sets in X,*
 (iii) $f_U : U \to U$ *is an eventually compact map.*

PROOF. Let A be a compact attractor for f. Since f is locally compact, there is an open set V containing A such that $\overline{f(V)} = K$ is compact. For each $x \in X$, there is a nbd U_x and an integer $n(x)$ such that $f^{n(x)}(y) \in V$ for all $y \in U_x$. Take a finite covering $\{U_{x_1}, \ldots, U_{x_k}\}$ of the compact set K and put $N = \max\{n(x_i) \mid 1 \leq i \leq k\}$.

Now we define

$$U = V \cup f^{-1}(V) \cup \ldots \cup f^{-(N-1)}(V) \cup f^{-N}(V).$$

Since for each $y \in K$ we have $y \in f^{-n(x)}(V)$ for some $n(x)$ with $0 \leq n(x) \leq N$, we infer that $K \subset U$.

We show that U is the required open set: we have

$$\begin{aligned} f(U) &\subset f(V) \cup V \cup f^{-1}(V) \cup \ldots \cup f^{-(N-1)}(V) \\ &\subset K \cup V \cup f^{-1}(V) \cup \ldots \cup f^{-(N-1)}(V) \subset U. \end{aligned}$$

Next, since U is invariant and $V \subset U$, it follows that U absorbs points in X; consequently, by (5.3), U also absorbs compact sets.

Finally, in view of the inclusions

$$\begin{aligned} f^{N+1}(U) &\subset f^{N+1}(V) \cup f^{N}(V) \cup f^{N-1}(V) \cup \ldots \cup f(V) \\ &\subset f^{N}(K) \cup \ldots \cup K \subset U, \end{aligned}$$

we infer that $f^{N+1}(U)$ is contained in the union of a finite number of compact sets, and hence f_U is eventually compact. □

As an immediate consequence of (6.3) and (6.4) we obtain

(6.5) THEOREM. *Let X be a regular space and $f : X \to X$ a map of compact attraction. Then f is compactly absorbing.* □

(6.6) COROLLARY. *Let X be a regular space. Then every map $f : X \to X$ of compact attraction has a stable compact attractor.*

PROOF. Let A_0 be a compact attractor for f. By (6.5) there is an invariant open nbd U of A_0 such that $\overline{f(U)} \subset U$ is compact. Letting $A = A_0 \cup \overline{f(U)} \subset U$, we have $f(A) \subset A$; and if $V \subset U$ is a smaller open nbd of A, then $\overline{f(V)} \subset \overline{f(U)} \subset A \subset V$, thus showing that A is a stable compact attractor for f. □

We now summarize the preceding discussion in the following

(6.7) THEOREM. *Assume that all topological spaces under consideration are regular. Then the basic classes of maps and the class $\mathscr{K}$ are related by the inclusions*

$$\mathscr{K} \subset \mathscr{K}_{\text{ev}} \subset \mathscr{K}_{\text{as}} \subset \mathscr{K}_{\text{at}} \subset \mathscr{K}_{\text{ca}}.$$

PROOF. The second inclusion is obvious; the third follows from (5.2), which implies that if $f \in \mathscr{K}_{\text{as}}$, then $\overline{C}_f$ is an attractor for f. Finally, the last inclusion was already established in (6.5). □

7. Asymptotic Lefschetz-Type Results in ANRs

The proof of the main result of this section relies on the following property of Lefschetz maps:

(7.1) LEMMA. *Let X be a space, $A \subset X$, and $f : (X, A) \to (X, A)$ a map such that A absorbs compact sets in X. Then:*

(i) *f is a Lefschetz map with $\Lambda(f) = 0$,*
(ii) *if one of the maps f_X, f_A is a Lefschetz map, then so is the other, and in that case $\Lambda(f_X) = \Lambda(f_A)$.*

PROOF. To prove (i), consider the induced endomorphism $f_{*q} : H_q(X, A) \to H_q(X, A)$ for some $q \geq 0$. Since A absorbs compact sets and singular homology has compact supports (see (14.1.6)), it follows that f_{*q} is weakly nilpotent, and hence by (1.9), $\operatorname{Tr} f_{*q} = 0$. This being true for every $q \geq 0$, our assertion follows.

To establish (ii), assuming that one of f_X and f_A is a Lefschetz map we conclude by (i) and (3.3) that so also is the other, and $\Lambda(f) = \Lambda(f_X) - \Lambda(f_A)$. Since $\Lambda(f) = 0$, the assertion follows. □

We are now ready to establish a general asymptotic fixed point theorem:

(7.2) THEOREM. *Let X be an ANR and $f : X \to X$ a compactly absorbing map. Then f is a strongly Lefschetz map.*

PROOF. Let U be as in (6.1)(iv). Since U is an ANR, it follows, because of (6.1)(iv)(c), that $\Lambda(f_U)$ is defined; consequently, by (7.1), f is a Lefschetz map and $\Lambda(f) = \Lambda(f_U)$. If $\Lambda(f) \neq 0$, then $\Lambda(f_U) \neq 0$; since $f_U : U \to U$ is a compact map of an ANR, we get by (4.3) a point $x_0 \in U$ such that $f(x_0) = x_0$. □

(7.3) COROLLARY (Browder). *Let X be an ANR, and let $f : X \to X$ be asymptotically compact. Then f is a strongly Lefschetz map.* □

(7.4) COROLLARY (Fournier). *Let X be an ANR and $f : X \to X$ a map of compact attraction. Then f is a strongly Lefschetz map.* □

To give some applications, we first establish the following

(7.5) LEMMA. *Let X be a pathwise connected space and $f : X \to X$ a map such that one of the following conditions is satisfied:*
(i) *there exists an acyclic set $A \subset X$ that absorbs all compact sets $K \subset X$,*
(ii) *for each compact set $K \subset X$ there exists an acyclic set $A_K \subset X$ such that $f^n(K) \subset A_K$ for all sufficiently large n.*
Then f is a Lefschetz map and $\Lambda(f) = 1$.

PROOF. Since singular homology has compact supports, condition (i) implies that for each $q \geq 1$ the map f_{*q} is weakly nilpotent, so $\operatorname{Tr} f_{*q} = 0$, and because f_{*0} is the identity on $H_0(X) \cong K$, and hence $\operatorname{Tr} f_{*0} = 1$, the conclusion follows.

The proof under assumption (ii) is left to the reader. □

With the aid of (7.5) and by taking into account general properties of ANRs, we obtain a general result embracing numerous fixed point theorems.

(7.6) THEOREM. *Let X be a connected ANR and, in particular, a connected metric space in one of the following classes:*
(i) *finite polyhedra,*
(ii) *finite-dimensional manifolds,*
(iii) *Banach or Fréchet manifolds,*
(iv) *finite unions of closed convex sets in metrizable locally convex spaces.*

Assume furthermore that $f : X \to X$ is a basic map such that one of the following conditions is satisfied:
(i) *there exists an acyclic set $A \subset X$ that absorbs all compact sets $K \subset X$,*
(ii) *for each compact set $K \subset X$ there exists an acyclic set $A_K \subset X$ such that $f^n(K) \subset A_K$ for all sufficiently large n.*

Then $\Lambda(f) = 1$ and f has a fixed point. □

8. Periodicity Index of a Map. Periodic Points

In this section the singular homology groups are taken over the field Q of rational numbers. Let X be a pathwise connected space and $f : X \to X$ a Lefschetz map. We recall that in this case we consider the automorphism $\tilde{f}_* = \{\tilde{f}_{*q}\}$ of the graded group $\widetilde{H_*(X)} = \{\widetilde{H_q(X)}\}$, and the Lefschetz number of f is $\Lambda(f) = \lambda(\tilde{f}_*)$, while the Euler number of f is $\chi(f) = \sum_{q=0}^{\infty}(-1)^q \dim \widetilde{H_q(X)}$.

We are concerned with the problem of detecting from f itself whether some iterate of it has Lefschetz number different from 0. We know that $\Lambda(f)$ can be expressed using only the eigenvalues of the automorphisms $\tilde{f}_{*q}$; indeed, from (9.1.2) we find that each $\operatorname{tr} \tilde{f}_{*q}$ appearing in $\lambda(\tilde{f}_*) = \Lambda(f)$ is simply the sum of the (possibly complex) eigenvalues of $\tilde{f}_{*q}$. It is therefore natural to introduce another algebraic invariant using only the number of eigenvalues that each $\tilde{f}_{*q}$ has (rather than their sum) and ask whether it too has a bearing on the fixed point problem for f.

For this purpose, let

$$\mathcal{E}_f = \{\lambda_{qi} \mid i = 1, \dots, n(q),\ q \geq 0\}$$

be the set of nonzero complex eigenvalues (counted with multiplicities as roots of the characteristic polynomials) of all maps $\tilde{f}_{*q}$ (where $n(0) = 1$ and of course, $\lambda_{01} = 1$). We find that

$$\Lambda(f) = \lambda(\tilde{f}_*) = \sum_{q=0}^{\infty}(-1)^q \sum_{i=1}^{n(q)} \lambda_{qi}.$$

or more generally that

$$\Lambda(f^s) = \lambda(\tilde{f}_*^s) = \sum_{q=0}^{\infty}(-1)^q \sum_{i=1}^{n(q)} \lambda_{qi}^s.$$

(8.1) DEFINITION. For each eigenvalue λ in the set $\mathcal{E}_f$, let

$n_e(\lambda) =$ the number of times λ is an eigenvalue for an $\tilde{f}_{*2i}$,

$n_o(\lambda) =$ the number of times λ is an eigenvalue for an $\tilde{f}_{*2i+1}$,

and

$$\beta(f) = \operatorname{card}\{\lambda \in \mathcal{E}_f \mid n_e(\lambda) - n_o(\lambda) \neq 0\};$$

$\beta(f)$ is called the *periodicity index* of the map f.

We now extend Theorem (9.3.3) and prove a general result about existence of periodic points.

(8.2) THEOREM (Bowszyc). *If the periodicity index $\beta(f)$ is nonzero, then $\Lambda(f^n) \neq 0$ for some $n \leq \beta(f)$.*

PROOF. We begin with the rational function

$$\begin{aligned} L_f(z) &= \sum_{q=0}^{\infty}(-1)^q \sum_{i=1}^{n(q)} \frac{1}{1-\lambda_{qi}z} \\ &= \frac{1}{1-z} - \left[\frac{1}{1-\lambda_{11}z} + \cdots + \frac{1}{1-\lambda_{1n(1)}z}\right] \\ &\quad + \left[\frac{1}{1-\lambda_{21}z} + \frac{1}{1-\lambda_{22}z} + \cdots + \frac{1}{1-\lambda_{2n(2)}z}\right] - \cdots . \end{aligned}$$

It is clearly analytic around $z = 0$ (certainly for $|z| < \min(1/|\lambda_{qi}|)$), so it has a power series expansion

$$L_f(z) = \sum_{q=0}^{\infty}(-1)^q n(q) + \sum_{n=1}^{\infty} \Lambda(f^n)z^n = \chi(f) + \sum_{n=1}^{\infty} \Lambda(f^n)z^n,$$

since $n(q) = \dim \widetilde{H_q(X)} =$ the number of nonzero eigenvalues in dimension q.

Now let us look at $L_f(z)$ in another way, this time gathering terms according to distinct eigenvalues, to get

$$L_f(z) = \sum_{\lambda \in \mathcal{E}_f} \frac{n_e(\lambda) - n_o(\lambda)}{1 - \lambda z} = \frac{p(z)}{q(z)}$$

with $\deg p < \deg q = \beta(f)$.

If $\beta(f) \neq 0$, then for at least one λ, we have $n_e(\lambda) - n_o(\lambda) \neq 0$, so $L_f(z) \neq 0$, because it has a pole at $1/\lambda$ and therefore at least one $\Lambda(f^n) \neq 0$.

To estimate the least n with this property assume

$$L_f(z) = a_0 + \Lambda(f^n)z^n + \cdots = a_0 + z^n(r(z)),$$

where r is analytic and $r(0) = \Lambda(f^n) \neq 0$. Then

$$p(z) - a_0 q(z) = z^n r(z) q(z);$$

the polynomial on the left has a zero of order n at $z = 0$, so since it is nontrivial,

$$n \leq \begin{cases} \deg q & \text{if } a_0 \neq 0, \\ \deg p < \deg q & \text{if } a_0 = 0. \end{cases}$$

Thus $n \leq \beta(f)$. □

We remark that the periodicity index is more efficient than the Euler number in detecting periodic points. For although $\chi(f) \neq 0$ clearly implies that $\beta(f) \neq 0$, the converse is not true: the reader is advised to construct a map $f : S^{2n+1} \to S^{2n+1}$ of degree $d(f)$ different from 0 and 1 for which $\beta(f) \neq 0$ but $\chi(f) = 0$.

We now draw a few consequences of Theorems (7.2) and (8.2).

(8.3) THEOREM. *Let X be a connected ANR and $f : X \to X$ a compactly absorbing map. If $\chi(f) \neq 0$ or $\beta(f) \neq 0$, then f has a periodic point with period $n \leq \beta(f)$.* □

(8.4) COROLLARY. *Let Y be a connected ANR with all odd-dimensional homology groups $H_{2n+1}(X; Q)$ zero. Then every compactly absorbing $f : X \to X$ has a periodic point.*

PROOF. We have $\chi(f) \geq 1$, since there are no offsetting negative terms. □

(8.5) COROLLARY (Fuller). *Let X be a connected compact ANR with Euler characteristic $\chi(X) \neq 0$. Then:*

(a) *Every $f : X \to X$ homotopic to the identity has a fixed point.*

(b) *Every homeomorphism $h : X \to X$ has a periodic point.*

PROOF. In case (a) we find $\lambda(f) = \lambda(\mathrm{id}) = \chi(\mathrm{id}) = \chi(X) \neq 0$; in case (b) we have $\chi(h) = \chi(\mathrm{id}) = \chi(X)$, which completes the proof. □

We conclude by presenting a simple reformulation of the Lefschetz theorem (obtained in a purely formal manner) that also conveys information about the existence of periodic points. It involves the integral group ring $Z(C)$ of the additive group C of complex numbers.

We recall that $Z(C)$ consists of the formal finite linear combinations $\sum n_i[\lambda_i]$, $n_i \in Z$, $\lambda_i \in C$; addition in this ring is the usual addition of linear combinations, and multiplication is defined by

$$\left(\sum n_i[\lambda_i]\right)\left(\sum m_j[\mu_j]\right) = \sum_{i,j} n_i m_j[\lambda_i + \mu_j].$$

We denote by $\varepsilon : Z(C) \to C$ the evaluation map defined by setting $\varepsilon([\lambda_i]) = \lambda_i$ on generators and extending by linearity.

(8.6) DEFINITION. Let $f : X \to X$ be a Lefschetz map, and let $\{\lambda_{qi} \mid i = 1, \dots, n(q)\}$ be the set of all eigenvalues (counted with multiplicities) of the endomorphism $\tilde{f}_q$. We let

$$[t_q(f)] = \sum_{i=1}^{n(q)} [\lambda_{qi}], \qquad [\Lambda(f)] = \sum_q (-1)^q [t_q(f)] \in Z(C).$$

(8.7) THEOREM. *Let X be a connected ANR and $f : X \to X$ a compact (or more generally a basic) map. Then:*
(i) *if $[\Lambda(f)] \neq 0$, then f has a periodic point of period at most the length of $[\Lambda(f)]$,*
(ii) *if $[\Lambda(f)] \notin \operatorname{Ker} \varepsilon$, then f has a fixed point.*

PROOF. (i) We observe that

$$[\Lambda(f)] = \sum (n_e(\lambda_i) - n_o(\lambda_i))[\lambda_i],$$

and this implies that the length of $[\Lambda(f)]$ is equal to the periodicity index $\beta(f)$. The assertion now follows from (8.3).

(ii) By definition of ε we have $\varepsilon([\Lambda(f)]) = \Lambda(f) \neq 0$, and thus f has a fixed point by (7.2).

9. Miscellaneous Results and Examples

A. Formal power series

Throughout the first two subsections all vector spaces are taken over a fixed algebraically closed field K. We denote by $K[[x]]$ the integral domain of all formal power series $s = \sum_{n=0}^{\infty} a_n x^n$ with $a_n \in K$.

(A.1) Let $s = \sum a_n x^n$ be a formal power series. Show:
(a) s is invertible if and only if $a_0 \neq 0$.
(b) If $s = 1 - \lambda x$, then $s^{-1} = \sum_{n=0}^{\infty} \lambda^n x^n$

(A.2) Let $K[x] \subset K[[x]]$ be the ring of polynomials and assume $s = \sum a_n x^n = uv^{-1}$, where $u, v \in K[x]$, $u, v \neq 0$ and $\deg u < \deg v = k$. Prove: There is no gap of length k in $\sum_{n=0}^{\infty} a_n x^n$

[Assume that there is such a gap: $uv^{-1} = [a_0 + \cdots + a_s x^s] + [a_{s+k+1}x^{s+k+1} + \cdots]$; letting $p_s = [a_0 + \cdots + a_s x^s]$, observe that the polynomial $u - vp_s$ has 0 as a root of order not less than $s + k + 1$, and thus $\deg(u - vp_s) \geq s + k + 1$; from this get a contradiction by showing that $\deg v \geq k + 1$.]

(A.3) Let $d : K[[x]] \to K[[x]]$ be the ordinary derivation, and for an invertible $s \in K[[x]]$, let $D(s) = s^{-1}d(s)$ be its logarithmic derivative. Show: If $s_1, \dots, s_k, s \in K[[x]]$ are invertible, then

$$D\Big(\prod_{i=1}^{k} s_i\Big) = \sum_{i=1}^{k} D(s_i) \quad \text{and} \quad D(s^{-1}) = -D(s).$$

(A.4) Given a rational function $s(x)$ define the conjugate s^* of s by

$$s^*(x) = \frac{1}{x}s\Big(\frac{1}{x}\Big).$$

Show:

(a) If $s = uv^{-1}$, where $u = \sum_{n=0}^{k-1} a_n x^n$, $v = \sum_{n=0}^{k} b_n x^n$, and $b_0, b_k \neq 0$, then

$$s^* = \Big(\sum_{n=0}^{k-1} a_n x^{k-1-n}\Big)\Big(\sum_{n=0}^{k} b_n x^{k-n}\Big)^{-1}$$

(b) If $s_1, \dots, s_m$ have conjugates, then so does their sum, and $(\sum_{i=1}^m s_i)^* = \sum_{i=1}^m s_i^*$.

B. The Lefschetz power series of a Leray endomorphism

Recall that for an endomorphism $\varphi : E \to E$ of a vector space E we let $\widetilde{\varphi} : \widetilde{E} \to \widetilde{E}$ be induced by φ on $\widetilde{E} = E/N_\varphi$, where $N_\varphi = \bigcup_{n\geq 1} \operatorname{Ker}\varphi^n$; if φ is admissible (i.e., $\dim \widetilde{E} < \infty$), then $\widetilde{\varphi}$ is an automorphism, and we let $w(\widetilde{\varphi})$ be its characteristic polynomial.

(B.1) Show: φ is admissible if and only if φ^n is admissible for some natural n.

(B.2) Let $\varphi : E \to E$ be an admissible endomorphism. Show: All the roots $\lambda_1, \dots, \lambda_m$ ($m = \dim \widetilde{E}$) of $w(\widetilde{\varphi})$ are different from zero, and for any natural n we have

$$\operatorname{Tr}(\varphi^n) = \operatorname{tr}(\widetilde{\varphi}^n) = \sum_{j=1}^{m} \lambda_j^n.$$

[Using Jordan's theorem, choose a basis in $\widetilde{E}$ in which $\widetilde{\varphi}$ has a triangular matrix form.]

(B.3) Let $\varphi = \{\varphi_q\}$ be a Leray endomorphism of a graded vector space $E = \{E_q\}$; we recall that in this case $\widetilde{E} = \{\widetilde{E}_q\}$ is of finite type, the Lefschetz number $\Lambda(\varphi)$ of φ is $\Lambda(\varphi) = \sum_{q=0}^{\infty}(-1)^q \operatorname{Tr}\varphi_q$, and its Euler number $\chi(\varphi)$ is $\chi(\varphi) = \chi(\widetilde{E}) = \sum_{q=0}^{\infty}(-1)^q \dim \widetilde{E}_q$. Show:

(a) φ is a Leray endomorphism if and only if so is φ^n for some n.
(b) If φ is a Leray endomorphism, then $\chi(\varphi) = \chi(\varphi^n)$ for all n.

(B.4) (*Rationality of the Lefschetz power series*) Let $\varphi = \{\varphi_q\}$ be a Leray endomorphism of $E = \{E_q\}$. The *Lefschetz power series* $L(\varphi) \in K[[x]]$ of φ is defined by

$$L(\varphi) = \chi(\varphi) + \sum_{n=1}^{\infty} \Lambda(\varphi^n)x^n = \sum_{n=0}^{\infty} \lambda(\widetilde{\varphi}^n)x^n = \sum_{n=0}^{\infty}\Big[\sum_{q=0}^{\infty}(-1)^q \operatorname{tr}(\widetilde{\varphi}_q^n)\Big]x^n,$$

and the *characteristic rational function* of φ is given by

$$w = w(\varphi) = \prod_{q=0}^{\infty} w_q^{(-1)^q},$$

where w_q is the characteristic polynomial of $\widetilde{\varphi}_q$. Show: $L(\varphi) = (D(w(\varphi)))^* = uv^{-1}$, where u and v are some relatively prime polynomials with $\deg u < \deg v$ (if $u \neq 0$).

[Denote by λ_{qj} ($j = 1, \ldots, \dim \widetilde{E}_q$) all the roots of the characteristic polynomial w_q of $\widetilde{\varphi}_q$. Using (A.1)–(A.3) and observing that $w_q = \prod_j (x - \lambda_{qj})$, establish and compare the following formulas:

$$(Dw)^* = \Big(\sum_{q,j} (-1)^q (x - \lambda_{qj})^{-1}\Big)^* = \sum_{q,j} (-1)^q (1 - \lambda_{qj} x)^{-1},$$

$$\begin{aligned} L(\varphi) &= \sum_{n=0}^{\infty} \lambda(\widetilde{\varphi}^n) x^n = \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} (-1)^q \operatorname{tr}(\widetilde{\varphi}_q^n) x^n \\ &= \sum_{n=0}^{\infty} \sum_{q,j} (-1)^q \lambda_{qj}^n x^n = \sum_{q,j} (-1)^q (1 - \lambda_{qj} x)^{-1}.] \end{aligned}$$

(B.5) Let $\varphi = \{\varphi_q\}$ be a Leray endomorphism and $L(\varphi) = uv^{-1}$ a rational representation with relatively prime polynomials u and v as in (B.4). Denote by $P(\varphi)$ the degree of v. Show:

(i) $\chi(\varphi) \neq 0$ implies $P(\varphi) \neq 0$.
(ii) $P(\varphi) \neq 0$ if and only if $\Lambda(\varphi^n) \neq 0$ for some natural n.
(iii) If $P(\varphi) = k \neq 0$, then for every natural m, one of $\Lambda(\varphi^{m+1}), \ldots, \Lambda(\varphi^{m+k})$ is different from 0.

(B.6) Let $\varphi = \{\varphi_q\}$ be a Leray endomorphism and assume that the characteristic rational function w of φ has the form $w = yz^{-1}$, where y, z are relatively prime polynomials. Let a and b be the numbers of different roots of y and z, respectively. Show:

$$\chi(\varphi) = \deg y - \deg z \quad \text{and} \quad P(\varphi) = a + b.$$

(The above results are due to Bowszyc [1969].)

C. Lefschetz-type results for LC^n-spaces

Throughout this subsection only metric spaces are considered. A space X is *n-locally connected* if for each $x \in X$ and each nbd U of x there is a nbd $V \subset U$ of x such that every $f : S^m \to V$, $m \leq n$, has an extension $F : K^{m+1} \to V$ The class of n-locally connected spaces is denoted by LC^n We recall that for a space X, we write $\dim X \leq n$ if each open covering $\{U\}$ of X has an open refinement $\{V\}$ in which no more than $n+1$ sets V have a nonempty intersection.

The proof of the main theorem in this subsection is based on the following results:

(*Kuratowski–Dugundji theorem*) For a space Y and some $n \geq 0$, the following properties are equivalent:

1° $Y \in LC^n$.
2° If $A \subset X$ is closed and $\dim(X - A) \leq n+1$, then each $f : A \to Y$ can be extended over a nbd U of A in X

(*Bothe embedding theorem*) For every $n = 1, 2, \ldots$ there exists a compact $(n+1)$-dimensional absolute retract $X_{\mathrm{B}}^{(n)}$ such that every separable metric space of dimension $\leq n$ has an embedding into $X_{\mathrm{B}}^{(n)}$

[For a proof, see Dugundji [1963] and Bothe [1958].]

(C.1) Let K be compact with $\dim K \leq n$ and assume that $f : K \to K$ can be factored as $K \xrightarrow{\alpha} X \xrightarrow{\beta} K$, where X is an LC^n-space. Prove: f is a strongly Lefschetz map.

[Using Bothe's theorem, embed K in the compact absolute retract $X_{\mathrm{B}}^{(n)}$ with $\dim X_{\mathrm{B}}^{(n)} = n + 1$. Consider the diagram

$$\begin{tikzcd}
X_{\mathrm{B}}^{(n)} & U \arrow[l, hook'] \arrow[dd, dashed, "\varphi"'] & & \widehat{K} \arrow[ll, hook', "j"'] \arrow[dl, shift left, "s^{-1}"] \\
& & K \arrow[ur, shift left, "s"] \arrow[dl, "\alpha"'] \arrow[d, "f"] & \\
& X \arrow[r, "\beta"] & K &
\end{tikzcd}$$

where $s : K \to \widehat{K}$ is a homeomorphism of K onto $\widehat{K} \subset X_{\mathrm{B}}^{(n)}$ Observe that since X is LC^n and $\dim(X_{\mathrm{B}}^{(n)} - \widehat{K}) \leq n+1$, the map $\alpha s^{-1} : \widehat{K} \to X$ extends (by the Kuratowski–Dugundji theorem) over an open nbd U of $\widehat{K}$ in $X_{\mathrm{B}}^{(n)}$ to a map $\varphi : U \to X$. Verify that f factors through U, and conclude the proof using (3.5) and the fact that U (being an ANR) is a Lefschetz space.]

(C.2) Let X be an LC^n-space and suppose $F : X \to X$ can be factored as $X \xrightarrow{\beta} K \xrightarrow{\alpha} X$, where K is a compact space with $\dim K \leq n$. Show: F is a strongly Lefschetz map.

(C.3) Let X be an LC^n-space, and let $F : X \to X$ be a compact map such that $\dim \overline{F(X)} \leq n$. Show: F is a strongly Lefschetz map.

10. Notes and Comments

The Leray trace and generalized Lefschetz number

The theory of generalized trace and generalized Lefschetz number is due to Leray [1959]; using this generalization, Leray was able to extend his fixed point index theory for convexoid spaces to compact subsets of such spaces without requiring that the subsets have cohomology of finite type.

The presentation of the Leray trace theory given in the text is close to that in the lecture notes [1969–1970] and [1980] of Granas. For the notion and properties of the relative Lefschetz number, the reader is referred to Bowszyc [1968].

The Leray functor

We now briefly comment on some refinements of the Leray trace theory. Let $\mathbf{E} = \mathbf{End}(\mathbf{V})$ be the category of endomorphisms and $\mathbf{E}_{\mathrm{mono}} \subset \mathbf{E}$ (respectively $\mathbf{E}_{\mathrm{iso}} \subset \mathbf{E}$) the subcategory of monomorphisms (respectively isomorphisms in $\mathbf{V}$; let $\lambda_1 : \mathbf{E} \to \mathbf{E}_{\mathrm{mono}}$ be the

functor $\tilde{}$ given in (1.4). Given a monomorphism $\varphi : E \to E$, let $I_\varphi = \bigcap_{n=0}^{\infty} \varphi^n(E)$ be the *generalized image* of φ and observe that $\varphi(I_\varphi) \subset I_\varphi$. Given now a pair (E, φ) in $\mathbf{E}_{\text{mono}}$, we let $\lambda_2((E, \varphi)) = (\hat{E}, \hat{\varphi})$, where $\hat{E}$ is the generalized image of φ and $\hat{\varphi}(x) = \varphi(x)$ for $x \in I_\varphi$. It is easily seen that $\hat{\varphi}$ is an isomorphism, and we define the *Leray functor* $\mathbf{L} : \mathbf{E} \to \mathbf{E}_{\text{iso}}$ as the composition $\lambda_2 \circ \lambda_1$.

The main property of the Leray functor (which can be regarded as an extension of (1.8)) is given in the following:

THEOREM. *Assume that we are given a commutative diagram as in* (1.8), *which can be rephrased as the commutativity of the diagram*

$$\begin{tikzcd} (E', \varphi') \arrow[r, "f"] \arrow[d, "\varphi'"'] & (E'', \varphi'') \arrow[d, "\varphi''"] \arrow[dl, "g"'] \\ (E', \varphi') \arrow[r, "f"] & (E'', \varphi'') \end{tikzcd}$$

in the category $\mathbf{E}$. *Then* $\mathbf{L}(f)$ *and* $\mathbf{L}(g)$ *are invertible, and* $\mathbf{L}(E', \varphi')$ *and* $\mathbf{L}(E'', \varphi'')$ *are isomorphic.*

The Leray functor appearing in Mrozek [1989] has found applications in the Conley index theory (Mrozek [1989] and Mrozek-Rybakowski [1991]).

Lefschetz theorem for ANRs

Lefschetz [1937] extended the Lefschetz–Hopf theorem for polyhedra to compact ANRs, and in his 1939 book also to the class of quasi-complexes, which contains the compact ANRs.

F. La Salle, J. Hale and S. Lefschetz, 1961

We give a direct and simple proof of the result for compact ANRs (as given in Hurewicz's lectures), based on the fact that every compact ANR is ε-dominated by a finite polyhedron. Let (X,d) be a compact ANR, $f : X \to X$, and $\varepsilon > 0$. There is a diagram

$$\begin{tikzcd} & P_\varepsilon \arrow[d, "g"] \\ X \arrow[ur, "k"] \arrow[r, "f"'] & X \end{tikzcd}$$

where P_ε is a finite polyhedron, $d(gk(x), f(x)) < \varepsilon$ for $x \in X$, and $f \simeq gk$. Assuming $\lambda(f) = \lambda(gk) = \lambda(kg) \neq 0$, we have $kg(p) = p$ for some $p \in P_\varepsilon$; hence $gkg(p) = g(p)$, and so $d(fg(p), g(p)) < \varepsilon$, i.e., f has an ε-fixed point. Since $\varepsilon > 0$ is arbitrary, the desired conclusion follows.

The proof of the Lefschetz theorem for compact maps of ANRs given in the text follows that of Granas [1967]; Jaworowski–Powers [1969] gave an independent proof of the same result, similar to the above proof of Hurewicz. Some more special but closely related results were established earlier by Göhde [1964], Browder [1965b], and Eells (see his survey [1966]).

Asymptotic fixed point theory

The first results in asymptotic fixed point theory are due to Leray [1945b], Berstein [1957], Bourgin [1957], and Deleanu [1959]. Later, the theory was developed by Browder [1970], [1974], Palais (unpublished), Nussbaum [1971], [1972], Fournier [1975], and Eells-Fournier [1976]. The notion of the core of a map in the context of compact spaces appeared for the first time in Leray [1945] and was later used by Deleanu [1959]. The classes of eventually compact and asymptotically compact maps were first studied by Browder [1965], [1970], [1974]. Maps of compact attraction and the Lefschetz theorem for such maps are due to Fournier [1975]; compactly absorbing maps are implicit in the proof of the above theorem presented in Granas's lecture notes [1980]. The presentation of the results in Sections 5–7 is close to that of Fournier [1975] and of the above-mentioned lecture notes.

The notion of an attracting fixed point goes back to Schreier–Ulam (Studia Math. 5 (1935), p. 155) and that of a compact attractor to Göhde [1964]. The notion of a stable attractor was introduced by Nussbaum [1971].

Existence of periodic points

The periodicity index defined in (8.1) and Theorem (8.2) are due to Bowszyc [1969]; Theorem (8.2) contains Theorem (9.3.3) as a special case. Theorem (8.7) is due to the authors and appears here for the first time.

There exists a related approach to periodicity based on using the Lefschetz zeta function. Let X be a space and $f : X \to X$ a Lefschetz map. By

the *Lefschetz zeta function* ζ_f of f is meant the formal power series

$$\zeta_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\Lambda(f^n)z^n}{n}\right),$$

where $\Lambda(f^n)$ is the generalized Lefschetz number of f^n (we use the singular homology with coefficients in Q).

THEOREM. *If $f : X \to X$ is a Lefschetz map, then the Lefschetz zeta function of f is a rational function, and*

$$\zeta_f(z) = \prod_{q=0}^{n} \det(I - \tilde{f}_{*q}z)^{(-1)^{q+1}},$$

where $\{\tilde{f}_{*q}\}$ is the endomorphism induced by $\{f_{*q}\}$ on the graded vector space $\{\tilde{H}_q(X;Q)\}$, and n satisfies $\tilde{H}_q(X;Q) = 0$ for $q > n$.

For various versions and special cases of this result the reader is referred to Smale [1967], Halpern [1968], Kelley–Spanier [1968], Franks [1980], and Fadell's survey [1970].

Lefschetz theory and cohomology

If P is a polyhedron, then $H_n(P;Q)$ and $H^n(P;Q)$ are dual vector spaces over Q. This implies that given a map $f : P \to P$, we may use cohomology with coefficients in Q to compute the Lefschetz number of f. In some cases, to detect fixed points of a map, it is preferable to use cohomology rather than homology, because the former has a ring structure.

Consider the complex projective space $P^n(C)$ with n even. With integer coefficients, the cohomology of $P^n(C)$ can be shown to be a graded polynomial ring on a generator $\beta \in H^2(P^n(C); Z)$ with $\beta^{n+1} = 0$. Given a map $f : P^n(C) \to P^n(C)$, there is an integer b such that $f^*(\beta) = b\beta$ in $H^2(P^n(C); Q)$. From the ring structure of $H^*(P^n(C); Q)$ it follows that

$$\lambda(f) = 1 + b + b^2 + \cdots + b^n$$

Since for n even, the polynomial $1 + x + x^2 + \cdots + x^n$ has no real roots, we find that $\lambda(f) \neq 0$, implying that f has a fixed point.

For more details and other examples the reader is referred to Lopez [1967], Bredon [1971], and Fadell's survey [1970].

We remark that the main results of this paragraph remain valid if in their formulation we use singular cohomology instead of singular homology. Moreover, in any category where the Čech and singular theories are naturally equivalent—such as ANRs—the Čech Lefschetz numbers and Čech Lefschetz spaces (defined using either the Čech homology or cohomology) coincide with the corresponding notions using the singular homology and cohomology.

§16. The Hopf Index Theorem

In Chapter IV we developed fixed point index theory in arbitrary ANRs. Our aim in this paragraph is to complete the above theory by establishing its relation to the Lefschetz number. Precisely, we are going to show that if X is an ANR and $f : X \to X$ is a compact map, then its index $I(f)$ is equal to the generalized Lefschetz number $\Lambda(f)$ of f. The general result will be reduced to the special case in which $X = K$ is the closure of a polyhedral domain in some $\boldsymbol{R}^n$ and the map f has only isolated fixed points. The proof of this special case will be based on the original approach of H. Hopf.

1. Normal Fixed Points in Polyhedral Domains

In this preliminary section we will deal only with a polyhedral domain $U \subset \boldsymbol{R}^n$ and a continuous map $f : \overline{U} \to \boldsymbol{R}^n$ with $\operatorname{Fix}(f|\partial U) = \emptyset$. We have $I(f, U) = d(\mathrm{id} - f, U)$. The approach to the index theorem we take requires reduction to the case where f has isolated fixed points in U or, in other words, $\mathrm{id} - f$ has isolated zeros.

Let p be an isolated fixed point of f, and let $V \subset U$ be an open set containing p and no other fixed point of f.

(1.1) DEFINITION. The *multiplicity* $I(f,p)$ of the fixed point p is given by $I(f,p) = I(f,V)$.

Clearly, by excision, $I(f,p)$ is well defined.

As we have seen in (14.6.6), the use of spherical nbds of p gives:

(1.2) PROPOSITION. *Let $f : U \to \boldsymbol{R}^n$ have an isolated fixed point p, and let $V \subset U$ be a closed ball centered at p and containing no other fixed point of f. Then the multiplicity $I(f,p)$ of p is the Brouwer degree $d(F_p)$ of the map $F_p : \partial V \to S^{n-1}$ given by*

$$F_p(x) = \frac{x - f(x)}{\|x - f(x)\|},$$

where ∂V is the boundary of the ball V. □

Because we will use the notion of multiplicity for polyhedra, it is more convenient to use simplices rather than balls. We shall denote a closed simplex by V (with a dimension superscript if needed), its interior by $\mathring{V}$, and its boundary by $\dot{V}$

Whenever the behavior of f around an isolated fixed point p satisfies a mild condition called normality, $I(f,p)$ can be calculated directly from f itself rather than from the associated field, a matter important in our subsequent work.

We now first show that f can always be converted, by a small deformation, to one having the required behavior around p, and then give two methods for calculating $I(f,p)$ for such maps.

(1.3) DEFINITION. An isolated fixed point p of $f : U \to \boldsymbol{R}^n$ is called *normal* if there is an n-simplex $V \subset U$ with $p \in V$ and containing no other fixed point of f, such that $V \cap f(\dot{V}) = \emptyset$.

Normality requires essentially that the vectors $f(x) - x$ on $\dot{V}$ all point away from V. A small deformation of f can always convert an isolated fixed point to a normal fixed point without changing its multiplicity.

(1.4) PROPOSITION. *Let $f : U \to \boldsymbol{R}^n$ have an isolated fixed point p, and let W be any open nbd of p. Let $V \subset W$ be any n-simplex containing p in its interior, with f fixed point free on $V - \{p\}$. Then there is a homotopy $f_t : U \to \boldsymbol{R}^n$ of f such that:*
(1) $f_t(x) = f(x)$ for all $x \in U - W$ and $t \in \boldsymbol{I}$,
(2) p is an isolated fixed point of each f_t ($\mathrm{Fix}(f_t|V) = \{p\}$),
(3) $V \cap f_1(\dot{V}) = \emptyset$ and $I(f,p) = I(f_1,p)$.

PROOF. Let δ be the diameter of $\dot{V}$, and let $\lambda(s) = \|f(s) - s\|$ for $s \in \dot{V}$. By hypothesis, $\lambda(s)$ is never zero, so $\dot{V}$ being compact, hence closed in U Tietze's theorem assures that there is a continuous $\varphi : U \to [0, \infty)$ such that

$$\varphi(x) = \begin{cases} 3\delta/\lambda(x), & x \in \dot{V}, \\ 0, & x \in \{p\} \cup (U - W). \end{cases}$$

Define $f_t : U \to \boldsymbol{R}^n$ by

$$f_t(x) = [-t\varphi(x)]x + [1 + t\varphi(x)]f(x),$$

which pushes $f(x)$ along the ray $\overline{xf(x)}$ away from x. Clearly $f_0 = f$ and $f_t(x) = f(x)$ for all t and all $x \in \{p\} \cup (U - W)$; in particular, $f_t(p) = f(p) = p$ for all $t \in \boldsymbol{I}$.

Each f_t is fixed point free on $V - \{p\}$: if $f_t(x) = x$ for some $x \in V$ and $t \in \boldsymbol{I}$, then $x = [-t\varphi(x)]x + [1 + t\varphi(x)]f(x)$, and therefore $[1 + t\varphi(x)]x = [1+t\varphi(x)]f(x)$; since $1+t\varphi(x) \geq 1$, this shows that $x = f(x)$, and therefore $x = p$. Thus, f_t is a homotopy of f that is fixed point free on $V - \{p\}$, so in particular, (1.2) shows that $I(f,p) = I(f_1,p)$.

To see that p is a normal fixed point of f_1, we note that if $s \in \dot{V}$, then $\|s - p\| \leq \delta$, so

$$\begin{aligned} \|f_1(s) - p\| &\geq \|f_1(s) - s\| - \|s - p\| \geq (1 + \varphi(s))\|f(s) - s\| - \delta \\ &= [1 + 3\delta/\lambda(s)]\lambda(s) - \delta \geq \lambda(s) + 2\delta \geq 2\delta, \end{aligned}$$

which shows $f_1(s) \notin V$ Thus, $V \cap f_1(\dot{V}) = \emptyset$, and the proof is complete. □

If p is a normal fixed point for $f : U \to \boldsymbol{R}^n$, then an n-simplex $V \subset U$ with $p \in \overset{\circ}{V}$ such that (1) $V \cap f(\dot{V}) = \emptyset$ and (2) f is fixed point free on $V - \{p\}$ is called a *normal simplex* for f at p. We give two ways for calculating $I(f, p)$ directly from the behavior of f itself on a normal simplex. The first involves $f|\dot{V}$:

(1.5) PROPOSITION. *Let $f : U \to \boldsymbol{R}^n$ have a normal fixed point p, and let V be a normal simplex for f at p. Let $\pi_p f : \dot{V} \to S^{n-1}$ be defined by*

$$\pi_p f(x) = \frac{f(x) - p}{\|f(x) - p\|}, \qquad x \in \dot{V}.$$

Then $I(f, p) = (-1)^n d(\pi_p f)$.

PROOF. Because $f(s) \notin V$ whenever $s \in \dot{V}$, we have $f(s) \neq (1-t)s + tp$ for all $s \in \dot{V}$ and $t \in \boldsymbol{I}$, so

$$\Phi_t(x) = \frac{[(1-t)x + tp] - f(x)}{\|[(1-t)x + tp] - f(x)\|}$$

gives a homotopy $\Phi_t : \dot{V} \to S^{n-1}$ of F_p (see (1.2)) to $-\pi_p f$. The degree of the antipodal map $\alpha : S^{n-1} \to S^{n-1}$ given by $x \mapsto -x$ is $(-1)^n$. Thus $\alpha \circ \Phi_1 = \pi_p f$, so from $I(f, p) = d(F_p) = d(\Phi_1) = d(\alpha)d(\pi_p f)$ the conclusion follows. □

The second description of $I(f, p)$ on a normal simplex is more important for our purposes; its formulation requires some preliminary notation.

Consider $\boldsymbol{R}^n$ embedded as the hyperplane $x_{n+1} = 0$ in $\boldsymbol{R}^{n+1}$ Given an n-simplex $V \subset \boldsymbol{R}^n$ and a point $p \in \overset{\circ}{V}$, let Δ be the cone over $\dot{V}$ with vertex $p^* = (p, 1)$, so that $V \cup \Delta = \Sigma^n$ can be visualized as a space homeomorphic to S^n attached to $\boldsymbol{R}^n - \dot{V}$ along the "equatorial $(n-1)$-sphere $\dot{V}$" The map of Σ^n into itself which, for each $s \in \dot{V}$, maps the segment ps linearly onto p^*s and the segment p^*s linearly onto ps, is a homeomorphism (corresponding to the reflection of S^n in its equatorial $(n-1)$-plane), and defined to be the identity map on $\boldsymbol{R}^n - V$, gives a homeomorphism $\varrho : \boldsymbol{R}^n \cup \Sigma^n \to \boldsymbol{R}^n \cup \Sigma^n$ We also observe that the radial retraction $r : \boldsymbol{R}^n \to V$ from p extends, by letting it be the identity on Δ, to a retraction

$$r : \boldsymbol{R}^n \cup \Delta \to \Sigma^n = V \cup \Delta.$$

Now let $f : U \to \boldsymbol{R}^n$ have a normal fixed point p, and let $V(p) = V$ be a normal n-simplex for f at p. Attaching the cone Δ to $\dot{V}$, we have $U \cup \Delta$ open in $\boldsymbol{R}^n \cup \Delta$, and we define the *normal extension* $\widehat{f} : U \cup \Delta \to \boldsymbol{R}^n \cup \Delta$ of f to be

$$\widehat{f}(x) = \begin{cases} \varrho^{-1} f \varrho(x), & x \in \Delta, \\ f(x) & x \in U. \end{cases}$$

Since $f(x) \notin V$ whenever $x \in \dot{V}$, the two definitions of f agree on $\Delta \cap U = \dot{V}$, so that $\widehat{f}$ is continuous.

(1.6) PROPOSITION. *Let $f : U \to \boldsymbol{R}^n$ have a normal fixed point p, let $\widehat{f} : U \cup \Delta \to \boldsymbol{R}^n \cup \Delta$ be its normal extension, and let $r : \boldsymbol{R}^n \cup \Delta \to \Sigma^n$ be the radial retraction from p. Then $d(r\widehat{f}|\Sigma^n) = d(\pi_p f)$.*

PROOF. Let $\widehat{f}_\Sigma = r\widehat{f}|\Sigma^n$. Using reduced homology groups, we first observe that because $H_i(\Delta) = 0$ for $i \geq 0$, the homology exact sequence for (Σ^n, Δ) shows that $j_* : H_n(\Sigma^n) \cong H_n(\Sigma^n, \Delta)$; by excision we find that

$$H_n(\Sigma^n, \Delta) = H_n(V \cup \Delta, \Delta) \underset{\cong}{\xleftarrow{e}} H_n(V, V \cap \Delta) = H_n(V, \dot{V}),$$

and finally, again because $H_i(V) = 0$ for $i \geq 0$, we conclude that $\partial : H_n(V, \dot{V}) \cong H_{n-1}(\dot{V})$.

Now consider the diagram

$$\begin{tikzcd}
H_n(\Sigma^n) \arrow[r, "j_*"] \arrow[d, "(\widehat{f}_\Sigma)_*"'] & H_n(\Sigma^n, \Delta) \arrow[d, "(\widehat{f}_\Sigma)_*"'] & H_n(V, \dot{V}) \arrow[l, "e"', "\cong"] \arrow[r, "\partial", "\cong"'] \arrow[d, "(\widehat{f}_\Sigma|V)_*"'] & H_{n-1}(\dot{V}) \arrow[d, "(\widehat{f}_\Sigma|\dot{V})_*"'] \\
H_n(\Sigma^n) \arrow[r, "j_*"] & H_n(\Sigma^n, \Delta) & H_n(V, \dot{V}) \arrow[l, "e"', "\cong"] \arrow[r, "\partial", "\cong"'] & H_{n-1}(\dot{V})
\end{tikzcd}$$

Since all the horizontal isomorphisms are induced by inclusions or boundary homomorphisms, this diagram is commutative. Note that if b is a generator for $H_n(\Sigma^n)$, then $c = \partial e^{-1} j_*(b)$ is a generator for $H_{n-1}(\dot{V})$; now the commutativity shows that if $(\widehat{f}_\Sigma)_*(b) = d \cdot b$, then $(\widehat{f}_\Sigma|\dot{V})_* c = d \cdot c$, and therefore $d(\widehat{f}_\Sigma) = d(rf|\dot{V})$. Letting $\varphi : \dot{V} \to S^{n-1}$ be the homeomorphism

$$x \mapsto \frac{x - p}{\|x - p\|},$$

we have

$$\pi_p f = \varphi \circ (rf|\dot{V}),$$

and so since $d(\varphi) = 1$, we find $d(\pi_p f) = d(rf|\dot{V}) = d(r\widehat{f}|\Sigma^n)$. □

In view of (1.5) and referring to the definition of the degree of a map $S^n \to S^n$, this result permits an interpretation of $(-1)^n I(f, p)$ as essentially the algebraic number of times that $f(V)$ covers a given point $x \in V - \{p\}$.

2. Homology of Polyhedra with Attached Cones

Let U be a polyhedral domain in $\boldsymbol{R}^n$ and $K = \overline{U}$. Our approach to the index theorem for a map $f : K \to K$ is based on an idea of Hopf; it relies on enlarging K to a polyhedron K_c by attaching suitable cones and relating

the Lefschetz number of the given f to that of a suitably chosen extension $f_c : K_c \to K_c$. In this section we derive the homological properties that will be needed.

Let $\sigma = (p_0, \ldots, p_n)$ be a closed n-simplex in K. Choose a vertex p^* in $R^{n+1} - K$, and recall that the cone $|\Delta|$ over $|\dot{\sigma}|$ consists of the simplices $\{p^* \cdot \partial\sigma\}$ together with all faces of such simplices (see §8, Section 5). The simplicial complex $K \cup |\Delta|$ is topologically $|K|$ with a cone over $|\dot{\sigma}|$; the subcomplex $|\Delta| \cup |\sigma|$ is denoted by Σ^n and is homeomorphic to S^n. Choose a point $p \in |\mathring{\sigma}|$; then the reflection $\Sigma^n \to \Sigma^n$ that interchanges $(|\sigma|, p)$ and $(|\Delta|, p^*)$, defined as in the previous section, extends to a homeomorphism $\varrho : (K \cup \Sigma^n, \Sigma^n) \to (K \cup \Sigma^n, \Sigma^n)$ by taking it to be the identity on $K - \Sigma^n$; in fact, if a central subdivision of $|\sigma|$ with p as vertex is extended to a subdivision of K, then ϱ is a simplicial homeomorphism of the resulting complex.

Letting Δ represent the chain $p^* \cdot \partial\sigma$ in $K \cup \Sigma^n$, we have $\partial\Delta = \partial\sigma - p^* \cdot \partial\partial\sigma = \partial\sigma$, so that the chain $\Delta - \sigma$ is an n-cycle in $K \cup \Sigma^n$.

In what follows, we work exclusively with homology over the rational field Q; to simplify the notation, we denote $H_*(K; Q)$ simply by $H_*(K)$.

(2.1) THEOREM. *Let $\widehat{K} = K \cup \Sigma^n$ be the polyhedron obtained by attaching a cone to K on the boundary of some n-simplex. Then*

$$H_i(\widehat{K}) = \begin{cases} H_i(K), & i \neq n, \\ Q, & i = n. \end{cases}$$

Precisely, the homology class $[\Delta - \sigma]$ of the cycle $\Delta - \sigma$ is a generator of $H_n(\widehat{K})$.

PROOF. Noting that

$$H_i(K \cup \Sigma^n, K) = H_i(K \cup |\Delta|, K) \underset{\cong}{\xleftarrow{e}} H_i(|\Delta|, |\Delta| \cap K) = H_i(|\Delta|, |\dot{\Delta}|)$$

by excision, and that $|\Delta|$ is homeomorphic to an n-ball, we deduce that $H_i(K \cup \Sigma^n, K) = 0$ for $i \neq n$, and is generated by Δ for $i = n$. The exact homology sequence for the pair $(K \cup \Sigma^n, K)$ therefore decomposes to

$$H_i(K \cup \Sigma^n) \cong H_i(K), \quad i \neq n, n-1,$$

and because $H_n(K) = 0$,

$$0 \to H_n(K \cup \Sigma^n) \xrightarrow{j_*} H_n(K \cup \Sigma^n, K)$$
$$\xrightarrow{\partial} H_{n-1}(K) \to H_{n-1}(K \cup \Sigma^n) \to 0.$$

Consider now $[\Delta - \sigma]$ in $H_n(K \cup \Sigma^n)$; since

$$j_*[\Delta - \sigma] = (\Delta - \sigma) + [B_n(K \cup \Sigma^n) + C_n(K)]$$
$$= \Delta + [B_n(K \cup \Sigma^n) + C_n(K)]$$

and Δ generates $H_n(K \cup \Sigma^n, K)$, it follows that j_* is an isomorphism; hence ∂ is the zero homomorphism and $H_{n-1}(K) \cong H_{n-1}(K \cup \Sigma^n)$. □

We now examine the behavior of certain maps on the generator $[\Delta - \sigma]$ of $H_n(K \cup \Sigma^n)$.

(2.2) COROLLARY. *Let $\varrho : K \cup \Sigma^n \to K \cup \Sigma^n$ be the reflection. Then $\varrho_*[\Delta - \sigma] = -[\Delta - \sigma]$.*

PROOF. We have already remarked that ϱ is simplicial if K is taken with a suitable subdivision, $\mathrm{Sd}\, K$. Since $\varrho_*(\Delta - \mathrm{Sd}\,\sigma) = \mathrm{Sd}\,\sigma - \Delta$, we conclude that $\varrho_*[\Delta - \sigma] = -[\Delta - \sigma]$. □

To study the behavior of arbitrary continuous maps $g : K \cup \Sigma^n \to K \cup \Sigma^n$, we begin by observing that because $|\sigma|$ is an absolute retract, there is a retraction $K \to |\sigma|$; this clearly extends, via the identity map on Σ^n, to a retraction $r : K \cup \Sigma^n \to \Sigma^n$.

(2.3) COROLLARY. *Let $g : K \cup \Sigma^n \to K \cup \Sigma^n$ be any continuous map, and let $g_\Sigma = g|\Sigma^n$. Then*

$$g_*[\Delta - \sigma] = d(rg_\Sigma)[\Delta - \sigma],$$

where $d(rg_\Sigma)$ is the degree of $rg_\Sigma : \Sigma^n \to \Sigma^n$.

PROOF. We certainly have $g_*[\Delta - \sigma] = q[\Delta - \sigma]$ for some $q \in Q$. To determine the value of q, consider the retraction $r : K \cup \Sigma^n \to \Sigma^n$. We have $r_*[\Delta - \sigma] = [\Delta - \sigma]$, since $r|\Sigma^n = \mathrm{id}$. Thus, $r_* g_*[\Delta - \sigma] = qr_*[\Delta - \sigma] = q[\Delta - \sigma]$, so that $q = d(rg_\Sigma)$. □

Repeated application of these results leads immediately to the main result, in the form we shall need.

(2.4) THEOREM. *Let $\{|\sigma_j| \mid j = 1, \dots, N\}$ be a pairwise disjoint family of closed n-dimensional simplices in a polyhedron K. Choose a point $p_j \in |\mathring{\sigma}_j|$ for each j, and let $K_c = K \cup \Sigma_1 \cup \cdots \cup \Sigma_N$ be the space K with cones attached on each $|\dot{\sigma}_j|$. Then:*

(1) *If $\varrho_j : \Sigma_j \to \Sigma_j$ is the reflection, then the map $\varrho : K_c \to K_c$ defined by*

$$\varrho(x) = \begin{cases} \varrho_j(x), & x \in \Sigma_j, \\ x, & x \in K - \bigcup \Sigma_j, \end{cases}$$

is consistently defined and continuous.

(2) *For each $i \neq n$, we have $H_i(K_c) \cong H_i(K)$.*

(3) *The vector space $H_n(K_c)$ has the basis $\{[\Delta_j - \sigma_j] \mid j = 1, \dots, N\}$.*

(4) *For any $g : K_c \to K_c$, if the endomorphism $g_* : H_n(K_c) \to H_n(K_c)$*

is expressed in this basis, then

$$g_*[\Delta_i - \sigma_i] = \sum_{j=1}^{N} d(r_j g|\Sigma_i)[\Delta_j - \sigma_j],$$

where $r_j : K_c \to \Sigma_j$ *is the retraction* [1]. □

3. The Hopf Index Theorem in Polyhedral Domains

We are now ready to prove the basic

(3.1) THEOREM. *Let K be the closure of a polyhedral domain U in $\mathbf{R}^n$, and assume that a map $f : K \to K$ has finitely many isolated fixed points $p_1, \ldots, p_s$ lying in different open simplices of dimension n. Then*

$$\lambda(f) = \sum_{j=1}^{s} I(f, p_j).$$

PROOF. According to (1.4) by a small modification of f confined entirely to an arbitrarily small neighborhood in the simplex containing it, each p_j can be enclosed in a small simplex normal for f at p_j, in such a way that the family of those normal simplices is pairwise disjoint. Taking K with a subdivision that does not disturb those simplices, we can therefore assume that the family of simplices $\{\sigma_j\}$ containing the $\{p_j\}$ is pairwise disjoint, each contains exactly one p_j, and each σ_j is a normal simplex for f at p_j.

For each p_j and σ_j, attach the cone Δ_j to K, to get the space $K_c = K \cup \Sigma_1 \cup \cdots \cup \Sigma_s$. Because each simplex is normal for f at p_j, the map $f : K \to K$ has a normal extension over each Δ_j, leading to $\widehat{f} : K_c \to K_c$; the only fixed points for $\widehat{f}$ are the p_j and the vertices p_j^* of the cones. Let $\varrho : K_c \to K_c$ be the reflection map of (2.4)(1); it follows that $\varrho\widehat{f}$ has no fixed points, and therefore $\lambda(\varrho\widehat{f}) = 0$.

We now relate $\lambda(\varrho\widehat{f})$ to $\lambda(f)$. Because for all dimensions $i < n$ we have $\varrho_{*i} = \mathrm{id}_{H_i(K_c)}$, and hence $\mathrm{tr}_i(\varrho\widehat{f}) \equiv \mathrm{tr}[(\varrho\widehat{f})_{*i}] = \mathrm{tr}_i f$, we need to worry only about dimension n. By (2.4)(3), $\{[\Delta_1 - \sigma_1], \ldots, [\Delta_s - \sigma_s]\}$ is a basis of $H_n(K_c)$.

By (2.4)(4), for each $j = 1, \ldots, s$,

$$(\varrho\widehat{f})_*[\Delta_j - \sigma_j] = d(r_j\varrho\widehat{f}|\Sigma_j)[\Delta_j - \sigma_j].$$

---

[1] This retraction is the composite of the obvious retraction $K_c \to K \cup \Sigma_j$ and a retraction ϱ from $K \cup \Sigma_j$ onto Σ_j obtained by extending the radial retraction from p_j of K onto σ_j over $K \cup \Sigma_j$ by letting $\varrho|\Delta_j = \mathrm{id}_\Delta$.

Thus,

$$\operatorname{tr}_n(\varrho\widehat{f}\,) = \sum_{j=1}^{s} d(r_j\varrho\widehat{f}|\Sigma_j) = \sum_{j=1}^{s} d(r_j\varrho\widehat{f}|\Sigma_j).$$

Now, letting $\varrho_j = \varrho|\Sigma_j : \Sigma_j \to \Sigma_j$ we have $r_j\varrho\widehat{f}|\Sigma_j = \varrho_j r_j\widehat{f}|\Sigma_j$, so this is the composition

$$\Sigma_j \xrightarrow{r_j\widehat{f}|\Sigma_j} \Sigma_j \xrightarrow{\varrho_j} \Sigma_j;$$

since $d(\varrho_j) = -1$ by (2.2), we find

$$d(r_j\varrho\widehat{f}|\Sigma_j) = -d(r_j\widehat{f}|\Sigma_j).$$

Moreover, since $\widehat{f}$ is a normal extension, (1.6) and (1.5) show that

$$\operatorname{tr}_n(\varrho\widehat{f}) = -\sum_{j=1}^{s} d(r_j\widehat{f}|\Sigma_j) = -\sum_{j=1}^{s} d(\pi_{p_j}f) = -(-1)^n\sum_{j=1}^{s} I(f,p_j).$$

Thus, by taking the alternating sum of traces, we find

$$0 = \lambda(\varrho\widehat{f}\,) = \sum_{i=0}^{n-1}(-1)^i \operatorname{tr}_i(\varrho\widehat{f}\,) + (-1)^n \operatorname{tr}_n(\varrho\widehat{f}\,).$$

We now claim that $\operatorname{tr}_i f = \operatorname{tr}_i(\varrho f)$ for $i < n$: indeed, if $l : K \hookrightarrow K_c$ is the inclusion and $r : K_c \to K$ the obvious retraction, it follows by (2.4)(2) that l_{*i} is an isomorphism. From $rl = \mathrm{id}_K$ we get $r_{*i} = l_{*i}^{-1}$, and from $f = r\varrho\widehat{f}l$ we obtain $f_{*i} = r_{*i}(\varrho\widehat{f})_{*i}l_{*i} = l_{*i}^{-1}(\varrho\widehat{f})_{*i}l_{*i}$; thus the desired conclusion follows.

From this and because $\operatorname{tr}_n f = 0$, we get

$$0 = \lambda(\varrho\widehat{f}) = \sum_{i=0}^{n}(-1)^i \operatorname{tr}_i(\varrho\widehat{f}) = \lambda(f) - (-1)^n(-1)^n\sum_{j=1}^{s} I(f,p_j);$$

this gives $\lambda(f) = I(f,K) = I(f,U) = \sum_{j=1}^{s} I(f,p_j)$. □

4. The Hopf Index Theorem in Arbitrary ANRs

We can now prove the basic normalization theorem, which we first establish in the following form:

(4.1) THEOREM. *Let $U \subset \boldsymbol{R}^n$ be open and $F : U \to U$ a compact map. Then*

$$I(F,U) = \Lambda(F),$$

where $\Lambda(F)$ is the (generalized) Lefschetz number of F and $I(F,U)$ is the fixed point index of F

PROOF. We begin by showing that $\Lambda(F)$ is well defined. By taking a smaller open set if necessary, we may assume that U is a polyhedral domain. Let $0 < \varepsilon < \operatorname{dist}(\partial U, \overline{F(U)})$; we can then find a polyhedral domain V with $K = \overline{V}$ such that $B(\overline{F(U)}, \varepsilon) \subset V \subset K \subset U$. Let $j : \dot{K} \hookrightarrow U$; then

$$\begin{aligned} I(F,U) &= I[jF,U] \quad (\text{since } U \xrightarrow{F} \dot{K} \xrightarrow{j} U) \\ &= I[Fj, j^{-1}(U)] = I[F|\dot{K}, \dot{K}]. \end{aligned}$$

Moreover, because $\Lambda(F) = \Lambda(jF) = \Lambda(Fj) = \Lambda(F|\dot{K}) = \lambda(F|\dot{K})$, $\Lambda(F)$ is defined.

For any $\delta < \varepsilon$, every δ-approximation of $F : \dot{K} \to K$ has values in $\dot{K}$. Now, let $f = \mathrm{id} - F : K \to \boldsymbol{R}^n$ There is a δ-approximation f_δ of f having finitely many zeros, and with $d(f_\delta, K) = d[\mathrm{id} - F, K] = I(F, \dot{K})$; we can assume that the zeros of f_δ are all in the interiors of n-simplices of $\dot{K}$.

Consider now $F_\delta = \mathrm{id} - f_\delta$; since F_δ is a δ-approximation to F, we have $F_\delta : \dot{K} \to \dot{K}$; and for sufficiently small δ, also $F_\delta \simeq F$ Furthermore, by definition

$$I(F_\delta, \dot{K}) = d[\mathrm{id} - F_\delta, K] = d(f_\delta, \dot{K}) = I(F, \dot{K}).$$

Now by (3.1) the assertion of the theorem holds for F_δ, so we get

$$I(F,U) = I(F,\dot{K}) = I(F_\delta, \dot{K}) = \Lambda(F_\delta) = \Lambda(F|\dot{K}) = \Lambda(F). \qquad \square$$

We are now ready to prove the main result.

(4.2) THEOREM. *Let X be an arbitrary ANR and $F : X \to X$ a compact map. Then $I(F,X) = \Lambda(F)$, where $\Lambda(F)$ is the (generalized) Lefschetz number of F and $I(F,X)$ is the fixed point index of F*

PROOF. Assume first that $X = U$ is an open subset of a normed linear space E. Given a compact map $F : U \to U$ let $0 < \varepsilon < \operatorname{dist}(\overline{F(U)}, \partial U)$, let $F_\varepsilon : U \to U$ be a Schauder ε-approximation of F with values in a finite-dimensional subspace E^n of E, and let $U_n = U \cap E$; clearly, F_ε is compact and $F_\varepsilon \simeq F$

Consider the following commutative diagram in which all the arrows represent either the obvious inclusions or contractions of the map F:

$$\begin{tikzcd} U_n \arrow[r] \arrow[d, "F'_\varepsilon"'] & U \arrow[d, "F_\varepsilon"] \arrow[dl] \\ U_n \arrow[r] & U \end{tikzcd}$$

By the definition of the index, (4.1), and (15.3.2), we have $I(F,U) = I(F'_\varepsilon, U_n) = \Lambda(F'_\varepsilon) = \Lambda(F_\varepsilon)$. Since $F_\varepsilon \simeq F$, this gives $I(F,U) = \Lambda(F)$, and the proof is complete.

Assume next that X is an arbitrary ANR and $F : X \to X$ is compact. Then there is an open subset U of a normed linear space E and maps $r : U \to X$ and $s : X \to U$ with $rs = 1_X$. By definition, $I(F, X) = I(sFr, U)$, and by the above special case, $I(sFr, U) = \Lambda(sFr)$; because $\Lambda(F) = \Lambda(sFr)$, the conclusion follows. □

5. The Lefschetz–Hopf Fixed Point Index for ANRs

In this section we return to the consideration of the fixed point index for ANRs and examine the concept from the viewpoint of algebraic topology.

We summarize now the main results of Chapters IV and V, and state the fundamental theorem on existence and uniqueness of the Lefschetz–Hopf index for compact and compactly fixed maps.

(5.1) THEOREM. *Let $\mathscr{F}$ be the class of maps defined by the condition: $f \in \mathscr{F}$ if and only if $f \in \mathscr{F}(U, X)$, where f is compact, X is an ANR, and U is open in X. Then there exists a unique index function $I : \mathscr{F} \to Z$ assigning $I(f) = I(f, U)$ to each $f \in \mathscr{F}(U, X)$ and satisfying the following properties:*

- (I) (Strong normalization) *If $U = X$ and $f : X \to X$ is in $\mathscr{F}$, then $I(f, X) = \Lambda(f)$.*
- (II) (Additivity) *For $f \in \mathscr{F}$ and any disjoint open $V_1, V_2 \subset U$, if* $\mathrm{Fix}(f) \subset V_1 \cup V_2$, *then $I(f, U) = I(f, V_1) + I(f, V_2)$.*
- (III) (Homotopy) *If $h_t : U \to X$ is a compact and compactly fixed homotopy, then $I(h_0, U) = I(h_1, U)$.*
- (IV) (Commutativity) *Let X, X' be ANRs, $U \subset X$, $U' \subset X'$ open, and $f : U \to X'$, $g : U' \to X$ continuous. Consider the maps*

$$gf : V = f^{-1}(U') \to X, \qquad fg : V' = g^{-1}(U) \to X'$$

If both gf and fg are in $\mathscr{F}$ and f is compact, then $I(gf, V) = I(fg, V')$.

PROOF. The existence of I is an immediate consequence of (12.5.2) and (4.2). We now show that the function $I : \mathscr{F} \to Z$ is in fact uniquely determined by properties (I)–(IV).

Let C_0 denote the category of open sets of finite-dimensional normed linear spaces, and $\mathscr{F}_0$ the class of all maps $f \in \mathscr{F}(U, X)$ with $X \in C_0$ and $U \subset X$ open. From Theorem (12.1.2), it follows that the function $I_0 : \mathscr{F}_0 \to Z$, $I_0 = I|\mathscr{F}_0$, is uniquely determined by properties (I)–(IV) (in fact, even by properties (I)–(III)); consequently, I_0 also has the excision and contraction properties as in (12.2.1).

Let C_1 denote the category of open subsets of normed linear spaces and $\mathscr{F}_1$ the class of all maps $f \in \mathscr{F}(U, X)$ with $X \in C_1$ and $U \subset X$ open.

Assume that $I_1 : \mathscr{F}_1 \to Z$ has properties (I)–(IV). Since I_1 has the excision property (by additivity), it follows by commutativity that I_1 also has the contraction property (the argument is similar to that in Section 12.2). Since every compact map is compactly homotopic to a finite-dimensional map, it follows by homotopy, excision, and contraction that I_1 is completely determined by its values on maps in $\mathscr{F}_0$. Thus, in view of the uniqueness of I_0, the function I_1 must coincide with the Leray–Schauder index.

Lastly, let I be defined on $\mathscr{F}$ and have properties (I)–(IV). Let $f \in \mathscr{F}(U,X)$ (X is an ANR, $U \subset X$ open), and let V be an open set in a normed linear space that r-dominates X with $s : X \to V$, $r : V \to X$, $rs = 1_X$. By commutativity applied to $s : X \to V$, $fr : r^{-1}(U) \to X$ (note: fr and sfr are compact), we get

$$I(frs, s^{-1}r^{-1}(U)) = I(f) = I(sfr, r^{-1}(U)).$$

Thus, if I has the commutativity property, then it is completely determined by its values on maps in $\mathscr{F}_1$. Consequently, if I also has properties (I)–(III), it is necessarily the unique extension of the Leray–Schauder index from $\mathscr{F}_1$ to $\mathscr{F}$. □

We now state the main theorem on the index for ANRs in a closely related and in fact equivalent setting of compact maps that are fixed point free on the boundary.

(5.2) THEOREM. *Let $\mathscr{K}$ be the class of maps defined by the condition: $f \in \mathscr{K}$ if and only if $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$, where X is an ANR and U is open in X. Then there exists a unique index function $i : \mathscr{K} \to Z$ assigning $i(f) = i(f,U)$ to each $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$ and satisfying the strong normalization, additivity, homotopy, and commutativity properties.*

PROOF. This follows easily from (12.6.2) and (5.1); the details are left to the reader. □

6. Some Consequences of the Index

We give a few consequences and applications of Theorem (5.1).

Relative Lefschetz theorem for pairs of ANRs

Given a pair (X,A) of spaces, a map $f : (X,A) \to (X,A)$ is said to be *compact* if both $f_X : X \to X$ and $f_A : A \to A$ are compact.

(6.1) THEOREM (Bowszyc). *Let (X, A) be a pair of ANRs, and let $f : (X,A) \to (X,A)$ be a compact map. Then:*

(a) *the relative Lefschetz number* $\Lambda(f) = \Lambda(f_*)$ *of* f *is defined and* $\Lambda(f) = \Lambda(f_X) - \Lambda(f_A)$,

(b) $\Lambda(f) \neq 0$ *implies that* f *has a fixed point in* $\overline{X - A}$.

PROOF. (a) follows at once from (15.3.3) and (15.4.3).

(b) Assume that $\Lambda(f) \neq 0$ and f has no fixed points in $\overline{X - A}$. Then $\operatorname{Fix}(f_X) \subset X - \overline{X - A} = U = \operatorname{Int} A$. By strong normalization and excision,

$$\Lambda(f_X) = I(f_X, X) = I(f_X, U).$$

On the other hand, because $\operatorname{Fix}(f) \subset U$,

$$I(f_X, U) = I(f_A, U) = I(f_A, A) = \Lambda(f_A).$$

Thus we get $\Lambda(f) = \Lambda(f_X) - \Lambda(f_A) = 0$, a contradiction. □

(6.2) COROLLARY. *Let* X *be an acyclic ANR,* $A = \bigcup_{i=1}^n A_i$ *a disjoint union of ARs, all closed or all open in* X, *and* $f : (X, A) \to (X, A)$ *a compact map. For each* $i \in [n]$ *define* $\tau(i)$ *by* $f(A_i) \subset A_{\tau(i)}$ *and set* $k = \operatorname{card}\{i \mid \tau(i) = i\}$. *If* $k \neq 1$, *then* f *has a fixed point in* $\overline{X - A}$.

PROOF. Since $\Lambda(f) = 1 - k$, the assertion follows from (6.1). □

(6.3) COROLLARY. *Let* X *be an acyclic ANR, and* U *the union of open sets* $U_1, \ldots, U_n$, $n \geq 2$, *whose closures are ARs and are pairwise disjoint. Let* $f : X - U \to X$ *be a compact map satisfying* $f(\partial U_i) \subset \overline{U}_i$ *for each* $i \in [n]$. *Then* f *has a fixed point.*

PROOF. Let $A = \bigcup_{i=1}^n \overline{U}_i$. In view of (6.2.5), there exists a compact map $g : (X, A) \to (X, A)$ such that $g(x) = f(x)$ for all $x \in X - U$. Since $\Lambda(g) = 1 - n \neq 0$, we infer from (6.1) that there is some $x \in \overline{X - A} = X - U$ such that $x = g(x) = f(x)$. □

The mod p theorem

The second consequence is a version of a result due to Steinlein–Krasnosel'skiĭ–Zabreĭko.

(6.4) THEOREM. *Let* X *be an ANR,* $f : X \to X$ *a compact map, and* p *a prime. Then*

$$I(f, X) = \Lambda(f) \equiv \Lambda(f^p) = I(f^p, X) \bmod p.$$

PROOF. In view of (5.1) we need only show that $\Lambda(f) \equiv \Lambda(f^p) \bmod p$. Assume first that $X = U$ is an open subset of a normed linear space E. Applying (12.3.1) to f, choose a sufficiently small $\varepsilon > 0$ and an ε-approximation $f_\varepsilon : U \to U$ of f such that $f_\varepsilon(U) \subset P_\varepsilon \subset U$, where P_ε is a finite polytope

and f_ε is homotopic to f. Consider the commutative diagrams

$$\begin{tikzcd} P_\varepsilon \arrow[r, hook] \arrow[d, "\widehat{f}_\varepsilon"'] & U \arrow[d, "f_\varepsilon"] \arrow[dl] \\ P_\varepsilon \arrow[r, hook] & U \end{tikzcd} \qquad \begin{tikzcd} P_\varepsilon \arrow[r, hook] \arrow[d, "\widehat{f}^p_\varepsilon"'] & U \arrow[d, "f^p_\varepsilon"] \arrow[dl] \\ P_\varepsilon \arrow[r, hook] & U \end{tikzcd}$$

and note that

$$\lambda(\widehat{f}_\varepsilon) = \Lambda(f_\varepsilon), \qquad \lambda(\widehat{f}^p_\varepsilon) = \Lambda(f^p_\varepsilon)$$

by (15.3.3). Since $\lambda(\widehat{f}_\varepsilon) \equiv \lambda(\widehat{f}^p_\varepsilon) \bmod p$ by Theorem (9.3.4), this gives

$$\Lambda(f_\varepsilon) \equiv \Lambda(f^p_\varepsilon) \bmod p,$$

and because $f \simeq f_\varepsilon$, the assertion follows in the special case $U = X$.

For an arbitrary ANR X, take an open set U in a normed linear space that r-dominates X, and let $s : X \to U$, $r : U \to X$ with $rs = 1_X$. Observe that by (15.3.5) we have

$$\Lambda(f) = \Lambda(sfr) \quad \text{and} \quad \Lambda(f^p) = \Lambda(sf^p r) = \Lambda((sfr)^p)$$

(since $(sfr)^p = (sfr) \circ \cdots \circ (sfr) = sf^p r$). Now, because by the established special case of the theorem, $\Lambda(sfr) \equiv \Lambda((sfr)^p) \bmod p$, the assertion follows. □

Repulsive and attractive fixed points

We now give two direct applications of the index to the study of repulsive and attractive fixed points.

Given a space X and $x_0 \in X$, we let $\mathscr{V}(x_0)$ denote the set of all open nbd's of x_0 in X.

(6.5) DEFINITION. Let $f : X \to X$, $x_0 \in \text{Fix}(f)$ and $U \in \mathscr{V}(x_0)$. We say that x_0 is a *repulsive fixed point* of f relative to U if for any $V \in \mathscr{V}(x_0)$ there exists a positive integer n_0 such that $f^n(X - V) \subset X - U$ whenever $n \geq n_0$.

The main results in the remaining part of this section rely on

(6.6) LEMMA. *Let X be a metric space, $f : X \to X$ a map, and A a closed f-invariant subset of X (i.e., $f(A) \subset A$). Assume that W is an open nbd of A in X such that $f^m(W) \subset A$ for some $m \geq 1$. Then there exists an open subset Y of X such that $A \subset Y \subset W$ and $f(\overline{Y}) \subset Y$.*

PROOF. Choose open sets $V_0, V_1, \ldots, V_{m-1}$ satisfying

$$A \subset V_{m-1} \subset \overline{V}_{m-1} \subset \cdots \subset V_1 \subset \overline{V}_1 \subset V_0 = W.$$

It is straightforward to verify that

$$Y = W \cap f^{-1}(V_1) \cap \cdots \cap (f^{m-1})^{-1}(V_{m-1})$$

has the desired properties. □

(6.7) THEOREM. *Let X be an ANR, $x_0 \in X$, and $f : X \to X$ be compact. Assume furthermore that:*

1° x_0 is a repulsive fixed point for f relative to $U \in \mathscr{V}(x_0)$,

2° f is fixed point free on ∂U,

3° there exists a $V \in \mathscr{V}(x_0)$ with $\overline{V} \subset U$ such that the inclusion $j : X - V \hookrightarrow X$ induces an isomorphism $j_* : H_*(X - V) \to H_*(X)$ (where H_* is the singular homology with coefficients in Q).

Then:

(a) $A(f) = i(f, X - \overline{U})$,

(b) $i(f, U) = 0$.

PROOF. (a) We begin by reducing the problem. Take $V \in \mathscr{V}(x_0)$ as in 3° and choose $n_0 \in N$ such that

(1) $$f^n(X - V) \subset X - U \subset X - V \quad \text{for all } n \geq n_0.$$

Let $A = (X - V) \cup \overline{f(X - V)} \cup \cdots \cup \overline{f^{n_0 - 1}(X - V)}$ and note that A is closed in X and f-invariant; since $x_0 \in V \cap \mathrm{Fix}(f)$, we see that $x_0 \notin A$, and hence there exists a nbd $O \in \mathscr{V}(x_0)$ such that $O \subset V \subset U$ and $A \subset X - \overline{O}$. Furthermore, since x_0 is repulsive, we can find a positive integer $m_0 \geq n_0$ such that

(2) $$f^n(X - O) \subset X - U \subset X - V \subset A \subset X - \overline{O} \quad \text{for all } n \geq m_0.$$

Define $W = X - \overline{O}$ and observe that (since $A \subset W$, $f^n(W) \subset A$ for $n \geq m_0$, and $f(A) \subset A$) the assumptions of Lemma (6.6) are satisfied for f. Applying (6.6) to the compact map $f : X \to X$, we find an open subset Y of X satisfying:

(i) $X - V \subset A \subset Y \subset X - \overline{O}$,

(ii) $f(Y) \subset Y$,

(iii) $f_Y : Y \to Y$ is compact.

We can now start the proof. Choose a prime $p \geq m_0 \geq n_0$; denote by $\widehat{f^p} : X - V \to X - V$ the map determined (in view of (1)) by f^p Observe that by (1), (2), (i)–(iv), the following diagram of graded homology vector spaces is commutative:

$$\begin{tikzcd}
H_*(X) \arrow[d, "(f^p)_*"'] & H_*(X - V) \arrow[l, "j_*"', "\cong"] \arrow[r, "i_*"] \arrow[d, "(\widehat{f^p})_*"'] & H_*(Y) \arrow[d, "(f^p_Y)_*"] \arrow[dl] \\
H_*(X) & H_*(X - V) \arrow[l, "j_*"'] \arrow[r, "i_*"] & H_*(Y)
\end{tikzcd}$$

Because j_* is an isomorphism and X, Y are ANRs, it follows that all three vertical arrows in the diagram are Leray endomorphisms and

$$\Lambda(f^p) = \Lambda((f^p)_*) = \Lambda((\widehat{f^p})_*) = \Lambda((f_Y^p)_*) = \Lambda(f_Y^p).$$

From this, in view of Theorem (6.4), we infer that for every sufficiently large prime p,

$$\Lambda(f) \equiv \Lambda(f^p) = \Lambda(f_Y^p) \equiv \Lambda(f_Y) \bmod p,$$

implying that

$$\Lambda(f) = \Lambda(f_Y).$$

Now, by strong normalization and excision, we get

$$\Lambda(f_Y) = i(f_Y, Y) = i(f_Y, X - \overline{V}) = i(f, X - \overline{V}) = i(f, X - \overline{U}),$$

and thus $\Lambda(f) = i(f, X - \overline{U})$.

To complete the proof, we observe that (b) follows from (a). □

Attractive fixed points

(6.8) DEFINITION. Let X be a space, $f : X \to X$, $x_0 \in \mathrm{Fix}(f)$, and $U \in \mathscr{V}(x_0)$. We say that x_0 is an *attractive fixed point* of f relative to U if for any $V \in \mathscr{V}(x_0)$ there exists a positive integer n_0 such that $f^n(U) \subset V$ for all $n \geq n_0$.

(6.9) THEOREM. *Let X be an ANR, $x_0 \in X$, and $f : X \to X$ a compact map. Assume furthermore that:*
1° x_0 is an attractive fixed point of f relative to $U \in \mathscr{V}(x_0)$,
2° f is fixed point free on ∂U,
3° there exists $W \in \mathscr{V}(x_0)$ with $\overline{W} \subset U$ and an acyclic subset K of X such that $W \subset K \subset U$.
Then $i(f, U) = 1$.

PROOF. We begin by reducing the problem. Take W as in 3° and choose $V \in \mathscr{V}(x_0)$ with $\overline{V} \subset W$ and a positive integer n_0 so that $f^n(U) \subset V$ for all $n \geq n_0$; observe that $f^n(W) \subset \overline{V} \subset W$ by 3°, and therefore $f^n : W \to W$ is compact. Let $O = V \cap f^{-1}(V) \cap \cdots \cap f^{-n_0}(V)$ and note that $O \in \mathscr{V}(x_0)$ and $f^n(O) \subset V$ for all $n \geq 0$. Using 1° and 3°, select a positive integer $m_0 \geq n_0$ such that

$$(*) \qquad f^n(W) \subset f^n(U) \subset O \quad \text{for all } n \geq m_0.$$

Let $A = \overline{O}$ and observe that A is closed in X and f-invariant. Moreover, since $f^n(O) \subset V$ for $n \geq 0$ and $\overline{V} \subset W$, we see that $A \subset W$, and therefore by $(*)$,

$$(**) \qquad f^n(W) \subset O \subset A \quad \text{for all } n \geq m_0.$$

Now, applying Lemma (6.6) to the compact map $f : X \to X$ we find an open subset Y of X satisfying:

(i) $A \subset Y \subset W$,

(ii) $f(Y) \subset Y$,

(iii) $f_Y : Y \to Y$ is compact.

We can now start the proof. Choose a prime $p \geq m_0 \geq n_0$. Because $f^p(K) \subset W \subset K$, $f^p(W) \subset Y \subset W$, $f^p(Y) \subset Y$ and $f^p(K) \subset Y$, we have the commutative diagram

$$\begin{tikzcd} K \arrow[d, "\widehat{f^p_K}"'] \arrow[dr, "f^p_K"] & Y \arrow[l, hook', "j"'] \arrow[d, "\widehat{f^p_Y}"] \\ K & Y \arrow[l, hook', "j"] \end{tikzcd}$$

where $\widehat{f^p_K}$, f^p_K, $\widehat{f^p_Y}$ are determined by f^p. Because K is acyclic and $\widehat{f^p_Y} = (f_Y)^p$, where $\widehat{f_Y} : Y \to Y$ is the contraction of f, we have

$$1 = \Lambda(\widehat{f^p_K}) = \Lambda(\widehat{f^p_W}) = \Lambda(\widehat{f^p_Y}) \equiv \Lambda(f_Y) \bmod p.$$

This being true for all sufficiently large primes p, it follows that $\Lambda(f_Y) = 1$, and hence $i(f_Y, Y) = 1$ by strong normalization. Lastly, by excision, we obtain $i(f, U) = i(f_Y, Y) = 1$, and the proof is complete. □

7. Miscellaneous Results and Examples

A. Some applications of the Mayer–Vietoris functor

Throughout this subsection X is a metrizable space, $\mathscr{A} = \{A_i \mid i \in [n]\}$ is a family of subsets of X, and $A = \bigcup_{i=1}^n A_i$. We denote by $M(\mathscr{A})$ the smallest lattice of sets containing all members of $\mathscr{A}$, and by $N(\mathscr{A})$ the nerve of $\mathscr{A}$.

Given a multi-index $J = (j_1, \dots, j_k)$ with $j_1 < \cdots < j_k$ and $|J| = k \leq n$, we let $A_J = A_{j_1} \cap \cdots \cap A_{j_k}$. If $f : A \to A$ is a map such that $f(A_i) \subset A_i$ for all $i \in [n]$, we let $f_J : A_J \to A_J$ be the obvious map determined by f. We say that the family $\mathscr{A}$ is *excisive* if for any X_1, X_2 in $M(\mathscr{A})$, the pair $\{X_1, X_2\}$ is excisive (cf. (14.3.2)).

(A.1) Let $\mathscr{A}$ be an excisive family, $f : A \to A$ a map with $f(A_i) \subset A_i$ for all $i \in [n]$, and assume that for each multi-index J with $|J| \leq n$ the map $f_J : A_J \to A_J$ determined by f is a Lefschetz map. Prove: f is a Lefschetz map and $\Lambda(f) = \sum_J (-1)^{|J|+1} \Lambda(f_J)$.

(A.2) Let X be an ANR and $\mathscr{A} = \{A_i\}_{i=1}^n$ an excisive family of closed subsets of X such that all members of the lattice $M(\mathscr{A})$ are ANRs. Let $f : (X, A) \to (X, A)$ be a compact map satisfying $f(A_i) \subset A_i$ for $i \in [n]$, and denote by $f_X : X \to X$ and $f_A : A \to A$ the maps determined by f. Prove:

(i) $\Lambda(f) = \Lambda(f_X) - \sum_J (-1)^{|J|+1} \Lambda(f_J)$.

(ii) If each A_J is either empty or acyclic, then $\Lambda(f) = \Lambda(f_X) - \chi(N(\mathscr{A}))$.

(iii) $\Lambda(f) \neq 0$ implies that f has a fixed point in $\overline{X - A}$.

(A.3) Let X be an ANR and $\mathscr{A} = \{A_i\}_{i=1}^n$ a family of open subsets of X. Let $f : (X, A) \to (X, A)$ be a compact map such that $f(A_i) \subset A_i$ and each $f_J : A_J \to A_J$ is compact. Prove:

(i) $\Lambda(f) = \Lambda(f_X) - \sum_J (-1)^{|J|+1} \Lambda(f_J)$.

(ii) If each A_J is either empty or acyclic, then $\Lambda(f) = \Lambda(f_X) - \chi(N(\mathscr{A}))$.

(iii) $\Lambda(f) \neq 0$ implies that f has a fixed point in $\overline{X - A}$.

(A.4) Let X be an AR and $A = \bigcup_{i=1}^n A_i$ $(n \geq 2)$ a disjoint union of ARs that are either all open or all closed in X. Let $f : (X, A) \to (X, A)$ be a map such that (a) $f(A_i) \subset A_i$ for all $i \in [n]$ and (b) $f_X : X \to X$ and all the $f_J : A_J \to A_J$ are compact. Show: f has a fixed point in $\overline{X - A}$.

[Observe that $\Lambda(f_X) = 1$ and $\chi(N(\mathscr{A})) = n$, and hence by (A.2) or (A.3), $\Lambda(f) = 1 - n \neq 0$.]

(A.5) Let X be a compact AR and U the union of $n \geq 2$ open sets U_i such that $\overline{U}_i \cap \overline{U}_j = \emptyset$ and each $\overline{U}_i$ is an AR. Assume further that $f : X - U \to X$ is a map satisfying $f(\partial U_i) \subset \overline{U}_i$ for all $i \in [n]$. Prove: f has a fixed point.

B. Common fixed points

Let X be a space. A family $\mathscr{F}_X = \{f\}$ of maps $f : X \to X$ is *divisible* if for any $f_1, f_2 \in \mathscr{F}_X$ there exists $h \in \mathscr{F}_X$ such that $f_1 = h^{n_1}$ and $f_2 = h^{n_2}$ for some $n_1, n_2 \in Z^+$

(B.1) Let $\mathscr{F}_X = \{f\}$ be a divisible family of compactly fixed maps, each having a fixed point. Show: $\bigcap\{\mathrm{Fix}(f) \mid f \in \mathscr{F}_X\} \neq \emptyset$.

[Prove that the family $\{\mathrm{Fix}(f) \mid f \in \mathscr{F}_X\}$ has the finite intersection property.]

(B.2) Let X be a space. A *semiflow* on X is a continuous family of maps $\{f_t : X \to X\}_{t \geq 0}$ such that (a) $f_{t_1+t_2} = f_{t_1} \circ f_{t_2}$ for all $t_1, t_2 \in R^+$; (b) $f_0 = 1_X$. A *fixed point* for the semiflow $\{f_t\}$ is a point $x_0 \in X$ such that $f_t(x_0) = x_0$ for all $t \geq 0$. Show: If $f_t : (X, A) \to (X, A)$, $t \in R^+$, is a semiflow, then (i) f_{t_1} is homotopic to f_{t_2} for all $t_1, t_2 \geq 0$; (ii) the family of those f_t indexed by positive rationals is divisible.

(B.3) Let (X, A) be a pair of ANRs such that A is either open or closed in X, and let $f_t : (X, A) \to (X, A)$, $t \in R^+$, be a semiflow such that f_t is compact for all $t > 0$. Prove: If the Euler characteristic $\chi(X, A)$ is finite and different from zero, then the semiflow $\{f_t\}$ has a fixed point in $\overline{X - A}$.

C. The Shub-Sullivan theorem

In this subsection we will use the following theorem of Shub-Sullivan [1974]: *Let M be a smooth compact manifold and $f : M \to M$ a C^1 map. If x_0 is an isolated fixed point for all f^n $(n \in N)$, then the sequence $\{J(f^n, x_0)\}$ of indices is bounded.*

(C.1) Let $f : S^m \to S^m$. Prove: $\lambda(f^n) = 1 + (-1)^m [d(f)]^n$

(C.2) Given a map $f : X \to X$, we let $P_n(f)$ denote the number of fixed points of f^n Prove: If $f : S^{2m} \to S^{2m}$, then $P_2(f) > 0$.

(C.3) Let $f : S^m \to S^m$ have $|d(f)| \geq 2$. Prove: $\lambda(f^n)$ grows exponentially with n.

(C.4) Let M be a smooth compact manifold and $f : M \to M$ a C^1 map such that $\limsup_{n\to\infty} \lambda(f^n) = \infty$. Show: f has infinitely many periodic points.

(C.5) Let $f: S^m \to S^m$ be a C^1 map. Prove: If $|d(f)| \geq 2$, then f has infinitely many periodic points.

8. Notes and Comments

Photo by I Namioka

F. Hirzebruch, P. Alexandroff, and H. Hopf. Moscow, 1966

The Hopf index theorem

There are various ways to prove this result. One is an elegant method due to Dold [1965], described in his book [1972] and in R.F Brown's book [1971] and in several articles of Dold [1974], [1975] extending the theory, with some simplifications suggested by Bowszyc. There are also two approaches by Hopf, one of which (Hopf [1928]) is fairly elegant and deserves to be better known (the other is presented in Alexandroff–Hopf's book [1935]). In this paragraph, we describe Hopf's original approach. It has the advantage of not dealing interminably with simplicial maps and their simplicial modifications; instead, cones are attached over simplices, and reflections in these cones are used to eliminate fixed points.

The Lefschetz-Hopf fixed point index

The Hopf index theorem (4.2) can be regarded as the algebraic supplement to the Leray Schauder index theory developed in §12. We remark that Theorem

(5.1), which combines (4.2) and (12.3.4), brings also a natural extension of the Lefschetz–Hopf index theory for compact ANRs. Therefore, it would be appropriate to call this result the Leray–Schauder- Lefschetz–Hopf theorem. However, to keep the terminology clear, we call the index given in (5.1) the "Lefschetz–Hopf index", and its "geometric part" given in (12.3.4) the "Leray–Schauder index".

There are two general methods for handling the Lefschetz–Hopf index. The first one is based on the chain approximation technique, and more precisely, on finding and applying appropriate algebraic analogues of the properties of the fixed point index for chain mappings. This method was used by several authors: Leray [1945], O'Neill [1953], Browder [1960], Fournier [1978], and also Eilenberg–Montgomery [1948], where the Lefschetz Hopf-type coincidence theorem is established; the use of the chain approximation method in the context of the fixed point index for set-valued maps will be discussed later on.

The chain approximation method originated and evolved from the work of Vietoris [1927]; in that article (where the Vietoris homology groups for compacta appeared in fact for the first time) the following theorem was proved (upon a suggestion of Brouwer, as Vietoris remarks in a footnote): *If X and Y are compacta and $f : X \to Y$ is surjective with acyclic fibers (with respect to Vietoris homology with coefficients in a field), then f induces an isomorphism of the corresponding Vietoris homology groups.* The proof of this theorem was based on constructing a chain map from the Vietoris chains of Y to those of X, which is (in a natural sense) an "approximation" of the inverse $f^{-1} : Y \to X$.

The second method, based on considerations on the level of homology groups, was discovered by Dold [1965] in the context of the index theory for ENRs; a closely related cohomological method in the context of the index for manifolds was found by Nakaoka [1969]. All the above methods utilize elaborate apparatus of algebraic topology.

Theorem (5.1), which is the central result of Chapters IV and V, was established in Granas [1972] on the basis of Dold's index theory for ENRs. Significant applications of the "geometric part" of the index theory have already been given in §13 (bifurcation theory, nonlinear PDEs).

We now briefly comment on some applications of the Lefschetz–Hopf index.

The relative Lefschetz theorem for pairs of ANRs, due to Bowszyc [1969], was initially established for pairs (X, A), where A is either open or closed in X; the proof was based on the relative Lefschetz theorem for polyhedra. The proof given in the text and using the index is due to Kucharski [1975], where the same type of argument was used to establish the relative Lefschetz theorem for set-valued maps. Applications of the index to the study of repulsive and attractive fixed points, given in Section 6, are due to Peitgen [1976b]. For more general results and other applications, the reader is re-

ferred to Nussbaum's lecture notes [1985] and also to Fenske-Peitgen [1976], Fournier-Peitgen [1977], [1978], and Benci-Degiovanni [1990].

The mod p theorem

Steinlein [1972] and independently Zabreĭko-Krasnosel'skiĭ [1971] established the following property of degree in $\boldsymbol{R}^n$: *Let $U \subset \boldsymbol{R}^n$ be open, $f : U \to \boldsymbol{R}^n$ continuous, and V an open subset of U such that f^m is defined on V, where $m = p^t$ and p is a prime. Assume that $S = \{x \in V \mid f^m(x) = x\}$ is compact and $f(S) \subset S$. Then $d(I - f^m, V) \equiv d(I - f, V)$* mod p. For the proof, the reader is referred to Nussbaum's lecture notes [1985].

Theorem (6.4) is a special case of the following result of Steinlein [1980], extending the above Steinlein–Zabreĭko-Krasnosel'skiĭ theorem to the setting of the index in ANRs: *Let X be an ANR, $U \subset X$ open, $f : U \to X$ continuous, and V an open subset of U such that f^m is defined on V, where $m = p^t$ and p is a prime. Assume that $S = \{x \in V \mid f^m(x) = x\}$ is compact (possibly empty), $f(S) \subset S$, and f is compact on some nbd of S. Then $I(f^m, V) \equiv I(f, V)$* mod p.

Vector fields on manifolds with boundary

Let M be a smooth manifold with boundary ∂M; for $p \in M$ we let $T_p(M)$ denote the tangent space to M at p. If $p \in \partial M$, then $T_p(\partial M)$ determines two closed half-spaces $T_p^+(M)$ and $T_p^-(M)$ of $T_p(M)$ with $T_p(\partial M) = T_p^+(M) \cap T_p^-(M)$. The tangent vectors in $T_p^+(M)$ (respectively in $T_p^-(M)$) are called *outward* (respectively *inward*) *directed*. Let $\xi = \{\xi_p\}_{p \in M}$ be a smooth vector field on M and assume that on each component of ∂M, either (i) all ξ_p are outward directed, or (ii) all ξ_p are inward directed. Denote by A (respectively B) the union of the components of ∂M of type (i) (respectively (ii)). Using the relative Lefschetz theorem, Bowszyc [1969] established the following generalization of the Hopf theorem: *If the relative Euler–Poincaré characteristic $\chi(M, A)$ is nonzero, then the vector field $\{\xi_p\}_{p \in M}$ vanishes at some point $p \in M$; the same holds if $\chi(M, B) \neq 0$.*

Lefschetz formula and critical point theory

We describe briefly the connection between the Lefschetz formula and the theory of critical points. Let $f : M \to \boldsymbol{R}$ be a C^∞ function on a compact C^∞ manifold M, and let p be a critical point of f. Then the second differential $d^2 f(p)$ (the *Hessian* of f at p) is a well defined symmetric bilinear form on the tangent space T_pM of vectors tangent to M at p; if this form is nonsingular, then p is called a *nondegenerate* critical point of f. The dimension of the maximal subspace of T_pM on which the Hessian is negative-definite is called the *index* of the critical point p and is denoted by $m_p(f) = m_p$.

We call f a *Morse function* if all its critical points are nondegenerate; for such an f (which has only a finite number of critical points) we let $\mathscr{M}_f(t) = \sum_{p:\, df_p=0} t^{m_p}$ and call it the *Morse polynomial* of f; it satisfies the following

THEOREM (Morse inequality). *Let $\mathscr{P}_M(t) = \sum_i b_i t^i$ be the Poincaré polynomial of M, where $b_q = \dim H_q(M; Q)$ is the qth Betti number of M. Then there exists a polynomial $Q(t)$ with nonnegative coefficients such that*

$$(*) \qquad \mathscr{M}_f(t) - \mathscr{P}_M(t) = (1 + t)Q(t).$$

A basic technique in the theory of critical points is that of gradient flows. Assume that M is equipped with a Riemannian metric and that f is a Morse function. Then the negative gradient vector field of f determines (via the differential equation $dx/dt = -\nabla f(x)$) a flow $\varphi_t : M \to M$ such that the value of f along any path $t \mapsto \varphi_t(p)$, $p \in M$, is never increasing. We observe that

$$\text{(a)} \qquad \begin{cases} \nabla f_p = 0 \Leftrightarrow p \in \mathrm{Fix}(\varphi_t) \text{ for all } t, \\ \varphi_t(p) \neq p \text{ for } t \neq 0 \Rightarrow \nabla f_p \neq 0, \end{cases}$$

and if p is a nondegenerate critical point, then

$$\text{(b)} \qquad \begin{cases} J(\varphi_1, p) = (-1)^k, \text{ where } J(\varphi_1, p) \text{ is the local index of } \varphi_1 \text{ at } p, \text{ and} \\ k \text{ is the index of the critical point } p. \end{cases}$$

From (a), (b), it follows that the Lefschetz formula coincides with the "Morse equality' asserting that the Euler characteristic $\chi(M) = \sum(-1)^k b_k$ equals $\sum(-1)^k \gamma_k$, where γ_k is the number of critical points of f of index k. (Observe that "Morse equality" follows from "Morse inequality", by taking $t = -1$ in (*) above.) This implies that the Morse theory, when applicable, contains more information than the Lefschetz formula. On the other hand, the Lefschetz formula has a wider range of applications to maps not homotopic to the identity, which do not appear in the deformation theory of critical points.

For more details the reader is referred to Smale [1967], Pitcher [1971], and also to Franks [1980] and Bott [1982], where some further references can be found.

Historical note

The sum of the indices of the fixed points of a given map, which, since Brouwer's first proofs of fixed point theorems, has been the object of many special investigations, was completely determined for maps of arbitrary manifolds by Lefschetz [1926]. For M a closed triangulable manifold and a map $f : M \to M$ having a finite number of fixed points $x_1, \ldots, x_s$, with the "local index" $i_f(x_j) \in Z$ for each $j \in [s]$, Lefschetz established the relation

$$\sum_{j=1}^{s} i_f(x_j) = \lambda(f),$$

where $\lambda(f)$ is the Lefschetz number of f. Hopf [1929], using the local index for isolated fixed points (defined with the aid of the Brouwer degree for self-maps of spheres), extended the Lefschetz formula to arbitrary finite polyhedra. These results can be regarded as the beginning of the fixed point index theory.

An early presentation of the fixed point index for polyhedra (using the tools of simplicial homology and due to H. Hopf) was given in the treatise of Alexandroff–Hopf [1935]. After the appearance of that book and also of Leray–Schauder's memoir [1934], the problem of generalizing and extending the Leray–Schauder theory and the fixed point index arose in a natural way. In the forties, Leray succeeded in developing an index theory that was applicable to quite general topological spaces. This required a new homology (or more precisely cohomology) theory, which he developed in [1945a,b] for

the category of "convexoid" spaces. A compact connected space X is called *convexoid* if any open cover $\{U_\alpha\}$ of X has a finite closed refinement $\{D_\beta\}$ such that any nonempty intersection $D_{\beta_1} \cap \cdots \cap D_{\beta_k}$ has trivial cohomology. These spaces can be regarded as generalizations of simplicial complexes, and Leray cohomology can be viewed as an extension of de Rham theory, expressing cohomology of smooth compact manifolds as the cohomology of the algebra of exterior differential forms. Leray's paper [1945c] generalizes the Leray–Schauder theory to the framework of convexoid spaces and of the new cohomology.

Let X be convexoid, $U \subset X$ open, and let $C_{\partial U}(\overline{U}, X)$ be the set of all maps $f : \overline{U} \to X$ such that $\mathrm{Fix}(f|\partial U) = \emptyset$. Leray [1945c] established:

THEOREM. *To any $f \in C_{\partial U}(\overline{U}, X)$ there corresponds an integer $i(f, U)$ (the index of fixed points of f belonging to U) with the following properties*:

(i) (Strong normalization) *If $U = X$ and the cohomology of X is finitely generated, then $i(f, U) = \lambda(f)$.*

(ii) (Homotopy) *If $h_t : \overline{U} \to X$ is a homotopy without fixed points on ∂U, then $i(h_0, U) = i(h_1, U)$.*

(iii) (Additivity) *If $U_1, U_2 \subset U$ are open and disjoint and* $\mathrm{Fix}(f) \subset U_1 \cup U_2$, *then $i(f, U) = i(f, U_1) + i(f, U_2)$.*

(iv) (Contraction) *If $X_0 \subset X$ is a convexoid subspace of X and $f(\overline{U}) \subset X_0$, then $i(f, U) = i(f, U \cap X_0)$.*

Leray established a form of the commutativity property of the index and the following result: *If $\bigcap_{n=1}^{\infty} f^n(X) = A \neq \emptyset$ and $\lambda(f|A)$ is defined, then $\lambda(f)$ is also defined and $\lambda(f) = \lambda(f|A)$.*

In his unpublished 1948–1949 lectures at the Collège de France (for a summary, see Leray [1950]), using the degree, J. Leray developed the index in locally convex spaces. By showing that this theory extends to nbd retracts of locally convex spaces, he also outlined the index theory for compact (metrizable and nonmetrizable) ANRs. The axioms of the index for polyhedra were studied for the first time in O'Neill's thesis, written under W. Hurewicz in 1952 (see O'Neill [1953]); O'Neill rederived—for the category of compact polyhedra—the principal results of Leray's theory (as well as Hopf's results), and moreover proved that the fixed point index is uniquely determined by the axioms. Bourgin [1955b], using Čech cohomology and O'Neill's results, introduced the index for compact (nonmetrizable) ANRs. Deleanu [1959] extended the Leray index to retracts of convexoid spaces. Browder [1960] formulated the axioms for the index on the category of compact spaces, and extended the index theory to "semicomplexes" by finding appropriate algebraic analogues of the properties of the fixed point index for chain mappings.

All these results use definitions of the fixed point index based on certain induced chain mappings.

§17. Further Results and Applications

This paragraph begins with a version of the index theory that is useful in problems dealing with globally defined maps of ANR's. Sections 2 through 6 are concerned with various extensions of the Lefschetz theory to wider classes of maps and spaces.

1. Local Index Theory for ANRs

In this section we describe a somewhat different version of the index theory for ANRs, which turns out to be useful in various problems. This version, called a "local index theory", can be regarded as a local form of the Lefschetz theorem ($\Lambda(f)$ may be regarded as a "global index").

For a given class $\mathscr{A}$ of self-maps of ANRs, let $\mathscr{A}^*$ denote the class of all triples (X, f, U), where $f : X \to X$ belongs to $\mathscr{A}$, $U \subset X$ is open, and $\mathrm{Fix}(f|\partial U) = \emptyset$. We call $\mathscr{A}$ *admissible* if for each $f \in \mathscr{A}$:

(i) f is compactly fixed,

(ii) f is a Lefschetz map (for homology with coefficients in $\boldsymbol{Q}$).

We will call $\mathscr{A}^*$ admissible if $\mathscr{A}$ is admissible.

(1.1) DEFINITION. Let $\mathscr{A}^*$ be an admissible class of triples. Then a *local index* on $\mathscr{A}^*$ is a function $i : \mathscr{A}^* \to \boldsymbol{Z}$ satisfying the following conditions:

(I) (*Strong normalization*) If a map $f : X \to X$ belongs to $\mathscr{A}$, then

$$\Lambda(f) = i(X, f, X).$$

(II) (*Homotopy*) If $h_t : X \to X$ is a homotopy and $U \subset X$ open such that the map $H : X \times \boldsymbol{I} \to X \times \boldsymbol{I}$ given by $(x, t) \mapsto (h_t(x), t)$ belongs to $\mathscr{A}$ and $(X \times \boldsymbol{I}, H, U \times \boldsymbol{I}) \in \mathscr{A}^*$, then

$$i(X, h_0, U) = i(X, h_1, U).$$

(III) (*Additivity*) If $(X, f, U) \in \mathscr{A}^*$ and V_1, V_2 are disjoint open subsets of U such that $\mathrm{Fix}(f) \subset V_1 \cup V_2$, then

$$i(X, f, U) = i(X, f, V_1) + i(X, f, V_2).$$

(IV) (*Commutativity*) If one of the maps $f : X \to Y$, $g : Y \to X$ is locally compact and both (X, gf, U) and $(Y, fg, g^{-1}(U))$ are in $\mathscr{A}^*$, then

$$i(X, gf, U) = i(Y, fg, g^{-1}(U)).$$

(V) (*Excision*) If $(X, f, U), (X, g, U) \in \mathscr{A}^*$ and $f|\overline{U} = g|\overline{U}$, then $i(X, f, U) = i(X, g, U)$.

As an immediate consequence of Theorem (16.5.1) we have

(1.2) THEOREM. *Let $\mathscr{A}$ be the class of all compact maps $f : X \to X$, where X is an ANR. Then there exists a local index $i : \mathscr{A}^* \to Z$ with properties* (I)–(V), *provided that in* (IV) *it is assumed that either f or g is a compact map.* □

As an obvious corollary we obtain

(1.3) THEOREM. *Let $\mathscr{A}$ be the class of all continuous maps $f : X \to X$, where X is a compact ANR. Then there exists a local index $i : \mathscr{A}^* \to Z$ with properties* (I)–(V). □

We now establish the existence of the local index for compactly absorbing maps.

(1.4) DEFINITION. Let $\mathscr{A}$ be the class of all compactly absorbing maps $f : X \to X$, where X is an ANR, and let $\mathscr{A}^*$ be the corresponding class of triples. Given $(X, f, U) \in \mathscr{A}^*$, let $W \subset X$ be open such that $f|W : W \to W$ is compact and W absorbs compact sets in X under the map f. The local index $i(X, f, U)$ is defined by

$$(*) \qquad i(X, f, U) = i(W, f|W, W \cap U),$$

where the right hand side is the index of the compact map $f|W$, given in Theorem (1.2); this definition does not depend on the choice of W.

We are now in a position to state the main result of this section.

(1.5) THEOREM. *Let $\mathscr{A}$ denote the class of all compactly absorbing maps $f : X \to X$, where X is an ANR. Then the function $i : \mathscr{A}^* \to Z$ defined by* $(*)$ *is a local index with properties* (I)–(V).

PROOF. The properties of the local index for compact maps given in Theorem (1.2) yield in a straightforward way the corresponding properties (I)–(V) of the index for compactly absorbing maps. As an example, we give the proof of commutativity. Let X, Y be ANRs, and let $f : X \to Y$, $g : Y \to X$ be maps such that f is locally compact and $gf : X \to X$ is compactly absorbing. We first show that $fg : Y \to Y$ is also compactly absorbing. To this end, choose $V \subset X$ open such that:

(i) V absorbs compact sets in X under the map gf,
(ii) $gf|V : V \to V$ is compact.

Using local compactness of f, cover the compact set $\overline{gf(V)} \subset V$ by a finite number of open sets $O_1, \ldots, O_n \subset V$ such that each $\overline{f(O_i)}$ is compact, and let $O = \bigcup_{i=1}^{n} O_i \subset V$. Since

$$\overline{gf(O)} \subset \overline{gf(V)} \subset O \subset V,$$

it follows that O is invariant under gf, the restriction $gf|O : O \to O$ is compact, and O absorbs compact sets in X.

Let $W = g^{-1}(O)$, and let K be a compact subset of Y. Since O absorbs compact sets under the map gf, there exists a positive integer m such that

$$(gf)^m g(K) = g(fg)^m(K) \subset O,$$

and hence $(fg)^m(K) \subset g^{-1}(O) = W$. This implies that W absorbs compact sets in Y under the map fg. On the other hand, the inclusion $gf(O) \subset O$ and compactness of $gf|O$ imply that $g(\overline{f(O)}) \subset O$, and therefore $\overline{f(O)} \subset g^{-1}(O) = W$. Hence

$$\overline{fg(W)} = \overline{fg(g^{-1}(O))} \subset \overline{f(O)} \subset W,$$

and thus $fg|W : W \to W$ is a compact map. Using Definition (1.4) and commutativity for compact maps, we obtain

$$\begin{aligned} i(X, gf, U) &= i(O, gf, U \cap O) = i(W, fg, g^{-1}(U \cap O)) \\ &= i(W, fg, g^{-1}(U) \cap W) = i(Y, fg, g^{-1}(U)), \end{aligned}$$

and thus the proof of commutativity is complete. The proof of the remaining properties is similar and is left to the reader. □

As an immediate consequence we obtain

(1.6) COROLLARY. *Let $\mathscr{A}$ be the class of maps $f : X \to X$ of compact attraction, where X is an ANR. Then the function $i : \mathscr{A}^* \to Z$ defined by* (∗) *in* (1.4) *is a local index with properties* (I)–(V).

2. Fixed Points for Self-Maps of Arbitrary Compacta

In this section, by concentrating on the nature of the map rather than on that of the underlying space, we determine fairly broad classes of self-maps of compacta for which the Lefschetz theorem remains valid. The results established embrace numerous fixed point theorems.

(2.1) DEFINITION. A map $f : X \to Y$ is called a $\mathcal{NES}$(*compact metric*)-*map* if for any pair (Z, A) of compacta and any $g : A \to X$ there exists a nbd U of A in Z and a map $\varphi : U \to Y$ such that $\varphi|A = fg$; the class of all such maps is denoted by $\mathcal{NES}$(compact metric).
A space X is called a *neighborhood extensor space for compact metric spaces* if $1_X \in \mathcal{NES}$(compact metric); the class of all such spaces is denoted by NES(compact metric).

We remark that the class of metric NES(compact metric) spaces contains all the ANRs.

Some simple properties of $\mathcal{NES}$(compact metric)-maps are given in

(2.2) PROPOSITION.

(i) *For any maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$, *the composite* $gf : X \to Z$ *is a* $\mathcal{NES}$*(compact metric)-map provided either* f *or* g *is a* $\mathcal{NES}$*(compact metric)-map.*

(ii) *If either* X *or* Y *is a* NES*(compact metric) space, then any* $f : X \to Y$ *is a* $\mathcal{NES}$*(compact metric)-map.*

PROOF. The straightforward verification is left to the reader. □

(2.3) THEOREM. *Let* K *be a compactum and* $f : K \to K$ *a* $\mathcal{NES}$*(compact metric)-map. Then* f *is strongly Lefschetz, that is,* $\Lambda(f)$ *is defined and* $\Lambda(f) \neq 0$ *implies that* f *has a fixed point.*

PROOF. Embed K into the Hilbert cube $\boldsymbol{I}^\infty$ and consider the diagram

$$\begin{tikzcd}
\boldsymbol{I}^\infty & U \arrow[l, hook'] \arrow[d, "\varphi"'] & \widetilde{K} \arrow[l, hook', "j"'] \arrow[dl, shift left, "s^{-1}"] \\
& K \arrow[r, "f"'] \arrow[ur, shift left, "s"] & K
\end{tikzcd}$$

where $s : K \to \widetilde{K}$ is a homeomorphism with inverse $s^{-1} : \widetilde{K} \to K$. Consider the pair $(\boldsymbol{I}^\infty, \widetilde{K})$ of compacta and the map $g = s^{-1} : \widetilde{K} \to K$. In view of Definition (2.1), and because $f \in \mathcal{NES}$(compact metric), there exists an open nbd U of $\widetilde{K}$ in $\boldsymbol{I}^\infty$ and a map $\varphi : U \to K$ such that $\varphi j = \varphi|\widetilde{K} = fs^{-1}$ Because $\varphi(js) = (\varphi j)s = (fs^{-1})s = f$ and U is an ANR, we see that f factorizes through a Lefschetz space, and hence in view of (15.3.6), our conclusion follows. □

(2.4) THEOREM.

(a) *Let* K *be a compact metric space and suppose* $f : K \to K$ *can be factored as* $K \xrightarrow{u} X \xrightarrow{v} K$, *where* X *is a* NES*(compact metric) space. Then* f *is strongly Lefschetz.*

(a)* *Let* X *be a* NES*(compact metric) space and suppose that* $F : X \to X$ *can be factored as* $X \xrightarrow{v} K \xrightarrow{u} X$, *where* K *is a compact metric space. Then* F *is strongly Lefschetz.*

PROOF. By (2.2), (a) is an immediate consequence of (2.3); and (a*) follows from (a). □

(2.5) COROLLARY. *Any metric* NES*(compact metric) space is a Lefschetz space.* □

(2.6) DEFINITION. A map $f : X \to Y$ is said to be an *approximate* $\mathcal{NES}$*(compact metric)-map* if for any pair (Z, A) of compacta and any $g : A \to X$ the following condition is satisfied: for each $\varepsilon > 0$ there

exists a nbd U_ε of A in Z and $\varphi_\varepsilon : U_\varepsilon \to Y$ such that $\varphi_\varepsilon|A$ and gf are ε-close and homotopic. The class of approximate $\mathcal{NES}$(compact metric)-maps is denoted by $\mathcal{ANES}$(compact metric).

(2.7) THEOREM. *Let K be a compactum and $f : K \to K$ an approximate $\mathcal{NES}$(compact metric)-map. Then f is strongly Lefschetz.*

PROOF. Embed K into the Hilbert cube I^∞ and consider the diagram

$$\begin{tikzcd}
I^\infty & U_\varepsilon \arrow[l, hook'] \arrow[d, dashed, "\varphi_\varepsilon"'] & \tilde{K} \arrow[l, hook', "j"'] \arrow[dl, shift left, "s^{-1}"] \\
& K \arrow[r, "f"'] \arrow[ur, shift left, "s"] & K
\end{tikzcd}$$

where $s : K \to \tilde{K}$ is a homeomorphism with inverse $s^{-1} : \tilde{K} \to K$. Consider the pair $(I^\infty, \tilde{K})$ of compacta and the map $g = s^{-1} : \tilde{K} \to K$, fix $n \in N$, and set $\varepsilon = 1/n$. By Definition (2.6), there exists an open nbd U_ε of $\tilde{K}$ in I^∞ and a map $\varphi_\varepsilon : U_\varepsilon \to K$ such that $\varphi_\varepsilon j = \varphi_\varepsilon|\tilde{K}$ and fs^{-1} are ε-close and homotopic; hence so are $f_\varepsilon = (\varphi_\varepsilon j)s$ and $(fs^{-1})s = f$, and f_ε factorizes through a Lefschetz space U_ε. We have thus found a sequence of maps $f_n : K \to K$ such that f and f_n are $(1/n)$-close and homotopic, and each f_n factorizes through a Lefschetz space. Because f_n is a Lefschetz map by (15.3.6), $f \simeq f_n$ implies that so also is f, and $\Lambda(f) = \Lambda(f_n)$ for all n. If $\Lambda(f) \neq 0$, then $\Lambda(f_n) \neq 0$, and therefore $\text{Fix}(f_n) \neq 0$ by (15.3.6). Letting $x_n = f_n(x_n)$, we see that x_n is a $(1/n)$-fixed point of f, and the conclusion follows from the compactness of K. □

3. Forming New Lefschetz Spaces from Old by Domination

In this section we turn to general Lefschetz-type results for compact maps in nonmetrizable spaces.

We first recall some terminology and notation. If Y is a space and $K \subset Y$, we denote by $\text{Cov}_Y(K)$ the set of all open coverings of K in Y. Given $\alpha, \beta \in \text{Cov}_Y(K)$ we write $\alpha \preceq \beta$ if β refines α. Clearly, $\preceq$ is a preorder relation converting $\text{Cov}_Y(K)$ into a directed set; we write $\text{Cov}_Y(Y) = \text{Cov}(Y)$. Let $f, g : X \to Y$ be two maps, and let $\alpha \in \text{Cov}(Y)$ be an open covering of Y. We say that f and g are *α-close* if for each $x \in X$ there exists a $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say that f and g are *α-homotopic* (written $f \overset{\alpha}{\simeq} g$) if there is a homotopy $h_t : X \to Y$ $(0 \leq t \leq 1)$ joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in \boldsymbol{I}$.

Let $f : X \to X$ and $\alpha \in \text{Cov}(X)$. A point $x \in X$ is said to be an *α-fixed point* for f provided x and $f(x)$ belong to a common $U_x \in \alpha$. Clearly, if

$\alpha, \beta \in \mathrm{Cov}(X)$ and α refines β, then every α-fixed point for f is also a β-fixed point for f.

(3.1) LEMMA. *Let $f : X \to X$ be a map and $K = f(X)$. The following statements are equivalent:*
(i) *f has a fixed point,*
(ii) *there is a cofinal family $\mathcal{D} \subset \mathrm{Cov}_X(K)$ such that f has an α-fixed point for every $\alpha \in \mathcal{D}$.*

PROOF. (i)$\Rightarrow$(ii) is evident. To prove (ii)$\Rightarrow$(i), assume that f has no fixed points. Then for each $x \in X$ there are neighborhoods V_x and $U_{f(x)}$ of x and $f(x)$, respectively, such that $f(V_x) \subset U_{f(x)}$ and $V_x \cap U_{f(x)} = \emptyset$. Setting $\beta = \{V_x\}$, we get a covering of X such that f has no β-fixed point. If $\alpha \in \mathcal{D}$ refines β, then f has no α-fixed point, contrary to (ii). □

(3.2) LEMMA. *Let $f : X \to X$ be a map, $K = f(X)$, and let $\mathcal{D} \subset \mathrm{Cov}_X(K)$ be a cofinal family of coverings of K. Assume that for each $\alpha \in \mathcal{D}$ there is an $f_\alpha : X \to X$ with the following properties:* (i) *f and f_α are α-close;* (ii) *f_α has a fixed point. Then f has a fixed point.*

PROOF. Because by (i) a fixed point for f_α is an α-fixed point for f, our assertion follows at once from (3.1). □

(3.3) DEFINITION. Let K be a compact space, $f : K \to K$ a map, and $\mathcal{D} = \mathrm{Cov}(K)$. A family $\{f_\alpha\}_{\alpha\in\mathcal{D}}$ of maps $f_\alpha : K \to K$ is called an *approximating family* for f provided for each $\alpha \in \mathcal{D}$ the maps $f, f_\alpha : K \to K$ are α-close and homotopic.

With this terminology we have the following useful property of Lefschetz maps:

(3.4) LEMMA. *Let K be a compact space and $f : K \to K$ a map. Assume that $\{f_\alpha\}_{\alpha\in\mathcal{D}}$ is an approximating family for f such that one of the following conditions is satisfied:*
(a) *each f_α is strongly Lefschetz,*
(b) *each f_α factors through a Lefschetz space.*
Then f is a strongly Lefschetz map.

PROOF. In view of (15.3.5) and (15.3.6), this is an obvious consequence of the definitions involved. □

Domination

(3.5) DEFINITION. A space P *dominates* a space X if there are maps $X \xrightarrow{s} P \xrightarrow{r} X$ such that $rs \simeq 1_X$. If $\alpha \in \mathrm{Cov}(X)$, we say that P *α-dominates* X if there are maps $X \xrightarrow{s_\alpha} P \xrightarrow{r_\alpha} X$ such that $r_\alpha s_\alpha \overset{\alpha}{\simeq} 1_X$.

Though domination is weaker than homotopy equivalence, it permits some transfer of information from P to X; for example, since every compact ANR is dominated by a finite polytope (cf. (7.14)), the singular homology of such an ANR is of finite type.

Let $\mathscr{P}$ be a class of spaces. We denote by $\mathscr{D}(\mathscr{P})$ the class of spaces defined as follows: $X \in \mathscr{D}(\mathscr{P})$ provided for each $\alpha \in \operatorname{Cov}(X)$ there is a space $P_\alpha \in \mathscr{P}$ that α-dominates X.

(3.6) THEOREM. *Let $\mathscr{P}$ be a class of Lefschetz spaces. Then any space in the class $\mathscr{D}(\mathscr{P})$ is a Lefschetz space.*

PROOF. Let $X \in \mathscr{D}(\mathscr{P})$ and $f : X \to X$ a compact map. Fix $\beta \in \operatorname{Cov}(X)$ and let $\alpha = \{f^{-1}(V)\}_{V \in \beta}$. Take a Lefschetz space that α-dominates X with maps $X \xrightarrow{s_\alpha} P_\alpha \xrightarrow{r_\alpha} X$ satisfying $r_\alpha s_\alpha \overset{\alpha}{\simeq} 1$ and consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "s_\alpha"] \arrow[d, "f r_\alpha s_\alpha"'] & P_\alpha \arrow[d, "s_\alpha f r_\alpha"] \arrow[dl, "f r_\alpha"'] \\ X \arrow[r, "s_\alpha"] & P_\alpha \end{tikzcd}$$

By compactness of f, both vertical maps are also compact. Since P_α is a Lefschetz space, (15.3.5) shows that the vertical maps are Lefschetz and $\Lambda(s_\alpha f r_\alpha) = \Lambda(f r_\alpha s_\alpha)$. Since $f \simeq f r_\alpha s_\alpha$, we infer that f is a Lefschetz map and

$$(*) \qquad \Lambda(s_\alpha f r_\alpha) = \Lambda(f r_\alpha s_\alpha) = \Lambda(f).$$

Assume that $\Lambda(f) \neq 0$. Since P_α is a Lefschetz space, $(*)$ shows that $\operatorname{Fix}(s_\alpha f r_\alpha) \neq \emptyset$, and hence $\operatorname{Fix}(f r_\alpha s_\alpha) \neq \emptyset$ (see (0.4.1)). To complete the proof, observe that by the choice of α, the relation $r_\alpha s_\alpha \overset{\alpha}{\simeq} 1$ implies $f \overset{\beta}{\simeq} f r_\alpha s_\alpha$, and therefore, since $\operatorname{Fix}(f r_\alpha s_\alpha) \neq \emptyset$, the map f, being β-close to $f r_\alpha s_\alpha$, has a β-fixed point. Because $\beta \in \operatorname{Cov}(X)$ was fixed arbitrarily, we conclude by (3.1) that f has a fixed point. □

REMARK. Denote by **Pol** the class of infinite polyhedra with the weak topology (cf. Section 7). From the properties of these spaces and from the Lefschetz–Hopf fixed point theorem (9.2.4), it follows easily that every $X \in$ **Pol** is a Lefschetz space. Because ANR $\subset \mathscr{D}(\mathbf{Pol})$, Theorem (3.6) gives another proof that every ANR is a Lefschetz space (see (15.4.3)).

4. Fixed Points in Linear Topological Spaces

This section develops some fixed point results for subsets of linear topological spaces. We first show that every open subset of an admissible linear topological space in the sense of Klee is a Lefschetz space.

Let E be a linear topological space and U a nbd of the origin in E. We say that U is *shrinkable* provided for all $x \in U$ and $0 < \lambda < 1$ the point λx lies in U. It is known (see Klee [1960]) that the shrinkable neighborhoods form a base of the topology of E at 0. It follows that for every nbd W of 0 there is a shrinkable nbd V of 0 such that $V + V \subset W$ and every interval $[x, y]$ with x and y in V is entirely contained in W This clearly implies

(4.1) PROPOSITION. *Let E be a linear topological space and $U \subset E$ open. Then for each $\alpha \in \operatorname{Cov}(U)$ there exists a refinement $\beta \in \operatorname{Cov}(U)$ such that any two β-close maps of any space into U are α-homotopic.* □

(4.2) DEFINITION. A linear topological space E is called *admissible* if for every compact $K \subset E$ and $\alpha \in \operatorname{Cov}(E)$ there is a map $\pi_\alpha : K \to E$ such that (i) $\pi_\alpha(K)$ is contained in a finite-dimensional subspace of E and (ii) π_α is α-close to the inclusion $i : K \hookrightarrow E$.

In view of Theorem (13.5.3) every locally convex linear topological space is admissible.

(4.3) THEOREM. *Any open subset of a locally convex and, more generally, of an admissible linear topological space E is a Lefschetz space.*

PROOF. Let E be admissible, $V \subset E$ open, and $f : V \to V$ a compact map sending V into a compact subset K of V Let $\alpha \in \mathcal{D} = \operatorname{Cov}(V)$ be given; take $\pi_\alpha : K \to E$ satisfying $\pi_\alpha(K) \subset E^n \subset E$ and α-close to the inclusion $i : K \hookrightarrow E$. Define a compact map $f_\alpha : V \to V$ by setting $f_\alpha(x) = \pi_\alpha f(x)$. Clearly, we have the commutative diagram

$$\begin{tikzcd}
V \cap E^n \arrow[r] \arrow[d, "\widehat{f}_\alpha"'] & V \arrow[d, "f_\alpha"] \arrow[dl] \\
V \cap E^n \arrow[r] & V
\end{tikzcd}$$

with the obvious inclusions and compact contractions. Since $V \cap E^n$ is a Lefschetz space, (15.3.5) shows that f_α is a Lefschetz map and $\Lambda(\widehat{f_\alpha}) = \Lambda(f_\alpha)$. In view of (4.1) we may assume without loss of generality that f is homotopic to f_α for each α; consequently, f is a Lefschetz map and $\Lambda(f) = \Lambda(f_\alpha)$. Now, if $\Lambda(f) \neq 0$, then $\Lambda(f_\alpha) \neq 0$ for each $\alpha \in \mathcal{D}$; as $V \cap E^n$ is a Lefschetz space, we deduce that each f_α has a fixed point, and hence f has a fixed point by (3.2), completing the proof. □

Open sets in Tychonoff cubes

We next show that an open subset of a Tychonoff cube is a Lefschetz space. Given a linear space E and a subset $K \subset E$, we denote by $L(K)$ the linear span of K, i.e., the smallest linear subspace of E that contains K.

(4.4) LEMMA. *Let E be a linear topological space, and let $K \subset E$ be compact. Then the linear span $L(K)$ is paracompact.*

PROOF. For any $m \in N$ let $X_m = [-m, m]^m \times K^m$ and define $f_m : X_m \to E$ by the assignment

$$(\alpha_1, \dots, \alpha_m, x_1, \dots, x_m) \mapsto \sum_{i=1}^{m} \alpha_i x_i.$$

Since f_m is continuous, $K_m = f_m(X_m)$ is compact. By observing that any element $\sum \alpha_j x_j$ of $L(K)$ belongs to K_m for some m, we see that $L(K) = \bigcup_{i=1}^{\infty} K_m$, i.e., the set $L(K)$ is σ-compact. Because $L(K)$ is regular, the assertion readily follows. □

(4.5) THEOREM. *Any open subset of a Tychonoff cube is a Lefschetz space.*

PROOF. Let $I^{\aleph}$ be a Tychonoff cube, and let $U \subset I^{\aleph}$ be open. We first show that U is r-dominated by an open subset of a locally convex linear topological space. Let E be a product of real lines that contains $I^{\aleph}$ and note that E equipped with the product topology is a locally convex linear topological space. We claim that $I^{\aleph}$ is a retract of $L(I^{\aleph})$: to see this, consider the diagram

$$\begin{tikzcd}
 & I^{\aleph} \\
I^{\aleph} \arrow[ur, "1_{I^{\aleph}}"] \arrow[r, hook] & L(I^{\aleph}) \arrow[u, dashed, "r"']
\end{tikzcd}$$

By (4.4), $L(I^{\aleph})$ is paracompact, hence normal. By the Tietze theorem, $1_{I^{\aleph}} : I^{\aleph} \to I^{\aleph}$ admits an extension $r : L(I^{\aleph}) \to I^{\aleph}$, which is the required retraction. In particular, U is a retract of $V = r^{-1}(U)$, and since the latter is an open subset in a locally convex space, we conclude the proof by applying (4.3) and (15.3.7). □

5. Fixed Points in NES(compact) Spaces

The main class of spaces introduced in this section is that of approximate NES(compact) spaces. For this class the most general Lefschetz-type theorems will be established.

(5.1) DEFINITION. A space Y is called a *neighborhood extensor space for compact spaces* if for any compact pair (X, A), every map $f : A \to Y$ has an extension $F : U \to Y$ over a nbd U of A in X; the class of neighborhood extensor spaces for compact spaces is denoted by NES(compact).

We list some basic properties of NES(compact) spaces:

(i) An open subset of a NES(compact) space is NES(compact).
(ii) A neighborhood retract of a NES(compact) space is NES(compact).
(iii) The product of a finite family of NES(compact) spaces is NES(compact).
(iv) Every paracompact local NES(compact) space is NES(compact).
(v) Every ANR is a NES(compact) space.

(5.2) THEOREM.
(a) *Let K be a compact space and suppose $f : K \to K$ can be factored as $K \xrightarrow{u} X \xrightarrow{v} K$, where X is a* NES(*compact*) *space. Then f is strongly Lefschetz.*
(a)* *Let X be a* NES(*compact*) *space and suppose that $F : X \to X$ can be factored as $X \xrightarrow{v} K \xrightarrow{u} X$, where K is a compact space. Then F is strongly Lefschetz.*

PROOF. In view of (15.3.5), it is enough to establish (a). To this end consider the diagram

$$\begin{tikzcd}
I^{\aleph} & U \arrow[l, hook'] \arrow[dd, dashed, "\varphi"'] & & \widehat{K} \arrow[ll, hook', "j"'] \arrow[dl, shift left, "s^{-1}"] \\
 & & K \arrow[ur, shift left, "s"] \arrow[dl, "u"'] \arrow[d, "f"] & \\
 & X \arrow[r, "v"] & K &
\end{tikzcd}$$

where X is a NES(compact) and $s : K \to \widehat{K}$ is a homeomorphism of K onto a subset $\widehat{K}$ of a Tychonoff cube $I^{\aleph}$. Since X is NES(compact), there is an extension $\varphi : U \to X$ of the map $us^{-1} : \widehat{K} \to X$ over an open nbd U of $\widehat{K}$ in $I^{\aleph}$, i.e., $\varphi j = us^{-1}$, where $j : \widehat{K} \to U$ is the inclusion. Now consider the composite

$$K \xrightarrow{js} U \xrightarrow{v\varphi} K.$$

We have $(v\varphi)(js) = v(\varphi j)s = v(us^{-1})s = vu = f$. Thus, in view of (4.5), f factors through a Lefschetz space, and hence by (15.3.6), we infer that f is a strongly Lefschetz map. □

As an immediate consequence we obtain

(5.3) THEOREM. *Any* NES(*compact*) *space is a Lefschetz space.* □

(5.4) DEFINITION. A space Y is called an *approximate neighborhood extensor space for compact spaces* if for any compact pair (Z, A), any map $f : A \to Y$, and each $\alpha \in \operatorname{Cov}(Y)$ there exists a nbd U_α of A in Z and a map $\varphi_\alpha : U_\alpha \to Y$ such that the maps $\varphi|A, f : A \to Y$ are

α-homotopic; the class of approximate neighborhood extensor spaces for compact spaces is denoted by ANES(compact).

We list some properties of ANES(compact) spaces:

(i) Any open subset of an ANES(compact) space is ANES(compact).
(ii) Any retract of an ANES(compact) space is ANES(compact).
(iii) Any convex set in a locally convex space is ANES(compact).

We are now ready to prove a general Lefschetz-type result containing as special cases many previously established fixed point theorems.

(5.5) THEOREM.

(a) *Let K be a compact space and suppose $f : K \to K$ can be factored as $K \xrightarrow{u} X \xrightarrow{v} K$, where X is an* ANES(*compact*) *space. Then f is strongly Lefschetz.*

(a)* *Let X be an* ANES(*compact*) *space and suppose that $F : X \to X$ can be factored as $X \xrightarrow{v} K \xrightarrow{u} X$, where K is a compact space. Then F is strongly Lefschetz.*

PROOF. In view of (15.3.5), it is enough to establish (a). To this end, we first define for f an approximating family $\{f_\alpha\}_{\alpha\in\mathrm{Cov}(K)}$ such that each $f_\alpha : K \to K$ factors through an open subset of a Tychonoff cube. We embed K into a Tychonoff cube $I^{\aleph}$, fix $\alpha \in \mathrm{Cov}(K)$, and consider the diagram

$$\begin{tikzcd}
I^{\aleph} & U_\alpha \arrow[l, hook'] \arrow[dd, dashed, "\varphi_\alpha"'] & & \widehat{K} \arrow[ll, hook', "j"'] \arrow[dl, shift left, "s^{-1}"] \\
& & K \arrow[ur, shift left, "s"] \arrow[dl, "u"'] \arrow[d, "f"] & \\
& X \arrow[r, "v"] & K &
\end{tikzcd}$$

where $s : K \to \widehat{K}$ is a homeomorphism onto $\widehat{K} \subset I^{\aleph}$, and U_α and φ_α are defined in the proof below. Consider the map $us^{-1} : \widehat{K} \to X$ and the covering $\beta = v^{-1}(\alpha)$ of X. As X is ANES(compact), there exists for $\beta = \beta(\alpha)$ an open nbd U_α of $\widehat{K}$ in $I^{\aleph}$ and a map $\varphi_\alpha : U_\alpha \to X$ such that

$$\varphi_\alpha j, us^{-1} : \widehat{K} \to X \quad \text{are } \beta\text{-homotopic.}$$

It follows that the maps $f_\alpha = v\varphi_\alpha js$ and $f = vus^{-1}s$ from K to K are α-homotopic. Since $\alpha \in \mathrm{Cov}(K)$ is arbitrary, it is clear, in view of the definition of f_α, that $\{f_\alpha\}_{\alpha\in\mathrm{Cov}(K)}$ is the desired approximating family for f Now, since open sets in Tychonoff cubes are Lefschetz spaces, our assertion follows at once from Lemma (3.4) and thus the proof is complete. □

As an immediate consequence, we obtain

(5.6) THEOREM. *Any* ANES(*compact*) *space is a Lefschetz space.* □

6. General Asymptotic Fixed Point Results

This section develops the asymptotic fixed point theory for spaces more general than ANRs.

(6.1) DEFINITION. Let $\mathscr{M}$ be a class of maps and X a space. We say that X is a *Lefschetz space for the class* $\mathscr{M}$ if every $f : X \to X$ belonging to $\mathscr{M}$ is a strongly Lefschetz map. The collection of all Lefschetz spaces for the class $\mathscr{M}$ is denoted by $\mathcal{L}_{\mathscr{M}}$.

Recall (see (15.6.1)) that by a basic class of locally compact maps is meant any of the following classes:

$$\mathscr{K}_{\text{ev}} = \text{the class of eventually compact maps,}$$

$$\mathscr{K}_{\text{ac}} = \text{the class of asymptotically compact maps,}$$

$$\mathscr{K}_{\text{at}} = \text{the class of maps of compact attraction,}$$

$$\mathscr{K}_{\text{ca}} = \text{the class of compactly absorbing maps.}$$

(6.2) THEOREM. *Let X be a regular space and assume that every nonempty open subset of X is a Lefschetz space. Then X is a Lefschetz space for any of the basic classes of locally compact maps.*

PROOF. Because $\mathscr{K}_{\text{ev}} \subset \mathscr{K}_{\text{ac}} \subset \mathscr{K}_{\text{at}} \subset \mathscr{K}_{\text{ca}}$, it is clearly enough to consider the case of the class $\mathscr{K}_{\text{ca}}$ of compactly absorbing maps. So let $f : X \to X$ be in $\mathscr{K}_{\text{ca}}$; we are going to show that f is strongly Lefschetz.

By (15.6.1), there exists an open f-invariant subset U of X such that U absorbs compact sets in X and $f_U : U \to U$ is compact. Because by assumption, U is a Lefschetz space, f_U is a Lefschetz map, and therefore so is f by (15.7.1), and $\Lambda(f) = \Lambda(f_U)$. If $\Lambda(f) = \Lambda(f_U) \neq 0$, then (again by assumption) $x_0 = f_U(x_0) = f(x_0)$ for some $x_0 \in U$, which completes the proof. □

From (2.4), (5.2), (5.3), and (6.2), we obtain the following general asymptotic fixed point theorem:

(6.3) THEOREM. *The following classes of spaces are Lefschetz spaces for compactly absorbing maps*:
(i) *metric* NES(*compact metric*) *spaces*,
(ii) *regular* NES(*compact*) *spaces*,
(iii) *regular* ANES(*compact*) *spaces.* □

As immediate consequences we obtain

(6.4) THEOREM (*Generalized Schauder–Tychonoff theorem*). *If C is a convex subset of a locally convex linear topological space, then any compactly absorbing map $f : C \to C$ has a fixed point.*

(6.5) THEOREM. *Let U be an open subset of a locally convex linear topological space, and let $f : U \to U$ be a compactly absorbing map. If f is nullhomotopic, then f has a fixed point.* □

7*. Domination of ANRs by Polytopes

Our aim in this section is to prove that every ANR is dominated by a polytope with the weak topology. We first discuss in some detail the concept and properties of infinite polyhedra and polytopes.

Infinite polyhedra are constructed by pasting together simplices along common faces. Precisely,

(7.1) DEFINITION. An *infinite polyhedron* K is a space that can be represented as the union of an infinite family $\mathscr{K} = \{\sigma\}$ of simplices, where $\mathscr{K}$ and the topology of K satisfy the following conditions:

(1) Each face of a $\sigma \in \mathscr{K}$ is also a member of $\mathscr{K}$
(2) If $\sigma, \tau \in \mathscr{K}$ have a nonempty intersection, then $\sigma \cap \tau$ is their common face.
(3) The space $K = \bigcup\{\sigma \mid \sigma \in \mathscr{K}\}$ has the weak topology determined by the family $\mathscr{K}$: $A \subset K$ is closed if and only if $A \cap \sigma$ is closed in σ for each $\sigma \in \mathscr{K}$

The family $\mathscr{K}$ is called the family of simplices of K; the *dimension* $\dim K$ of K is the supremum of the dimensions of its simplices; K is *locally finite* if each vertex belongs to at most finitely many simplices. By a *subpolyhedron* of K is meant the union of any subset of its simplices that forms a polyhedron. The union of all simplices of dimension $\leq n$ is called the *n-skeleton* of K and denoted by K^n; it is clearly a subpolyhedron, as are the boundary of any simplex of K and any simplex of K together with all its faces.

Because of the weak topology, every subpolyhedron of K is closed in K; in particular, since the set K^0 of vertices, and every subset of K^0, is a subpolyhedron, it follows that K^0 is a discrete closed set. Moreover, we have the following characterization of compact subsets of a polyhedron:

(7.2) PROPOSITION. *A closed subset $A \subset K$ is compact if and only if it is contained in a finite subpolyhedron of K.*

PROOF. Since the union of finitely many compact spaces is compact, the "if" part is obvious. Conversely, if A is not contained in a finite subpolyhedron, choose an $a \in A \cap \sigma$ for each σ meeting A; then $\{a\}$ is an infinite discrete closed subset of A, so A cannot be compact. □

We shall need some open sets. For any vertex $p \in K^0$, define $C(p) = \bigcup\{\sigma \in \mathscr{K} \mid p \notin \sigma\}$; $C(p)$ is clearly a subpolyhedron of K, and $\operatorname{St} p = K - C(p)$ is an open set containing p, called the *star* of p. It is useful to observe that

$$(*) \qquad \bigcap_{i=0}^{s} \operatorname{St} p_i \neq \emptyset \quad \text{if and only if} \quad [p_0, p_i, \dots, p_s] \text{ is a simplex of } K.$$

Indeed, it is enough to note that $\bigcap \operatorname{St} p_i = \emptyset$ if and only if $\bigcup C(p_i) = K$, and the latter equality can occur if and only if for each $\sigma \in \mathscr{K}$ there is at least one $p_i \notin \sigma$, i.e., if and only if there is no simplex in K having all the $p_0, \dots, p_s$ as vertices.

Because of the weak topology, a map f of K into any space Y is continuous if and only if $f|\sigma$ is continuous for each simplex σ of K; and similarly, a map $H : K \times I \to Y$ is continuous if and only if $H|\sigma \times I : \sigma \times I \to Y$ is continuous for each $\sigma \in \mathscr{K}$.

(7.3) DEFINITION. A map $f : K \to L$ of a polyhedron K into a polyhedron L is called *simplicial* if:

(1) for each simplex $[p_0, p_1, \dots, p_n]$ of K, the points $[f(p_0), f(p_1), \dots, f(p_n)]$ are vertices of a (possibly lower-dimensional) simplex of L,

(2) if $x = \sum \lambda_i p_i$, then $f(x) = \sum \lambda_i f(p_i)$, i.e., f is "linear" on each simplex.

A simplicial map $f : K \to L$ is therefore uniquely determined by its values on K^0; and being continuous on each $\sigma \in \mathscr{K}$, it is always continuous.

A homeomorphism of a space X onto a polyhedron is called a *triangulation* of X.

(7.4) DEFINITION. A space X together with a triangulation $h : X \to K$ is called a *polytope* (or a *triangulated space*) and is denoted by $(X; h, K)$.

The sets $h^{-1}(\sigma)$ are called *simplices* of the polytope $(X; h, K)$, and for each subpolyhedron $L \subset K$, the set $h^{-1}(L)$ is called a *subpolytope* of $(X; h, K)$. It is clear that $h^{-1}(\sigma) \approx \sigma$ for each simplex σ of K, and that X has the weak topology with respect to the family $\{h^{-1}(\sigma) \mid \sigma \in \mathscr{K}\}$. In general, a triangulable space has many triangulations; when there is no ambiguity about the particular triangulation, we denote the polytope $(X; h, K)$ simply by X, and call K the triangulation of X.

Abstract complexes. Vertex schemes

Definition (7.1) suggests that a polyhedron can be described by prescribing its vertices and those sets of vertices that span its simplices. Since this method of definition is used frequently, we describe it in detail.

(7.5) DEFINITION. An *abstract complex* $\mathscr{K}$ is a set $\mathscr{A}$ of elements (called "vertices") together with a family $\mathscr{F}$ of finite subsets of $\mathscr{A}$ (called "simplices") which satisfies the condition that if $s \in \mathscr{F}$, then every subset of s also belongs to $\mathscr{F}$.

To each polyhedron K, there corresponds an abstract complex called its *vertex scheme*: $\mathscr{A}$ is the set K^0 of vertices of K, and $\mathscr{F}$ consists of those subsets of K^0 that span simplices of K. Any polyhedron with vertex scheme $\mathscr{K}$ is called a *realization* of $\mathscr{K}$. In these terms, each polyhedron is a realization of an abstract complex. We now show that, conversely, each abstract complex $\mathscr{K}$ has a realization: Let $L(\mathscr{K})$ be a real vector space with basis $\{p_\alpha \mid \alpha \in \mathscr{A}\}$ which is in a fixed one-to-one correspondence with the vertices of $\mathscr{K}$, and with the weak topology determined by the Euclidean topology on each finite-dimensional flat. Then the polyhedron

$$|\mathscr{K}| = \bigcup\{\mathrm{Conv}(p_{\alpha_0}, \ldots, p_{\alpha_n}) \mid (\alpha_0, \ldots, \alpha_n) \in \mathscr{F}\}$$

with the subspace topology is a realization of $\mathscr{K}$, called the *standard realization*.

It follows from this description of $|\mathscr{K}|$ that each $x \in |\mathscr{K}|$ is uniquely representable as $x = \sum \lambda_\alpha(x) p_\alpha$, where for each x almost all $\lambda_\alpha(x)$ are zero, all satisfy $0 \leq \lambda_\alpha(x) \leq 1$, and $\sum \lambda_\alpha(x) = 1$; in fact, the system $\{\lambda_\alpha(x) \mid \alpha \in \mathscr{A}\}$ is simply the coordinates of x in $L(\mathscr{K})$, and so each $\lambda_\alpha : |\mathscr{K}| \to I$ is continuous. The system of functions $\{\lambda_\alpha(x) \mid \alpha \in \mathscr{A}\}$ is called the *barycentric coordinate system* for $|\mathscr{K}|$; note, in particular, that $\mathrm{St}(p_\alpha) = \{x \in |\mathscr{K}| \mid \lambda_\alpha(x) > 0\}$.

The justification for considering vertex schemes, which also indicates the force of the weak topology, is given in the following

(7.6) THEOREM. *Any two realizations of an abstract complex are simplicially homeomorphic.*

PROOF. If K_1 and K_2 are two realizations of $\mathscr{K}$, there is a one-to-one correspondence $p_\alpha \leftrightarrow q_\alpha$, in which a set $\{p_{\alpha_i}\}$ spans a simplex of K_1 if and only if the corresponding $\{q_{\alpha_i}\}$ span a simplex of K_2. We can therefore define a simplicial map $f : K_1 \to K_2$ by setting $f(p_\alpha) = q_\alpha$ and extending linearly over each simplex. This map is clearly bijective, and since both f and f^{-1} are continuous on each simplex, both maps are continuous, so f is a homeomorphism. □

It now follows that one can define a polyhedron essentially uniquely by simply specifying its vertex scheme. Observe that because of (8.4.2) and the standard realization, a finite subpolyhedron L of any polyhedron is simplicially homeomorphic to a polyhedron in R^s, where $s = \mathrm{card}\, L^0$.

Contiguous maps into polytopes

Let K be a polyhedron and Y an arbitrary space. The continuity of an $h : K \to Y$ being equivalent to that of $h|\sigma$ for each $\sigma \in K$, we next seek conditions under which a map $f : Y \to K$ is continuous. Clearly, if f is continuous, then so also is $\lambda_\alpha \circ f : Y \to \boldsymbol{I}$ for each barycentric coordinate function λ_α; but without some restriction on K, such as local finiteness, the converse need not be true. It is therefore convenient to introduce another topology in the set K that does not have this drawback. If $\{\lambda_\alpha \mid \alpha \in \mathscr{A}\}$ are the barycentric coordinate functions for K, the function $d : K \times K \to \boldsymbol{R}$ given by $d(x,y) = \sum_\alpha |\lambda_\alpha(x) - \lambda_\alpha(y)|$ is easily seen to be a metric, and K with the topology induced by this metric (called the *strong topology* of K) is written K_{m}. Each $\lambda_\alpha : K_{\mathrm{m}} \to \boldsymbol{I}$ is continuous; and because $x_n \to x$ in K_{m} if and only if $\lambda_\alpha(x_n) \to \lambda_\alpha(x)$ for each α, it follows that $f : Y \to K_{\mathrm{m}}$ is continuous if and only if each $\lambda_\alpha f$ is continuous.

The relation between the strong and weak topologies on K is discussed in the following

(7.7) PROPOSITION. *The identity map $i : K \to K_{\mathrm{m}}$ is a homotopy equivalence; and if K is locally finite, then i is in fact a homeomorphism.*

PROOF. The map $i : K \to K_{\mathrm{m}}$ is continuous, being continuous on each simplex. To construct a homotopy inverse, note that the family $\{\mathrm{St}\, p_\alpha \mid \alpha \in \mathscr{A}\}$ is an open covering of the metric space K_{m}, so has a subordinate partition of unity $\{\kappa_\alpha \mid \alpha \in \mathscr{A}\}$. Define $\kappa : K_{\mathrm{m}} \to K$ by $\kappa(x) = \sum \kappa_\alpha(x) p_\alpha$. This is actually a map into K: for if $\kappa_0, \ldots, \kappa_n$ are all the finitely many functions κ_α that do not vanish at a given $x \in K_{\mathrm{m}}$, then $x \in \bigcap_{i=0}^n \mathrm{supp}\, \kappa_i \subset \bigcap_{i=0}^n \mathrm{St}\, p_i$, so by (*), $(p_0, \ldots, p_n)$ is a simplex of K, and $\kappa(x)$ belongs to that simplex; in fact, both x and $\kappa(x)$ belong to the same simplex of the underlying set K. Moreover, κ is continuous: each $x \in K_{\mathrm{m}}$ has a nbd U on which at most finitely many κ_α are not identically zero, so that the image of U lies in a finite subpolytope L of K; since L is simplicially homeomorphic to a subspace of some $\boldsymbol{R}^s$, and since addition is continuous in $\boldsymbol{R}^s$, it follows that κ is continuous on U, and therefore at x.

Now let $\{\lambda_\alpha\}$ be the barycentric coordinate system of K, and for $(x,t) \in K \times \boldsymbol{I}$ let

$$H(x,t) = \sum_\alpha [t\lambda_\alpha(x) + (1-t)\kappa_\alpha(x)] p_\alpha.$$

Each $H(x,t)$ lies on the line segment joining x to $\kappa(x)$, so that H maps $K \times \boldsymbol{I}$ into K.

Regarded as a map $K_{\mathrm{m}} \times \boldsymbol{I} \to K_{\mathrm{m}}$, H is continuous, since the barycentric coordinates vary continuously; because $H(x,0) = i \circ \kappa(x)$, we find that $H : i \circ \kappa \simeq \mathrm{id}$.

Regarded as a map $K \times \boldsymbol{I} \to K$, H is also continuous, because $H|\sigma$ is continuous for each σ; since $H(y,0) = \kappa \circ i(y)$, this shows that $H : \kappa \circ i \simeq \mathrm{id}$. Thus, i is a homotopy equivalence with inverse κ. The proof that i^{-1} is continuous whenever K is locally finite is left to the reader. □

Continuous maps of a space into K can therefore be constructed as mappings into K_{m} followed by κ; this will be illustrated in the following definition.

(7.8) DEFINITION. Two maps f, g of a space X into a polytope K are called *contiguous* if $f(x)$ and $g(x)$ belong to a common simplex $\sigma(x)$ of K for each $x \in K$.

(7.9) THEOREM. *Let K be a polytope and X an arbitrary space. If $f, g : X \to K$ are continuous contiguous maps, then $f \simeq g$.*

PROOF. It is no restriction to assume that K is a standard polyhedron. Let $i : K \to K_{\mathrm{m}}$ be the identity map, and κ a homotopy inverse. Because $i \circ f$ and $i \circ g$ are contiguous in K_{m}, the function

$$H(x,t) = \sum [t\lambda_\alpha(i \circ f(x)) + (1-t)\lambda_\alpha(i \circ g(x))]p_\alpha$$

maps $X \times \boldsymbol{I}$ into K_{m} and is continuous because the barycentric coordinates vary continuously. This shows that $i \circ g \simeq i \circ f$. Thus, $\kappa \circ i \circ g \simeq \kappa \circ i \circ f$, and since $\kappa \circ i \simeq \mathrm{id}$, the proof is complete. □

Maps into nerves

Information about the structure of a space can be obtained by studying the nature of polyhedra "approximating" the space in a sense we now make precise.

(7.10) DEFINITION. Let X be a topological space and $A = \{A_\alpha \mid \alpha \in \mathscr{A}\}$ any indexed family of subsets of X. Let K be the abstract complex with the set $\mathscr{A}$ as vertices and with $\{\alpha_0, \ldots, \alpha_n\} \in \mathscr{F}$ if and only if $A_{\alpha_0} \cap \cdots \cap A_{\alpha_n} \neq \emptyset$. The standard realization of K is called the *nerve* of the family $\{A_\alpha \mid \alpha \in \mathscr{A}\}$ and is denoted by $N(A)$.

The intuitive idea is that the nerves of open coverings of a space X are polyhedra approximating the space, with a finer covering giving rise to a "better" approximation. Note, in particular, that if K is a polyhedron, then because of (∗), the open covering by the stars of its vertices has the same vertex scheme as K itself, so by (7.6), the nerve of this covering is homeomorphic to K.

For paracompact spaces, the relation of the space to the nerves of its open covers can be given in more concrete terms:

(7.11) THEOREM. *Let $U = \{U_\alpha \mid \alpha \in \mathscr{A}\}$ be an open covering of a paracompact space Y. Then there exists a continuous map $\kappa : Y \to N(U)$ such that $\kappa^{-1}(\operatorname{St}\alpha) \subset U_\alpha$ for each vertex α of $N(U)$.*

PROOF. Let $\{\lambda_\alpha \mid \alpha \in \mathscr{A}\}$ be a partition of unity subordinate to the covering $\{U_\alpha \mid \alpha \in \mathscr{A}\}$, and let

$$\kappa(y) = \sum \lambda_\alpha(y)\alpha.$$

Then κ is a map of Y into $N(U)$: for if $\lambda_{\alpha_0}, \ldots, \lambda_{\alpha_n}$ are all the (finitely many) functions that do not vanish at a given $y \in Y$, then $y \in \bigcap \operatorname{supp} \lambda_{\alpha_i} \subset \bigcap U_{\alpha_i}$, so that $(\alpha_0, \ldots, \alpha_n)$ is a simplex of $N(U)$, and y and $\kappa(y)$ belong to that simplex. The map κ is continuous at each $y_0 \in Y$: for y_0 has a nbd V meeting at most finitely many sets $\operatorname{supp}\lambda_\alpha$, so that κ maps V into a finite subpolyhedron L of $N(U)$; since L is simplicially homeomorphic to a polyhedron in some $\boldsymbol{R}^s$, and since addition is continuous on $\boldsymbol{R}^s$, the map κ is continuous on V, and therefore at y_0. Finally, since $\kappa(y) \in \operatorname{St}\alpha$ if and only if $\lambda_\alpha(y) > 0$, we find that $\kappa^{-1}(\operatorname{St}\alpha) \subset U_\alpha$. □

These simple maps are called the *standard maps* of Y into $N(U)$; they have the universal property that any continuous map of Y into any polytope factors up to homotopy through a standard map. Precisely, we have

(7.12) THEOREM. *Let Y be paracompact, and let $h : Y \to P$ be a continuous map into some polytope. Then h can be homotopy factored through a standard map into the nerve of an open covering. In fact, if $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ is any open refinement of the open cover $\{h^{-1}(\operatorname{St}p) \mid p \in P^0\}$, then $h \simeq \mu \circ \kappa$, where $\kappa : Y \to N(U)$ is the standard map and $\mu : N(U) \to P$ is a simplicial map.*

PROOF. For brevity, set $N = N(U)$. If $\{\lambda_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$, the standard map $\kappa : Y \to N$ is given by $\kappa(y) = \sum \lambda_\alpha(y)\alpha$.

To construct μ, for each U_α select an $h^{-1}(\operatorname{St}p)$ such that $U_\alpha \subset h^{-1}(\operatorname{St}p)$ and define $\mu : N^0 \to P^0$ by $\mu(\alpha) = p$. Note that if $(\alpha_0, \ldots, \alpha_n)$ is a simplex of N, then $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \emptyset$, so $\emptyset \neq \bigcap_i h^{-1}(\operatorname{St}\mu(\alpha_i)) = h^{-1}(\bigcap_i \operatorname{St}\mu(\alpha_i))$, and hence $\bigcap_i \operatorname{St}\mu(\alpha_i) \neq \emptyset$, which implies that $(\mu(\alpha_0), \ldots, \mu(\alpha_n))$ is a simplex of P The linear extension over each simplex defined by

$$\mu\Bigl(\sum \xi_\alpha \alpha\Bigr) = \sum \xi_\alpha \mu(\alpha)$$

therefore determines a simplicial map $\mu : N \to P$.

We now have $\mu \circ \kappa(y) = \sum_\alpha \lambda_\alpha(y)\mu(\alpha)$. This expression shows that if y belongs (only) to $\operatorname{supp}\lambda_{\alpha_i}$ $(i = 0, \ldots, n)$, then $\mu \circ \kappa(y)$ belongs to $(\mu(\alpha_0), \ldots, \mu(\alpha_n))$; and since $y \in \bigcap_{i=0}^n U_{\alpha_i} \subset h^{-1}(\bigcap_{i=0}^n \operatorname{St}\mu(\alpha_i))$, the point $h(y)$ lies in some simplex having $\mu(\alpha_0), \ldots, \mu(\alpha_n)$ among its vertices. Thus,

$\mu \circ \kappa$ and h are contiguous as maps of Y into P, so by (7.9) they are homotopic. □

Applications

(A) *Let Y be paracompact with covering dimension* $\dim Y \leq n$. *Then any continuous map $f : Y \to P$, where P is any polytope, is homotopic to an $f' : Y \to P^n \subset P$.*

PROOF. Since $\{h^{-1}(\operatorname{St} p)\}$ has a refinement $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ of order $\leq n$, it follows that $\dim N(U) \leq n$. Now $f \simeq \mu \circ \kappa$, where μ is simplicial; and a simplicial map cannot increase dimension. Thus, μ maps into P^n, and the proof is complete. □

(B) *Let Y be paracompact with* $\dim Y < n$. *Then every $f : Y \to S^n$ is homotopic to a constant.*

PROOF. Regard S^n is a polytope. By (A), f can be deformed into $(S^n)^k$ for some $k < n$. Choosing a point ξ in the interior of an n-simplex and contracting $S^n - \{\xi\}$ along great circles to the antipode ξ^* completes the proof. □

(C) *Let Y be paracompact and $A \subset Y$ closed. If* $\dim A < n$, *then every continuous $f : A \to S^n$ can be extended over Y.*

PROOF. Since $\dim A < n$, we have $f \simeq 0$ by (B). But S^n is NES(paracompact), so by Borsuk's theorem, extension is a problem in the homotopy category. Since the constant map is extendable over Y, so also is f. □

Domination

Topological information is transferred by maps. We have seen that we can, to an extent, transfer information from a space to polytopes "relatively accurately" in the paracompact case. We now wish to transfer information from a polytope to a space; again the map used must be "accurate" in some sense—the constant map back, for example, says very little for an arbitrary space. In general, this cannot be done. But for ANRs it is indeed possible: specifically, for every ANR Y we can construct a polytope P and maps $Y \xrightarrow{\kappa} P \xrightarrow{g} Y$ such that $g \circ \kappa \simeq \mathrm{id}$. This leads to the expectation that the algebraic properties of ANRs will be analogous to those of polytopes.

(7.13) LEMMA. *Let Y be a locally convex set in a normed linear space. Then there is a polytope P and continuous $Y \xrightarrow{\kappa} P \xrightarrow{g} Y$ such that $g \circ \kappa \simeq \mathrm{id}$ (we say that Y is dominated by P).*

PROOF. Cover Y by convex open sets $\{C_\alpha \mid \alpha \in \mathscr{A}\}$; this has a nbd-finite refinement $\{W_\beta \mid \beta \in \mathscr{B}\}$ such that for each y, $\operatorname{St}(y, \{W_\beta\})$ is contained in

some C'_α (a barycentric refinement). Let N be the nerve of $\{W_\beta\}$, and let

$$\kappa(y) = \sum \lambda_\beta(y)\beta$$

be the standard map into the nerve.

We now construct $g : N \to Y$. In each W_β choose a point y_β. Define g as follows:

$$g\Big(\sum_\beta \xi_\beta \beta\Big) = \sum_\beta \xi_\beta y_\beta.$$

This g is indeed a map of N into Y: for if $(\beta_0, \dots, \beta_n)$ is a simplex of N, then $W_{\beta_0} \cap \dots \cap W_{\beta_n} \neq \emptyset$, so by the barycentric refinement property, $\bigcup_{i=0}^n W_{\beta_i}$ is contained in some C_α, and every linear combination of points of C_α lies in $C_\alpha \subset Y$ Moreover, g is continuous, since it is continuous on each simplex.

It remains to show that $g \circ \kappa \simeq \text{id}$. Fix $y \in Y$; then $\text{St}(y, \{W_\beta\})$ is contained in some C_α, and because

$$g \circ \kappa(y) = g\Big(\sum \lambda_\beta(y)\beta\Big) = \sum \lambda_\beta(y) y_\beta$$

is a linear combination of points in C_α, we find that $g \circ \kappa(y)$ also lies in C_α. Thus,

$$(y, t) \mapsto ty + (1 - t) g \circ \kappa(y)$$

provides a homotopy $Y \times I \to Y$ showing that $g \circ \kappa \simeq \text{id}$. □

(7.14) THEOREM. *Every ANR is dominated by a polytope.*

PROOF. Let Y be an ANR; embed it as a closed subset of a normed linear space L. Then there is a retraction $r : V \to Y$ of some open subset $V \subset L$; we can assume that V is a union of balls, all having their centers in Y, so in particular, V is locally convex. By (7.13), we have the maps

$$Y \xrightarrow{i} V \xrightarrow{\kappa} P \xrightarrow{g} V \xrightarrow{r} Y,$$

where i is inclusion and $g \circ \kappa \simeq \text{id}$. Therefore,

$$(r \circ g) \circ (\kappa \circ i) = r \circ (g \circ \kappa) \circ i \simeq r \circ i = \text{id}. \ \square$$

More precise information about the nature of this domination—in fact, all such dominations—can be obtained from the following

(7.15) LEMMA. *Let Y be a paracompact space dominated by a polytope P. Then each open cover $\{U_\alpha \mid \alpha \in \mathscr{A}\}$ of Y has a refinement $\{V_\beta \mid \beta \in \mathscr{B}\}$ such that Y is dominated by the nerve of $\{V_\beta\}$, as well as by the nerve of any refinement of $\{V_\beta\}$.*

PROOF. We are given $Y \xrightarrow{h} P \xrightarrow{g} Y$ with $g \circ h \simeq \text{id}$. Consider any refinement of $\{h^{-1}(\text{St}\, p) \cap U_\alpha\}$, say $\{V_\beta\}$. This $\{V_\beta\}$ refines $\{U_\alpha\}$; but it also refines

$\{h^{-1}(\mathrm{St}\,p)\}$, so by (7.12), we have a diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "\kappa"'] & P \arrow[r, "g"] & Y \\ N(V) \arrow[ur, "\mu"'] & & \end{tikzcd}$$

with $h \simeq \mu \circ \kappa$. Thus $(g \circ \mu) \circ \kappa = g \circ (\mu\kappa) \simeq g \circ h \simeq \mathrm{id}$. □

(7.16) THEOREM. *Let Y be a paracompact space dominated by a polytope. Then Y is dominated by the nerve P of a suitable open covering. Moreover:*

(a) *If Y is Lindelöf, then P can be assumed countable.*
(b) *If Y is compact, then P can be assumed finite.*
(c) *If $\dim Y \le n$, then P can be assumed of dimension $\le n$.*

PROOF. This is a direct consequence of (7.15). □

8. Miscellaneous Results and Examples

A. Neighborhood extensor spaces

In this subsection Q stands for any subclass of $\mathscr{N}$ = normal spaces; frequent ones will be: $\mathscr{P}$ = paracompact spaces, $\mathscr{K}$ = compact spaces, $\mathscr{M}$ = metric spaces, $\mathscr{K}\mathscr{M}$ = compact metric spaces. A space Y is called a *neighborhood extensor space* for the class Q [written: Y is a NES(Q)] if for each $X \in Q$ and every closed $A \subset X$, each continuous $f : A \to Y$ is extendable over some open nbd U of A.

(A.1) Establish the following properties of NES spaces:

(i) If $Q \subset Q'$, then NES(Q') $\subset$ NES(Q).
(ii) If Y is a NES(Q), then so is every open subset of Y
(iii) If Y is a NES(Q) and $B \subset Y$ is a retract of some open nbd in Y, then B is also a NES(Q).
(iv) A finite cartesian product $\prod Y_k$ is a NES(Q) if and only if each Y_k is a NES(Q).

(A.2) (*Kuratowski lemma*) Let Y be a space that can be represented as a union $Y = Y_1 \cup Y_2$, where Y_1, Y_2 and $Y_1 \cap Y_2$ are NES(Q) for some class $Q \subset \mathscr{N}$ Let $X \in Q$, $A \subset X$ closed, and let $f : A \to Y$. Prove: If the entire space X can be represented as the union $B_1 \cup B_2$ of two closed sets such that $f(A \cap B_1) \subset Y_1$, $f(A \cap B_2) \subset Y_2$, then f is extendable over a nbd of A in X.

[Follow the proof of (11.4.1).]

(A.3) Let Y be a space with $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are closed. Prove: If Y_1, Y_2, and $Y_1 \cap Y_2$ are NES($\mathscr{M}$), then Y is a NES($\mathscr{M}$).

[Let $X \in \mathscr{M}$, $A \subset X$ closed, and $f : A \to Y$ be given. Define

$$B_1 = \{x \mid \mathrm{dist}(x, f^{-1}(Y_1)) \le \mathrm{dist}(x, f^{-1}(Y_2))\},$$
$$B_2 = \{x \mid \mathrm{dist}(x, f^{-1}(Y_2)) \le \mathrm{dist}(x, f^{-1}(Y_1))\}.$$

Show that $X = B_1 \cup B_2$ with each B_i closed in X and $f^{-1}(Y_i) = A \cap B_i$ ($i = 1, 2$); and apply (A.2).]

(A.4) Let Y be any NES($\mathcal{M}$) and $B \subset Y$ a closed NES($\mathcal{M}$) subset. Show:

(i) $Y \times \{0\} \cup B \times I$ is a NES($\mathcal{M}$).

(ii) Any finite union of closed convex sets in a locally convex space is a NES($\mathcal{M}$).

(A.5) Let $Y = U_1 \cup U_2$, where U_1, U_2 are open. Show: If U_1, U_2 are NES(Q) for some $Q \subset \mathcal{M}$, then Y is a NES(Q).

[Let $X \in \mathcal{M}$, $A \subset X$ closed, and $f : A \to Y$ be given. Consider the disjoint closed sets $F_1 = A - f^{-1}(U_1)$, $F_2 = A - f^{-1}(U_2)$ in X and choose $\lambda : X \to I$ with $\lambda|F_1 = 1$, $\lambda|F_2 = 0$. Then for the closed sets $B_1 = \{x \in X \mid \lambda(x) \leq 1/2\}$, $B_2 = \{x \in X \mid \lambda(x) \geq 1/2\}$ with $X = B_1 \cup B_2$, show $f(A \cap B_i) \subset U_i$ ($i = 1, 2$). Apply the Kuratowski lemma (A.2).]

(A.6) (*Michael theorem*) Let Y be a topological space. A property $\mathcal{H}$ of subsets of Y (or equivalently a subset $\mathcal{H} \subset 2^Y$) is called *G-hereditary* whenever it satisfies the following three conditions:

(a) If $A \in \mathcal{H}$ and if $W \subset A$ is open in A, then $W \in \mathcal{H}$.

(b) If U, V are open, and if $U, V \in \mathcal{H}$, then $U \cup V \in \mathcal{H}$.

(c) If $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ is any family of pairwise disjoint open sets in Y, and if each U_α is in $\mathcal{H}$, then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{H}$.

Let Y be a metric space, and let $\mathcal{H} \subset 2^Y$ be a G-hereditary property. Prove: If each $y \in Y$ has an open nbd $U \in \mathcal{H}$, then $Y \in \mathcal{H}$. (This result, due to E.A. Michael, is also true for any paracompact space Y.)

[Let $\{U(y) \mid y \in Y\}$ be an open covering of Y with each $y \in U(y) \in \mathcal{H}$. Take a σ-discrete open refinement $\{U_{\alpha,n} \mid (\alpha, n) \in \mathcal{A} \times Z^+\}$ of $\{U(y) \mid y \in Y\}$, where for each fixed n, the family $\{U_{\alpha,n}\}$ is pairwise disjoint (cf. Appendix). Letting $V_n = \bigcup\{U_{\alpha,n} \mid \alpha \in \mathcal{A}\}$, conclude that $Y = \bigcup V_n$ with each $V_n \in \mathcal{H}$. Next assuming, without loss of generality, that $V_n \subset V_{n+1}$ and $Y \neq V_n$, for all n, shrink the covering $\{V_n\}$ to an open covering $\{W_n\}$, where $W_n = \{y \in Y \mid \operatorname{dist}(y, Y - V_n) > 1/2^n\}$ for $n \geq 1$. Then prove successively that each of the following sets belongs to $\mathcal{H}$:

(1) Each W_n, $n \geq 1$.

(2) Each $R_n = W_n - \overline{W}_{n-2}$, $n \geq 1$, where $W_i = \emptyset$ for $i \leq 0$.

(3) $Y = (\bigcup R_{2n}) \cup (\bigcup R_{2n-1})$.]

(A.7) (*Hanner theorem*) Let $Q \subset \mathcal{P}$. Prove: A metric space Y is a NES(Q) if and only if Y is locally a NES(Q). (This result, due to Hanner [1952], is also true for any paracompact space Y.)

(A.8) Show: Every ANR is a NES(compact).

B. Relative Lefschetz theorem for NES(*compact*) *spaces*

Throughout this subsection we use the notation and terminology of (15.3.1). Let (X, A) be a pair of spaces and $f : (X, A) \to (X, A)$ a map. Then:

(i) f is a *Lefschetz map* if the Lefschetz numbers $\Lambda(f)$, $\Lambda(f_X)$ and $\Lambda(f_A)$ are defined,

(ii) f is *compact* if f_X and f_A are compact,

(iii) f is *partially extendable* if f is compact and f_A extends to an $\widehat{f}_A \in \mathcal{K}(U, A)$, where $U \subset X$ is open and $A \subset U$. (In what follows, given such a partially extendable f, we always write $i : A \to U$ for the inclusion, so that $f_A = \widehat{f}_A \circ i$.)

(B.1) Let (X, A), (Y, B) be two pairs and assume that the diagrams

$$\begin{tikzcd} & (Y,B) \arrow[dl, "\alpha"'] \arrow[d, "f"] \\ (X,A) \arrow[r, "\beta"] & (Y,B) \end{tikzcd} \qquad \begin{tikzcd} & (X,A) \arrow[dl, "\beta"'] \arrow[d, "F"] \\ (Y,B) \arrow[r, "\alpha"] & (X,A) \end{tikzcd}$$

are commutative. Show:

(a) If there exists a map $r : Y \to X$ sending homeomorphically $\mathrm{Fix}(\beta\alpha)$ onto $\mathrm{Fix}(\alpha\beta)$ and $r^{-1}(A) = B$, then

$$r[\mathrm{Fix}(f) \cap \overline{Y - B}] \subset \mathrm{Fix}(F) \cap \overline{X - A}.$$

(b) If $\alpha^{-1}(A) = B$ and $\beta^{-1}(B) = A$, then the sets $\mathrm{Fix}(f) \cap \overline{Y - B}$ and $\mathrm{Fix}(F) \cap \overline{X - A}$ are homeomorphic.

(B.2) (*Allowable pairs*) By an *allowable pair* is meant a pair (X, A) such that each partially extendable $f : (X, A) \to (X, A)$ is a Lefschetz map, and $\Lambda(f) \neq 0$ implies that f has a fixed point in $\overline{X - A}$. Prove: If $r : (Y, B) \to (X, A)$ is a retraction, $B = r^{-1}(A)$ and (Y, B) is an allowable pair, then so also is (X, A).

[Take a partially extendable $f \quad (X, A) \to (X, A)$ with extension $\widehat{f}_A \in \mathscr{K}(U, A)$. Consider the commutative diagram

$$\begin{tikzcd} (X,A) \arrow[r, hook, "\beta"] \arrow[d, "f"'] & (Y,B) \arrow[dl, "\alpha = fr"'] \arrow[d, "\beta f r = F"] \\ (X,A) \arrow[r, hook, "\beta"] & (Y,B) \end{tikzcd}$$

and observe that $\beta\alpha = \beta f r$ is partially extendable (since $r^{-1}(U) \supset B$ is open in Y and $\beta \widehat{f}_A r : r^{-1}(U) \to B$ extends $\beta f r|B \quad B \to B$). By (15.2.2) and the assumption, $\Lambda(f) = \Lambda(\alpha\beta) = \Lambda(\beta\alpha)$. If $\Lambda(f) \neq 0$, then because $\beta\alpha$ is partially extendable, $\mathrm{Fix}(\beta\alpha) \cap \overline{Y - B} \neq \emptyset$. Using (B.1)(a), conclude that $\mathrm{Fix}(\alpha\beta) \cap \overline{X - A} \neq \emptyset$.]

(B.3) Let X be an open subset of a locally convex space E, and let $A \subset X$ be arbitrary. Show: (X, A) is allowable.

[Let $f : (X, A) \to (X, A)$ be partially extendable with extension $\widehat{f}_A \in \mathscr{K}(U, A)$; using the index I for compactly fixed maps in E, observe that (i) $I(i\widehat{f}_A, U) = \Lambda(i\widehat{f}_A) = \Lambda(\widehat{f}_A i) = \Lambda(f_A)$, and (ii) $\Lambda(f) = \Lambda(f_X) - \Lambda(f_A)$, because $\Lambda(f_A)$, $\Lambda(f_X)$ are defined, and so also is $\Lambda(f)$ by (15.2.3). To conclude, assume $\mathrm{Fix}(f) \cap \overline{X - A} = \emptyset$; then $I(f_X, X) = I(i\widehat{f}_A, U)$ by excision and $\Lambda(f) = I(f_X, X) - I(i\widehat{f}_A, U) = 0$, by (i), (ii), and strong normalization.]

(B.4) Let X be an open subset of a Tychonoff cube $I^{\aleph}$, and let $A \subset X$ be arbitrary. Show: (X, A) is allowable.

[Prove that there exists a retraction $r : (Y, B) \to (X, A)$, where Y is open in a locally convex space, $B = r^{-1}(A)$, and then apply (B.3) and (B.2).]

(B.5) Let X be a NES(compact) space, and let $A \subset X$ be arbitrary. Show: (X, A) is allowable.

[Given a partially extendable map $f : (X, A) \to (X, A)$ with extension $\widehat{f}_A \in \mathscr{K}(U, A)$, let (K, L) be a compact pair such that f factorizes as $(X, A) \xrightarrow{\widehat{f}} (K, L) \xhookrightarrow{j} (X, A)$ and L

contains $\hat{f}_A(U)$. Embed K into a Tychonoff cube $I^{\aleph}$ and consider the diagram

$$\begin{tikzcd}
(K,L) \arrow[ddr, hook, "j"'] & & \\
& (Y,B) \arrow[d, shift right, "\alpha"'] & (\hat{K},\hat{L}) \arrow[l, hook', "s"'] \arrow[ull, "h^{-1}", "\approx"'] \\
& (X,A) \arrow[u, shift right, "\beta"'] \arrow[r, "\hat{f}"] & (K,L) \arrow[u, "\approx", "h"']
\end{tikzcd}$$

where: (i) h is a homeomorphism; (ii) j, s are inclusions and α is an extension of jh^{-1} $\hat{K} \to X$ over Y open in $I^{\aleph}$ (existing since $X \in$ NES(compact)); and (iii) $B = \alpha^{-1}(A)$ and $\beta = sh\hat{f}$. Observe that

$$\alpha\beta = \alpha(sh\hat{f}) = jh^{-1}h\hat{f} = j\hat{f} = f$$

and that $h\hat{f}_A\alpha \in \mathscr{K}(\alpha^{-1}(U), B)$ extends $(\beta\alpha)|B$ $B \to B$ over an open nbd $\alpha^{-1}(U)$ of B in Y Thus, $\beta\alpha : (Y,B) \to (Y,B)$ is partially extendable, and since Y is open in a Tychonoff cube, we have $\Lambda(\beta\alpha) = \Lambda(\alpha\beta) = \Lambda(f)$ by (B.4) and (15.2.2); together with (15.2.3), this implies that f is a Lefschetz map, since $\Lambda(f_X)$ is defined. Assuming $\Lambda(f) = \Lambda(\beta\alpha) \neq 0$, we obtain $\mathrm{Fix}(\beta\alpha) \cap \overline{Y - B} \neq \emptyset$ by (B.4), and (since $\alpha^{-1}(A) = B$) the desired conclusion follows from (B.1)(a) with $r = \alpha$.]

(B.6) (*Lefschetz pairs*) By a *Lefschetz pair* is meant a pair (X,A) such that each compact $f : (X,A) \to (X,A)$ is a Lefschetz map, and $\Lambda(f) \neq 0$ implies that f has a fixed point in $\overline{X - A}$. Prove: If $r : (Y,B) \to (X,A)$ is a retraction, $r^{-1}(A) = B$, and (Y,B) is a Lefschetz pair, then so also is (X,A).

(B.7) Let E be a locally convex space and (Y,B) a pair of open sets in E. Show: (Y,B) is a Lefschetz pair.

(B.8) Let (X,A) be a pair of NES(compact) spaces. Show: (X,A) is a Lefschetz pair.

[Let f $(X,A) \to (X,A)$ be compact; choose a compact pair (K,L) such that $(f(X), f(A)) \subset (K,L) \subset (X,A)$ and an extension $\hat{j}$ $W \to A$ of j $L \hookrightarrow A$ over an open nbd $W \supset L$ in K, which exists because $A \in$ NES(compact). Letting $V \subset K$ be open with $L \subset V \subset \overline{V} \subset W$ and $U = f^{-1}(V)$, observe that $\hat{f}_A = \hat{j} \circ f \cdot U \to A$ is a compact extension of f_A over U, since $\hat{f}_A(U) \subset \hat{j}(\overline{V})$ and $\hat{j}(\overline{V})$ is compact. This shows that f is partially extendable, and the conclusion follows from (B.5).]

(B.9) Let X be a NES(compact) space, and let $A \subset X$ be open. Show: (X,A) is a Lefschetz pair.

(B.10) Let (X,A) be a pair of metric NES(compact metric) spaces. Show: (X,A) is a Lefschetz pair.

(The above results are due to C. Bowszyc.)

C. Lefschetz-type results for self-maps of compact spaces

This group of problems is concerned with certain general classes of maps of compact spaces for which the Lefschetz-type theorem holds.

(C.1) ($\mathcal{NES}$(*compact*)-*maps*) We say that $f : X \to Y$ is a $\mathcal{NES}$(*compact*)-*map* if for any compact pair (Z,A) and any g $A \to X$ there exists a nbd U of A in Z and a

map $\varphi : U \to Y$ such that $\varphi|A = fg$; the class of all $\mathcal{NES}$(compact)-maps is denoted by $\mathcal{NES}$(compact). Prove:

(i) $X \in$ NES(compact) $\Leftrightarrow 1_X \in \mathcal{NES}$(compact).

(ii) $g \circ f \in \mathcal{NES}$(compact) whenever either f or g is a $\mathcal{NES}$(compact)-map.

(iii) If either X or Y is a NES(compact) space, then every $f : X \to Y$ is a $\mathcal{NES}$(compact)-map.

(C.2) Let K be a compact space and $f : K \to K$ a $\mathcal{NES}$(compact)-map. Show: f is strongly Lefschetz.

[Embed K into a Tychonoff cube $I^{\aleph}$, and then prove that f factors through an open subset of $I^{\aleph}$ Because U is a Lefschetz space, conclude using (15.3.5).]

(C.3) (*Approximate $\mathcal{NES}$(compact)-maps*) We say that $f : X \to Y$ is an *approximate $\mathcal{NES}$(compact)-map* if for any compact pair (Z, A) and any $g : A \to X$, the following condition is satisfied: for each $\alpha \in \mathrm{Cov}(Y)$ there exists a nbd U_α of A in Z and a map $\varphi_\alpha : U_\alpha \to Y$ such that $\varphi_\alpha|A, fg : A \to Y$ are α-homotopic; the class of all approximate $\mathcal{NES}$(compact)-maps is denoted by $\mathcal{ANES}$(compact). Prove:

(i) $X \in$ ANES(compact) $\Leftrightarrow 1_X \in \mathcal{ANES}$(compact).

(ii) $g \circ f \in \mathcal{ANES}$(compact) whenever either f or g is an $\mathcal{ANES}$(compact)-map.

(iii) If either X or Y is an ANES(compact) space, then every f $X \to Y$ is an $\mathcal{ANES}$(compact)-map.

(C.4) Let K be a compact space and $f : K \to K$ an $\mathcal{ANES}$(compact)-map. Show: f is strongly Lefschetz.

[Show that f admits an approximating family $\{f_\alpha\}_{\alpha \in \mathrm{Cov}(K)}$ (cf. (3.3)) such that each f_α factors through an open subset of a Tychonoff cube. Conclude the proof using (3.4).]

(C.5) Show: Theorems (5.3) and (5.6) are special cases of (C.4).

(The above results are due to Gauthier–Granas [2003].)

D. Fixed point classes and the Nielsen number

Throughout this subsection (X, d) is an ANR and $f : X \to X$ is a compact map.

Given $x_0, x_1 \in \mathrm{Fix}(f)$, we say that x_0 and x_1 are *f-equivalent* (and we write $x_0 \sim_f x_1$) if there is a path $p = \{x_t\}_{t \in I}$ from x_0 to x_1 that is homotopic to the path $f \circ p = \{f(x_t))\}_{t \in I}$ by a homotopy keeping x_0 and x_1 fixed. The relation $\sim_f$ is an equivalence relation on $\mathrm{Fix}(f)$, and its equivalence classes are called *fixed point classes* of f.

(D.1) Given a homotopy $h_t : X \to X$, we let $\widehat{h}(x,t) = (h_t(x), t)$ for $(x,t) \in X \times I$. Thus $\widehat{h} : X \times I \to X \times I$. Prove: (x,t) and (y,t) are in $\mathrm{Fix}(\widehat{h})$ and are $\widehat{h}$-equivalent if and only if x and y are in $\mathrm{Fix}(h_t)$ and are h_t-equivalent.

(D.2) Let $x \in \mathrm{Fix}(f)$, and let $\Phi(x)$ be the fixed point class of f containing x. Prove: $\Phi(x)$ is open in $\mathrm{Fix}(f)$.

[Use the Arens–Eells embedding and show that there exists a $\delta = \delta(x) > 0$ such that whenever $x' \in \mathrm{Fix}(f)$ and $d(x, x') < \delta$, then $x \sim_f x'$.]

(D.3) Prove: f has a finite number of compact fixed point classes.

(D.4) Let Φ be a fixed point class of f. Show:

(a) There exists an open subset $U \subset X$ such that $\Phi \subset U$ and $U \cap \mathrm{Fix}(f) = \Phi$.

(b) The index $i(\Phi)$ of Φ defined by $i(\Phi) = I(f, U)$ is independent of the choice of U.

[For (a), use (D.2); for (b), use the excision property of the index.]

(D.5) Call a fixed point class Φ of f *essential* if $i(\Phi) \neq 0$. The *Nielsen number* $N(f)$ of f is defined to be the number of essential fixed point classes of f. Prove: f has at least $N(f)$ fixed points.

(D.6) Let $h : X \times I \to X$ be a compact homotopy and $\widehat{\Phi}$ a fixed point class of the compact map $\widehat{h} : X \times I \to X \times I$; denote by $\widehat{\Phi}_t = \{x \mid (x, t) \in \widehat{\Phi}\}$ the t-slice of $\widehat{\Phi}$ for $t \in I$. For each given $t \in I$, prove:

(a) $\widehat{\Phi}_t$ is either empty or a fixed point class of $h_t : X \to X$,

(b) if Φ is a fixed point class of h_t, then it is the t-slice of a (unique) fixed point class $\widehat{\Phi}$ of $\widehat{h}$.

(D.7) With the notation of (D.6), prove: $i(\widehat{\Phi}_0) = i(\widehat{\Phi}_1)$, where $i(\widehat{\Phi}_t) = 0$ if $\widehat{\Phi}_t = \emptyset$.

[(i) Fix $r \in I$. Choose an open $U \subset X \times I$ with $\widehat{\Phi} \subset U$, $\overline{U} \cap \operatorname{Fix}(\widehat{h}) = \widehat{\Phi}$ and observe that $\widehat{\Phi}_r \subset U_r$, $\overline{U}_r \cap \operatorname{Fix}(h_r) = \widehat{\Phi}_r$ and $i(\widehat{\Phi}_r) = i(h_r|\overline{U}_r, U_r)$.

(ii) Use the general homotopy invariance (12.6.3).]

(D.8) Let $h : X \times I \to X$ be a compact homotopy. Prove: $N(h_0) = N(h_1)$

[Observe, using (D.6) and (D.7), that if a given fixed point class of h_0 is the 0-slice of a fixed point class $\widehat{\Phi}$ of $\widehat{h}$, then $i(\widehat{\Phi}_0) = i(\widehat{\Phi}_1)$; this implies $N(h_0) \leq N(h_1)$. A similar argument gives the reverse inequality.]

(The above results are due to R.F. Brown [1969].)

9. Notes and Comments

Fixed points for $\mathscr{F}^*$-maps of ANRs

Let X be a space, $U \subset X$ open, and $f : U \to X$ a map. We say that f is an *$\mathscr{F}^*$-map* if f is compactly fixed and for some nbd $V \subset U$ of $\operatorname{Fix}(f)$ the restriction $f|V : V \to X$ is compact. In a similar way, we define the notion of *$\mathscr{F}^*$-homotopy* and we let $\mathscr{F}^*(U, X)$ denote the set of all $\mathscr{F}^*$-maps from U to X.

Let now X be an ANR, and let $U \subset X$ be open. Given a map $f \in \mathscr{F}^*(U, X)$, we define the index $\operatorname{Ind}(f)$ of f by

$$\operatorname{Ind}(f) = \operatorname{Ind}(f, U) = I(f|V, V),$$

where $V \subset U$ is a neighborhood of $\operatorname{Fix}(f)$ such that $f|V$ is compact; by the excision property of I, $\operatorname{Ind}(f)$ is independent of the choice of V.

With this definition, we have the following generalization of (16.5.1) (cf. Granas [1972]):

THEOREM. *Let $\mathscr{F}^*$ be the class of maps defined by the condition: $f \in \mathscr{F}^*$ if and only if $f \in \mathscr{F}^*(U, X)$, where X is an ANR and $U \subset X$ is open. Then the index function* $\operatorname{Ind} : \mathscr{F}^* \to Z$ *assigning* $\operatorname{Ind}(f)$ *to each* $f \in \mathscr{F}^*(U, X)$ *has the following properties*:

(I) (Strong normalization) *If $U = X$ and $f : X \to X$ is a compactly absorbing map, then* $\Lambda(f) = \operatorname{Ind}(f, X)$.

(II) (Additivity) *For* $f \in \mathscr{F}^*(U,X)$ *and every pair of disjoint open* $V_1, V_2 \subset U$, *if* $\mathrm{Fix}(f) \subset V_1 \cup V_2$, *then*

$$\mathrm{Ind}(f,U) = \mathrm{Ind}(f,V_1) + \mathrm{Ind}(f,V_2).$$

(III) (Homotopy) *If* $h_t : U \to X$ *is an* $\mathscr{F}^*$*-homotopy, then*

$$\mathrm{Ind}(h_0,U) = \mathrm{Ind}(h_1,U).$$

(IV) (Commutativity) *Let* X, Y *be ANRs, let* $U \subset X$, $V \subset Y$ *be open, and let* $f : V \to X$, $g : U \to Y$ *be two maps. Assume that the set*

$$\mathrm{Fix}(gf) = \{x \in f^{-1}(U) \mid x = gf(x)\} \subset V$$

is compact and that for some neighborhood $W \subset V$ *of* $\mathrm{Fix}(gf)$ *the restriction* $f|W : W \to X$ *is compact. Then:*

(i) $\mathrm{Fix}(fg) = \{y \in g^{-1}(V) \mid y = fg(y)\}$ *is compact,*
(ii) *both* $gf : f^{-1}(U) \to Y$ *and* $fg : g^{-1}(V) \to X$ *are* $\mathscr{F}^*$*-maps,*
(iii) $\mathrm{Ind}(gf, f^{-1}(U)) = \mathrm{Ind}(fg, g^{-1}(V))$.

For various extensions of this theorem to more general classes of maps see Nussbaum [1977], [1993], and also his lecture notes [1985].

Various extensions of the Lefschetz theorem

The presentation of the results in Section 2 is close to that of Gauthier–Granas [2003]; for more general related results, see "Miscellaneous Results and Examples" (subsection B). For (3.1) and (3.2), see Dugundji's book [1965], p. 414, and also Fournier–Granas [1973]. Theorem (3.6) is due to Jaworowski–Powers [1969]. General results of the type of those in Sections 4 and 5 can be found in Granas [1967], Fournier–Granas [1973], and the lecture notes of Granas [1980]. For (4.3) and (4.5), see Granas [1967] and Fournier–Granas [1973]. Lemma (4.4) is due to J.H. Michael [1957]. Extensions of the Lefschetz theorem to various classes of NES-spaces, in particular Theorem (5.3), are due to Fournier–Granas [1973]. We remark that it is not known whether all NES(metric) spaces are Lefschetz spaces; for some partial results see Fournier–Granas [1973]. Approximate NES(compact) spaces and Theorem (5.6) are discussed in the lecture notes of Granas [1975].

A good exposition of Lefschetz-type results in the context of Čech homology is given in Gauthier [1983]; in particular, the reader will find there Lefschetz theorems for approximate ANRs in the sense of Noguchi [1953] and Clapp [1971]. Some further extensions of the Lefschetz theorem, due to K. Borsuk and J. Dugundji, will be given in §20.

Cauty [2002] established recently that any compact equiconnected space has the fixed point property (and more generally, any compact locally equiconnected space is a Lefschetz space).

Fixed points for differentiable maps

We now give a few examples of results showing that frequently one gets more refined fixed point assertions by assuming that the map f is continuously differentiable.

THEOREM (Cartan [1986]). *If M is a C^n Banach manifold and $f : M \to M$ is a C^n map such that $f \circ f = f$, then* Fix(f) *is a submanifold of M.*

THEOREM (Eells–Fournier [1975]). *Let M be a C^1 Banach manifold and $f : M \to M$ a C^1 map such that $f^n : M \to M$ is compact for some $n \geq 2$. Then $\Lambda(f)$ is defined and $\Lambda(f) \neq 0$ implies that f has a fixed point* ([1]).

For similar and more general results the reader is referred to Eells–Fournier [1976] and Nussbaum [1977].

Degree for self-maps of manifolds and topological entropy

Let (X, d) be a compact metric space and $f : X \to X$ a map. Define a sequence $\{d^f_n\}_{n=1}^{\infty}$ of metrics on X by $d^f_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y))$ and let $B_f(x, \varepsilon, n) = \{y \in X \mid d^f_n(x, y) < \varepsilon\}$. A set $A \subset X$ is called (n, ε)-*spanning* if $X = \bigcup_{a \in A} B_f(a, \varepsilon, n)$. We let $S_d(f, \varepsilon, n)$ denote the smallest cardinality of an (n, ε)-spanning set, and let

$$h_d(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log S_d(f, \varepsilon, n).$$

The quantity $h_d(f) = \lim_{\varepsilon \to 0} h_d(f, \varepsilon)$ (depending only on the topology of X) is the *topological entropy* of f and is denoted by $h_{\text{top}}(f)$; this invariant describes the dynamical complexity of a given map.

Let M be an n-dimensional compact connected orientable manifold without boundary, $f : M \to M$ a map, and $f_* : H_n(M; Z) \to H_n(M; Z)$ the induced homomorphism. Fix an orientation on M, i.e., an isomorphism $H_n(M; Z) \cong Z$. Then the *degree* $d(f)$ of f is $d(f) = f_*(1) \in Z$. When $M = S^n$, this coincides with the Brouwer degree described in §9.

THEOREM (Misiurewicz–Przytycki [1977]). *If M is a smooth compact connected orientable manifold and $f : M \to M$ is a C^1 map, then*

$$\log |d(f)| \leq h_{\text{top}}(f).$$

This result ensures that a map with $|d(f)| \geq 2$ is dynamically complex.

([1]) We remark that the following problem remains open: Let E be a Banach space and $f : E \to E$ a map such that f^n is compact for some $n \geq 2$. Does f have a fixed point?

VI.

Selected Topics

This chapter is concerned with a few selected topics related to the Leray–Schauder theory. It presupposes on the part of the reader some knowledge of the Čech homology and cohomology theory; occasionally, an important background theorem is included without proof, whenever such a theorem is needed.

§18. Finite-Codimensional Čech Cohomology

Let E be a fixed infinite-dimensional normed linear space. The *Leray-Schauder category* associated with E is the category $\mathfrak{L}_E$ having closed bounded subsets of E as objects and compact fields as morphisms. This paragraph is concerned with extending some basic notions and results of algebraic topology to the framework of the category $\mathfrak{L}_E$. The main method to be developed here is that of "finite-codimensional cohomology" We will show that with this new tool, one can efficiently handle certain geometric problems which previously seemed inaccessible by application of the Leray–Schauder degree theory.

Throughout the paragraph we assume that the Leray–Schauder category $\mathfrak{L} = \mathfrak{L}_E$ associated with the normed linear space E is equipped with the relation of homotopy of compact fields.

By a *finite-codimensional cohomology theory* (or simply a cohomology theory) $H^{\infty-*}$ on $\mathfrak{L}^2$ is meant a sequence $\{H^{\infty-n}(X,A)\}_{n=1}^{\infty}$ of contravariant functors from pairs (X, A) in $\mathfrak{L}^2$ to abelian groups together with a sequence of natural transformations $\delta^{\infty-n} : H^{\infty-n}(A) \to H^{\infty-n+1}(X, A)$ satisfying the following conditions:

(1) (*Homotopy*) If $f, g : (X, A) \to (Y, B)$ are compact fields homotopic in $\mathfrak{L}$, then

$$H^{\infty-n}(f) = H^{\infty-n}(g) \quad \text{for all } n.$$

(2) (*Exactness*) For each pair (X, A) in $\mathfrak{L}$, the sequence

$$\cdots \to H^{\infty-n}(X, A) \to H^{\infty-n}(X) \to H^{\infty-n}(A) \xrightarrow{\delta^{\infty-n}} H^{\infty-n+1}(X, A) \to \cdots$$

is exact, where all unmarked homomorphisms are induced by inclusions.

(3) (*Strong excision*) The inclusion $e : (A, A \cap B) \to (A \cup B, B)$ induces an isomorphism

$$H^{\infty-n}(e) : H^{\infty-n}(A \cup B, B) \cong H^{\infty-n}(A, A \cap B)$$

for all n.

The *coefficient group* of this theory is defined to be the graded group $\{H^{\infty-n}(S)\}_{n=1}^{\infty}$, where S is the unit sphere in E. In this paragraph we construct one such cohomology theory which corresponds to ordinary Čech cohomology for compact spaces and whose coefficients satisfy $H^{\infty-1}(S) = G$ and $H^{\infty-n}(S) = 0$ for $n \neq 1$.

1. Preliminaries

After some preliminary definitions, we first assemble the basic categories that will enter later on in our study of cohomology theories. The remaining part of the section is concerned with a review of Čech cohomology theory for compact spaces.

h-categories

(1.1) DEFINITION. An *h-category* $(\mathbf{K}, \sim)$ is a category $\mathbf{K}$ such that for each pair of objects A, B in $\mathbf{K}$, the morphism set $\mathbf{K}(A, B)$ is equipped with an equivalence relation $\sim$ (called *homotopy*) satisfying the following (compatibility) condition:

$$f_1 \sim f_2,\ g_1 \sim g_2 \ \Rightarrow\ g_1 f_1 \sim g_2 f_2$$

whenever the compositions are defined.

A morphism $f \in \mathbf{K}(A, B)$ is called *homotopy invertible* (or simply *h-invertible*) if there exists a $g \in \mathbf{K}(B, A)$ such that $gf \sim 1_A$ and $fg \sim 1_B$; we then say that the objects A and B are *homotopy equivalent* (or simply *h-equivalent*).

We say that an h-category $(\mathbf{L}, \approx)$ is an *h-subcategory* of $(\mathbf{K}, \sim)$ if $\mathbf{L}$ is a subcategory of $\mathbf{K}$ and $f \approx g$ in $\mathbf{L}$ implies $f \sim g$ in $\mathbf{K}$.

A functor λ from an h-category $(\mathbf{K}, \sim)$ to an h-category $(\mathbf{L}, \approx)$ is called *homotopy invariant* (or briefly an *h-functor*) if $f \sim g$ implies $\lambda(f) \approx \lambda(g)$.

REMARK. We always consider **Ab** as an h-category with the trivial homotopy relation $f \sim g \Leftrightarrow f = g$.

The categories $(\mathfrak{L}, \sim)$ *and* $(\mathfrak{L}_0, \approx)$

We shall use the terminology and notation pertaining to compact fields as in (6.7.1) and (6.7.3).

Recall that whenever a compact field between two subsets of E is denoted by a lowercase letter, as in $f : X \to Y$, the compact map $x \mapsto x - f(x)$ is denoted by the corresponding capital letter, i.e., $F(x) = x - f(x)$; we then call $F : X \to E$ the compact map associated with the given field. Compact fields are further described by the terminology used for compact maps: thus the field f is finite-dimensional if the associated compact map is finite-dimensional.

We now list the main h-categories that will appear in our study of the Čech cohomology theory $H^{\infty-*}$.

1° *The category of compact fields* $(\mathfrak{C}, \sim)$. This category has as its objects all subsets of E, and as its morphisms all compact fields ($\mathfrak{C}(X, Y) =$ the set of compact fields from X to Y); the relation $\sim$ is the homotopy of compact fields.

2° *The category* $(\mathfrak{C}_0, \approx)$. This category is obtained from $\mathfrak{C}$ by restricting the morphisms to be the finite-dimensional fields ($\mathfrak{C}_0(X, Y) =$ the set of finite-dimensional fields from X to Y), and $\approx$ is the relation of finite-dimensional homotopy between finite-dimensional fields. Thus $(\mathfrak{C}_0, \approx)$ is a dense (1) h-subcategory of $(\mathfrak{C}, \sim)$.

3° *The Leray–Schauder category* $(\mathfrak{L}, \sim)$. Here the objects are all closed bounded subsets of E, and the morphisms are all compact fields between the objects ($\mathfrak{L}(X, Y) =$ the set of all compact fields from X to Y); the relation $\sim$ is the homotopy of compact fields. Thus $(\mathfrak{L}, \sim)$, being obtained from $(\mathfrak{C}, \sim)$ by restricting the type of objects under consideration, is a full h-subcategory of $(\mathfrak{C}, \sim)$.

4° *The category* $(\mathfrak{L}_0, \approx)$. This category has as objects all closed bounded subsets of E and as morphisms all finite-dimensional fields ($\mathfrak{L}_0(X, Y) =$ the set of all finite-dimensional fields from X to Y); the relation $\approx$ is the finite-dimensional homotopy between morphisms. Thus, $(\mathfrak{L}_0, \approx)$ is a dense h-subcategory of $(\mathfrak{L}, \sim)$.

We remark that the h-categories described in 2°, 3°, and 4° are all h-subcategories of the category $(\mathfrak{C}, \sim)$.

---

(1) A subcategory $\mathbf{L}$ of $\mathbf{K}$ is called *dense* if it has the same objects as $\mathbf{K}$; and it is called *full* if it has the same morphisms as $\mathbf{K}$, i.e., $\mathbf{L}(X, Y) = \mathbf{K}(X, Y)$ for all X, Y in $\mathbf{L}$.

In what follows, when there is no risk of misunderstanding, we use the following abbreviations:

$$\mathfrak{C} = (\mathfrak{C}, \sim), \quad \mathfrak{C}_0 = (\mathfrak{C}_0, \approx), \quad \mathfrak{L} = (\mathfrak{L}, \sim), \quad \mathfrak{L}_0 = (\mathfrak{L}_0, \approx).$$

The Leray–Schauder category $(\mathfrak{L}, \sim)$ is of our primary interest: we shall be concerned with those properties of its objects that remain invariant under equivalences and homotopy equivalences in $\mathfrak{L}$.

The directed set $\mathcal{L}_E$ and the categories $(\mathfrak{L}_\alpha, \sim_\alpha)$

We denote by $\mathcal{L} = \mathcal{L}_E = \{L_\alpha, L_\beta, L_\gamma, \ldots\}$ the directed set of all finite-dimensional subspaces of E with the natural order relation $\preceq$ defined by $L_\alpha \preceq L_\beta \Leftrightarrow L_\alpha \subset L_\beta$. For notational convenience, we assume that there is a one-to-one correspondence $\alpha \leftrightarrow L_\alpha$ between the symbols $\alpha, \beta, \gamma, \ldots$ and subspaces $L_\alpha, L_\beta, L_\gamma, \ldots$, and in formulas to occur, we frequently replace one kind of symbol with the other; thus we write, for example, $\alpha \preceq \beta$ instead of $L_\alpha \preceq L_\beta$.

Given $\alpha \in \mathcal{L}$, we let $d(\alpha) = \dim L_\alpha$. If $X \subset E$ is nonempty, we denote by $\mathcal{L}_X$ the cofinal subset of $\mathcal{L}$ consisting of those $\alpha \in \mathcal{L}$ for which $X_\alpha = X \cap L_\alpha$ is nonempty. If $X, Y \subset E$ and $f : X \to Y$ is a map such that $f(X_\alpha) \subset Y_\alpha$, then we denote by $f_\alpha : X_\alpha \to Y_\alpha$ the contraction of f to the pair (X_α, Y_α).

(1.2) Definition. Let X and Y be subsets of E, and let $\alpha \in \mathcal{L}$.

(i) A field $f \in \mathfrak{C}_0(X, Y)$ is called an *α-field* if the finite-dimensional map F associated with f sends X into L_α.

(ii) Two α-fields $f, g : X \to Y$ are called *α-homotopic* (written $f \sim_\alpha g$) if there is a homotopy $h_t : X \to Y$ with $h_0 = f$, $h_1 = g$ such that

$$h_t(x) = x - H(x, t), \quad x \in X, \ t \in \boldsymbol{I},$$

where $H : X \times \boldsymbol{I} \to E$ is finite-dimensional and $H(X \times \boldsymbol{I}) \subset L_\alpha$.

(1.3) Proposition.

(a) *For each $\alpha \in \mathcal{L}$, there is an h-category $(\mathfrak{L}_\alpha, \sim_\alpha)$ having as objects all closed bounded subsets of E and as morphisms all α-fields ($\mathfrak{L}_\alpha(X, Y) =$ the set of all α-fields from X to Y); the relation $\sim_\alpha$ is the α-homotopy of α-fields.*

(b) *$(\mathfrak{L}_\alpha, \sim_\alpha)$ is a dense h-subcategory of $(\mathfrak{L}_0, \approx)$ for each $\alpha \in \mathcal{L}$, and if $\alpha \preceq \beta$ then $(\mathfrak{L}_\alpha, \sim_\alpha)$ is a dense h-subcategory of $(\mathfrak{L}_\beta, \sim_\beta)$.*

(c) *For any objects X and Y in $\mathfrak{L}_0$,*

$$\mathfrak{L}_0(X, Y) = \bigcup \{\mathfrak{L}_\alpha(X, Y) \mid \alpha \in \mathcal{L}\},$$

and given $f \in \mathfrak{L}_\alpha(X, Y)$, $g \in \mathfrak{L}_\beta(X, Y)$, we have

$$[f \approx g] \Rightarrow [f \sim_\gamma g \text{ for some } \gamma \succeq \alpha, \beta].$$

PROOF. The verification of these statements is straightforward and is left as an exercise to the reader. $\square$

Preliminaries on Čech cohomology

In what follows, our study is largely based on Čech cohomology for compact spaces. The details of construction of the Čech cohomology theory do not concern us here, but we shall describe explicitly its general properties and also some special results that will be needed.

Let $\mathbf{K}$ be the category of compact pairs and G an abelian group. The *Čech cohomology theory* on $\mathbf{K}$ is a sequence $H^* = \{H^n\}_{n \in Z}$ of cofunctors $H^n : \mathbf{K} \to \mathbf{Ab}$ together with a family of natural transformations

$$\delta^n : H^n(A) \to H^{n+1}(X, A),$$

one for each n, such that the following properties hold:

(1) (*Homotopy*) If the maps $f, g : (X, A) \to (Y, B)$ are homotopic, then $H^n(f) = H^n(g)$.

(2) (*Exactness*) If (X, A) is a compact pair, then the cohomology sequence

$$\cdots \to H^{n-1}(A) \xrightarrow{\delta^{n-1}} H^n(X, A) \to H^n(X) \to H^n(A) \xrightarrow{\delta^n} H^{n+1}(X, A) \to \cdots$$

of (X, A) is exact, where the unmarked homomorphisms are induced by inclusions.

(3) (*Strong excision*) If $(X; A, B)$ is a triad [1] and

$$e : (A, A \cap B) \hookrightarrow (X, B)$$

is the inclusion, then $H^n(e) : H^n(X, B) \cong H^n(A, A \cap B)$ for all n.

(4) (*Dimension*) If p_0 is a point, then

$$H^n(p_0) = \begin{cases} 0 & \text{for } n \neq 0, \\ G & \text{for } n = 0. \end{cases}$$

(5) (*Continuity*) If $\{(X_\alpha, A_\alpha)\}$ is a system of compact pairs in some space, downward directed by inclusion, and $X = \bigcap X_\alpha$, $A = \bigcap A_\alpha$, then the inclusion maps $i_\alpha : (X, A) \hookrightarrow (X_\alpha, A_\alpha)$ induce an isomorphism

$$H^n(X, A) \cong \varinjlim_{\alpha} \{H^n(X_\alpha, A_\alpha), i^*_{\alpha\beta}\},$$

where $i_{\alpha\beta} : (X_\beta, A_\beta) \hookrightarrow (X_\alpha, A_\alpha)$ is the inclusion for $\alpha \preceq \beta$.

---

[1] By a *triad* is meant an ordered triple $T = (X; X_1, X_2)$ of compact spaces such that $X = X_1 \cup X_2$, and by a *map* $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ *of triads* a map $f : X \to Y$ that sends X_i to Y_i for $i = 1, 2$.

Given a triad $(X; X_1, X_2)$ with $A = X_1 \cap X_2$, denote by

$$\begin{aligned} j_{01} &: A \hookrightarrow X_1, & j_{02} &: A \hookrightarrow X_2, \\ i_1 &: X_1 \hookrightarrow X, & i_2 &: X_2 \hookrightarrow X, \\ j_1 &: X \hookrightarrow (X, X_1), & j_2 &: X \hookrightarrow (X, X_2) \end{aligned}$$

the corresponding inclusions. The *Mayer Vietoris cohomology sequence* of the triad $(X; X_1, X_2)$ is the sequence of abelian groups

$$\cdots \to H^{n-1}(A) \xrightarrow{\Delta^{n-1}} H^n(X) \xrightarrow{\varphi} H^n(X_1) \oplus H^n(X_2) \xrightarrow{\psi} H^n(A) \to \cdots$$

in which φ and ψ are given by

$$\begin{aligned} \varphi(\gamma) &= (i_1^*(\gamma), i_2^*(\gamma)) & &\text{for } \gamma \in H^n(X), \\ \psi(\gamma_1 + \gamma_2) &= j_{01}^*(\gamma_1) - j_{02}^*(\gamma_2) & &\text{for } \gamma_i \in H^n(X_i)\ (i = 1, 2), \end{aligned}$$

and the *Mayer Vietoris homomorphism*

$$\Delta^{n-1} : H^{n-1}(A) \to H^n(X)$$

is defined by

$$\Delta^{n-1} = j_1^* \circ (k^*)^{-1} \circ \delta^{n-1},$$

where k^* is the isomorphism induced by the excision $k : (X_2, A) \hookrightarrow (X, X_1)$ and δ^{n-1} is the coboundary homomorphism of the pair (X_2, A). When there is no risk of confusion, we shall frequently omit the superscript n on Δ^n.

The *cohomology sequence of a triple* $B \subset A \subset X$ with inclusions

$$A \xrightarrow{k} (A, B) \xrightarrow{i} (X, B) \xrightarrow{j} (X, A)$$

is the sequence of abelian groups

$$\cdots \to H^{n-1}(A, B) \xrightarrow{\delta} H^n(X, A) \xrightarrow{j^*} H^n(X, B) \xrightarrow{i^*} H^n(A, B) \to \cdots$$

in which the coboundary homomorphism δ is defined as the composite

$$H^{n-1}(A, B) \xrightarrow{k^*} H^{n-1}(A) \xrightarrow{\delta^{n-1}} H^n(X, A).$$

The following results are deduced from the axioms by purely formal arguments.

(1.4) THEOREM. *The Mayer-Vietoris sequence of a triad is exact. If $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ is a map of triads, then f induces a homomorphism of the corresponding Mayer-Vietoris sequences.* □

(1.5) THEOREM. *The cohomology sequence of a triple is exact. If $f : (X, A, B) \to (X', A', B')$ is a map of triples, then f induces a homomorphism of the corresponding cohomology sequences.* □

Let $T_0 = (A; A_1, A_2)$ and $T = (X; X_1, X_2)$ be two triads. Then T_0 is said to be a *subtriad* of T, written $T_0 \subset T$, if $A \subset X$ and $A_i \subset X_i$ for $i = 1, 2$; T_0 is said to be a *proper subtriad* of T, written $T_0 \Subset T$, if $A_i = A \cap X_i$ for $i = 1, 2$.

If $T_0 \Subset T$, then clearly $T_0 \subset T$ and $A_0 = A_1 \cap A_2 = A \cap X_0$, where $X_0 = X_1 \cap X_2$; moreover, the inclusions

$$q : (X_0, A_0) \hookrightarrow (X_0 \cup A_2, A_2), \qquad k : (X_2, X_0 \cup A_2) \hookrightarrow (X, X_1 \cup A)$$

are excisions.

(1.6) DEFINITION. Given triads $(A; A_1, A_2) \Subset (X; X_1, X_2)$, we define the *relative Mayer–Vietoris homomorphism*

$$\Delta : H^{n-1}(X_0, A_0) \to H^n(X, A)$$

to be

$$\Delta = j^* \circ (k^*)^{-1} \circ \delta \circ (q^*)^{-1},$$

where

$$\delta : H^{n-1}(X_0 \cup A_2, A_2) \to H^n(X_2, X_0 \cup A_2)$$

is the coboundary homomorphism of the triple $(X_2, X_0 \cup A_2, A_2)$ and $j : (X, A) \hookrightarrow (X, X_1 \cup A)$ is the inclusion.

The following proposition is an immediate consequence of the definitions involved:

(1.7) PROPOSITION. *To a commutative diagram of triads*

$$\begin{tikzcd} (B; B_1, B_2) \arrow[r, phantom, "\Subset"] & (Y; Y_1, Y_2) \\ (A; A_1, A_2) \arrow[r, phantom, "\Subset"] \arrow[u, "g"] & (X; X_1, X_2) \arrow[u, "f"'] \end{tikzcd}$$

corresponds the following commutative diagram of abelian groups:

$$\begin{tikzcd} H^{n-1}(Y_0, B_0) \arrow[r, "\Delta"] \arrow[d, "f_0^*"'] & H^n(Y, B) \arrow[d, "f^*"] \\ H^{n-1}(X_0, A_0) \arrow[r, "\Delta"] & H^n(X, A) \end{tikzcd}$$

where $Y_0 = Y_1 \cap Y_2$ and $B_0 = B_1 \cap B_2$. □

Consecutive pairs of triads

Given a triad $T = (X; X_1, X_2)$, let $-T = (X; X_2, X_1)$ and denote by $\Delta^n(T)$, or simply $\Delta(T)$, the Mayer–Vietoris homomorphism

$$\Delta(T) : H^n(X_1 \cap X_2) \to H^{n+1}(X)$$

of the triad T. We note that

$$(*) \qquad \Delta(T) = -\Delta(-T).$$

(1.8) DEFINITION. Let $T_0 = (Y; Y_1, Y_2)$ and $T = (X; X_1, X_2)$ be two triads. The pair (T_0, T) is called a *consecutive pair of triads* if

$$Y_1 \cup Y_2 = Y = X_1 \cap X_2;$$

we then write $T_0 \Rightarrow T$ and say that the pair (T_0, T) *starts* at $Y_1 \cap Y_2$ and *ends* at $X_1 \cup X_2$.

Observe that for every consecutive pair of triads (T_0, T) we may form the composition

$$\Delta(T) \circ \Delta(T_0) = \Delta^{n+1}(T) \circ \Delta^n(T_0) : H^n(Y_1 \cap Y_2) \to H^{n+2}(X)$$

of the corresponding Mayer–Vietoris homomorphisms.

(1.9) LEMMA. *Assume that in the diagram*

$$\begin{tikzcd} T_0 \arrow[r, Rightarrow] & T \\ T_0' \arrow[r, Rightarrow] \arrow[u, hook] & T' \arrow[u, hook] \end{tikzcd}$$

the consecutive pairs

$$(T_0, T) = ((Y; Y_1, Y_2), (X; X_1, X_2)),$$
$$(T_0', T') = ((Y'; Y_1', Y_2'), (X'; X_1', X_2'))$$

both start at $Y_1 \cap Y_2 = Y_1' \cap Y_2'$ *and end at* $X = X'$. *Then*

$$\Delta(T) \circ \Delta(T_0) = \Delta(T') \circ \Delta(T_0').$$

PROOF. This is an immediate consequence of Theorem (1.4). □

(1.10) LEMMA. *Let* $((Y; Y_1, Y_2), (X; X_1, X_2))$ *and* $((Z; Z_1, Z_2), (X; W_1, W_2))$ *be two consecutive pairs of triads, both starting at*

$$Y \cap Z = Y_1 \cap Y_2 = Z_1 \cap Z_2$$

and ending at X. *Assume, moreover, that*

$$Z_i = Z \cap X_i \quad \text{and} \quad Y_i = Y \cap W_i \quad \text{for } i = 1, 2.$$

Then

$$\Delta(X; X_1, X_2) \circ \Delta(Y; Y_1, Y_2) = -\Delta(X; W_1, W_2) \circ \Delta(Z; Z_1, Z_2).$$

PROOF (¹). Consider the following triads:

$$
\begin{aligned}
T_1 &= (Y; Y_1, Y_2), & T_2 &= (X; X_1, X_2), \\
T_3 &= ((W_1 \cap X_2) \cup Y_2; W_1 \cap X_2, Y_2), & T_4 &= (X; X_1 \cup W_1, X_2), \\
T_5 &= (Y_2 \cup Z_2; Z_2, Y_2), & T_6 &= (X; X_1 \cup W_1, X_2 \cap W_2), \\
T_7 &= ((X_1 \cap W_2) \cup Z_2; X_1 \cap W_2, Z_2), & T_8 &= (X; X_1 \cup W_1, W_2), \\
T_9 &= (Z; Z_1, Z_2), & T_{10} &= (X; W_1, W_2).
\end{aligned}
$$

We claim that every pair (T_{2i-1}, T_{2i}) for $i = 1, 2, 3, 4, 5$ is a consecutive pair of triads starting at $Y \cap Z$ and ending at X.

For $i = 1$ and $i = 5$, this is true by assumption. Assume now that $i = 2$. Since $Y_2 = Y \cap W_2 \subset X_2$ and $Y_1 \subset W_1$, we have

$$
\begin{gathered}
(W_1 \cap X_2) \cap Y_2 = W_1 \cap Y_2 = W_1 \cap W_2 \cap Y = Z \cap Y, \\
(X_1 \cup W_1) \cap X_2 = (X_1 \cap X_2) \cup (W_1 \cap X_2) = Y \cup (W_1 \cap X_2) = Y_2 \cup (W_1 \cap X_2),
\end{gathered}
$$

proving the statement for $i = 2$. For $i = 4$, the proof is analogous.

Finally, for $i = 3$ we have

$$
\begin{gathered}
Z_2 \cap Y_2 = (Z \cap X_2) \cap (Y \cap W_2) = (Z \cap W_2) \cap (Y \cap X_2) = Z \cap Y, \\
(X_1 \cup W_1) \cup (X_2 \cap W_2) = (X_1 \cup W_1 \cup X_2) \cap (X_1 \cup W_1 \cup W_2) = X, \\
(X_1 \cup W_1) \cap (X_2 \cap W_2) = (X_1 \cap X_2 \cap W_2) \cup (W_1 \cap X_2 \cap W_2) \\
= (Y \cap W_2) \cup (Z \cap X_2) = Y_2 \cup Z_2,
\end{gathered}
$$

completing the proof of the claim. Now, by taking into account the inclusions

$$T_1, T_5 \subset T_3, \quad T_2, T_6 \subset T_4, \quad -T_5, T_9 \subset T_7, \quad T_6, T_{10} \subset T_8,$$

the various relations between the triads may be exhibited in the diagrams

$$
\begin{tikzcd}
T_5 \arrow[r, Rightarrow] \arrow[d, hook] & T_6 \arrow[d, hook] & -T_5 \arrow[r, Rightarrow] \arrow[d, hook] & T_6 \arrow[d, hook] \\
T_3 \arrow[r, Rightarrow] & T_4 & T_7 \arrow[r, Rightarrow] & T_8 \\
T_1 \arrow[r, Rightarrow] \arrow[u, hook] & T_2 \arrow[u, hook] & T_9 \arrow[r, Rightarrow] \arrow[u, hook] & T_{10} \arrow[u, hook]
\end{tikzcd}
$$

Letting $\Delta_i = \Delta(T_i)$ for $i = 1, \ldots, 10$, we apply Lemma (1.9) and property $(*)$ to obtain

$$\Delta_2 \Delta_1 = \Delta_4 \Delta_3 = \Delta_6 \Delta_5 = -\Delta_8 \Delta_7 = -\Delta_{10} \Delta_9,$$

and thus the proof of the lemma is complete. □

(¹) The reader may skip this proof on first reading and return to it when necessary for the proofs of consequences of (1.10)

2. Continuous Functors

The basic constructions of this paragraph depend essentially on the continuity property of the functors under consideration. This section develops the above property, and its main result is that every continuous h-cofunctor $\lambda_0 : \mathfrak{L}_0 \to \mathbf{Ab}$ defined on the subcategory $\mathfrak{L}_0$ of $\mathfrak{L}$ admits a unique extension over $\mathfrak{L}$.

Throughout this section we let λ_0 denote a homotopy invariant cofunctor from the category $\mathfrak{L}_0 = (\mathfrak{L}_0, \approx)$ to the category of abelian groups; $\mathfrak{L}_0$ being dense in $\mathfrak{L}$, we let

$$\lambda(X) = \lambda_0(X) \qquad \text{for } X \in \mathfrak{L}_0$$

and

$$f^* = \lambda_0(f) : \lambda(Y) \to \lambda(X) \qquad \text{for } f \in \mathfrak{L}_0(X, Y).$$

Approximating sequences

We now describe some technical tools that will be needed for the proof of the main result of this section.

Given an object Y in $\mathfrak{L}$ and $k \in N$, we let

$$Y^{(k)} = \{x \in E \mid d(x, Y) \leq 1/k\}.$$

To a sequence $\{X_k\}$ of objects we associate the *enlarged sequence* $\{\widetilde{X}_k\}$ by setting

$$\widetilde{X}_k = X_k^{(k)} = \{x \in E \mid d(x, X_k) \leq 1/k\}.$$

(2.1) DEFINITION. Let Y be an object of $\mathfrak{L}$. An *approximating sequence* $\{Y_n\}$ for Y is a descending sequence $Y_1 \supset Y_2 \supset \cdots$ of objects in $\mathfrak{L}$ such that $Y = \bigcap_{k=1}^{\infty} Y_k$.

Some useful properties of approximating sequences are collected in

(2.2) PROPOSITION. *Let $\{X_k\}$ and $\{Y_k\}$ be approximating sequences for X and Y, respectively, and let $f : X_1 \to E$ be a compact field. Then:*
  (i) *$\{X_k \cup Y_k\}$ is an approximating sequence for $X \cup Y$,*
  (ii) *$\{\widetilde{Y}_k\}$ is an approximating sequence for Y,*
  (iii) *for every $\alpha \in \mathcal{L}_Y$, $\{Y_k \cap L_\alpha\}_{k \in N}$ is an approximating sequence for Y_α,*
  (iv) *$\{f(X_k)\}$ is an approximating sequence for $f(X)$.*

PROOF. Properties (i)–(iii) are evident. To establish (iv), it is sufficient to show that $\bigcap_{k=1}^{\infty} f(X_k) \subset f(\bigcap_{k=1}^{\infty} X_k)$. Let $y \in \bigcap_{k=1}^{\infty} f(X_k)$; we have $y = x_k - F(x_k)$, where $x_k \in X_k$. Since F is compact, we may assume without loss of generality that $\lim_{k\to\infty} x_k = x$ and consequently $y = \lim_{k\to\infty} f(x_k) = f(x)$. Since $x \in \bigcap_{k=1}^{\infty} X_k$, this completes the proof. $\square$

We are now ready to formulate the definition of continuity for the cofunctor $\lambda_0 : (\mathfrak{L}_0, \approx) \to \mathbf{Ab}$.

Let Y be an object of $\mathfrak{L}$. Take an approximating sequence $\{Y_k\}$ for Y and consider the homomorphisms

$$i_{kn}^* = \lambda_0(i_{kn}) : \lambda(Y_k) \to \lambda(Y_n),$$
$$j_k^* = \lambda_0(j_k) : \lambda(Y_k) \to \lambda(Y),$$

induced by the inclusions $i_{kn} : Y_n \hookrightarrow Y_k$, $k < n$, and $j_k : Y \hookrightarrow Y_k$. Taking into account the obvious commutativity relations in $\mathbf{Ab}$,

$$i_{km}^* = i_{nm}^* \circ i_{kn}^* \qquad \text{for } k < n < m,$$
$$j_k^* = j_n^* \circ i_{kn}^* \qquad \text{for } k < n,$$

we see that $\{\lambda(Y_k), i_{kn}^*\}$ is a direct system of abelian groups and $\{j_k^* : \lambda(Y_k) \to \lambda(Y)\}$ is a direct family of homomorphisms.

(2.3) DEFINITION. A functor $\lambda_0 : \mathfrak{L}_0 \to \mathbf{Ab}$ is said to be *continuous* if for each object Y and any approximating sequence $\{Y_k\}$ for Y the direct limit

$$\varinjlim_k j_k^* : \varinjlim_k \{\lambda(Y_k), i_{kn}^*\} \to \lambda(Y)$$

is an isomorphism.

Approximating systems

The proof of the main theorem of this section requires some preparatory considerations concerning a suitable refinement of the approximation technique for compact fields.

(2.4) DEFINITION. Let $f \in \mathfrak{L}(X, Y)$ be a compact field. A sequence $\{Y_k, f_k\}$ of objects Y_k in $\mathfrak{L}$ and fields $f_k \in \mathfrak{L}_0(X, Y_k)$ is called an *approximating system* for f if:

(i) $\{Y_k\}$ is an approximating sequence for Y,
(ii) $f_k \sim j_k f$ in $\mathfrak{L}$, where $j_k : Y \hookrightarrow Y_k$ is the inclusion,
(iii) $f_k \approx i_{kn} f_n$ in $\mathfrak{L}_0$, where $i_{kn} : Y_n \hookrightarrow Y_k$ is the inclusion $(k \le n)$.

(2.5) PROPOSITION. *Let $f \in \mathfrak{L}(X, Y)$ be a compact field. Then for every $k \in N$ there is a field $f_k \in \mathfrak{L}_0(X, Y^{(k)})$ such that*

$$\|f(x) - f_k(x)\| \le 1/k \qquad \textit{for all } x \in X;$$

moreover, $\{Y^{(k)}, f_k\}$ is an approximating system for f. □

In what follows, any system $\{Y^{(k)}, f_k\}$ as in (2.5) will be called a *standard approximating system.*

In the rest of this section we assume that the cofunctor $\lambda_0 : \mathfrak{L}_0 \to \mathbf{Ab}$ is continuous.

Let $f \in \mathfrak{L}(X, Y)$ be a compact field and $\{Y_k, f_k\}$ an approximating system for f. In view of (2.4)(iii), every h-commutative diagram in $\mathfrak{L}_0$

$$\begin{tikzcd}
Y_k & & Y_n \arrow[ll, "i_{kn}"'] \\
& X \arrow[ul, "f_k"] \arrow[ur, "f_n"'] &
\end{tikzcd}$$

where $k \leq n$, yields a commutative diagram in **Ab**:

$$\begin{tikzcd}
\lambda(Y_k) \arrow[rr, "i_{kn}^*"] \arrow[dr, "f_k^*"'] & & \lambda(Y_n) \arrow[dl, "f_n^*"] \\
& \lambda(X) &
\end{tikzcd}$$

Consequently, $\{f_k^*\}$ is a direct sequence of homomorphisms, and therefore we have the map

$$\varinjlim_k f_k^* : \varinjlim_k \{\lambda(Y_k), i_{kn}^*\} \to \lambda(X).$$

(2.6) PROPOSITION. *Let $\{Y_k, f_k\}$, $\{\widehat{Y}_k, \widehat{f}_k\}$ be two approximating systems for a compact field $f \in \mathfrak{L}(X, Y)$, and let $j_k^* : \lambda(Y_k) \to \lambda(Y)$ and $\widehat{j}_k^* : \lambda(\widehat{Y}_k) \to \lambda(Y)$ be induced by inclusions. Then*

$$(*) \qquad (\varinjlim f_k^*) \circ (\varinjlim j_k^*)^{-1} = (\varinjlim \widehat{f}_k^*) \circ (\varinjlim \widehat{j}_k^*)^{-1}.$$

PROOF. The proof will be carried out in three steps.

STEP 1. Assume first that $Y_k \subset \widehat{Y}_k$ for each k, and $\widehat{f}_k = l_k \circ f_k$, where $l_k : Y_k \hookrightarrow \widehat{Y}_k$ is the inclusion. Since for each k the diagram

$$\begin{tikzcd}
& Y_k \arrow[dd, "l_k"] & \\
Y \arrow[ur, "j_k"] \arrow[dr, "\widehat{j}_k"'] & & X \arrow[ul, "f_k"'] \arrow[dl, "\widehat{f}_k"] \\
& \widehat{Y}_k &
\end{tikzcd}$$

commutes in $\mathfrak{L}_0$, its image under λ_0 in **Ab** is also commutative:

$$\begin{tikzcd}
& \lambda(Y_k) \arrow[dl, "j_k^*"'] \arrow[dr, "f_k^*"] & \\
\lambda(Y) & & \lambda(X) \\
& \lambda(\widehat{Y}_k) \arrow[uu, "l_k^*"] \arrow[ul, "\widehat{j}_k^*"] \arrow[ur, "\widehat{f}_k^*"'] &
\end{tikzcd}$$

By considering the corresponding commutative diagram in the category of direct systems of abelian groups and by applying to it the direct limit functor we obtain the following commutative diagram of abelian groups:

$$\begin{tikzcd}
& \varinjlim \lambda(Y_k) \arrow[dl, "\varinjlim j_k^*"'] \arrow[dr, "\varinjlim f_k^*"] & \\
\lambda(Y) & & \lambda(X) \\
& \varinjlim \lambda(\widehat{Y}_k) \arrow[ul, "\varinjlim \widehat{j}_k^*"] \arrow[ur, "\varinjlim \widehat{f}_k^*"'] \arrow[uu, "\varinjlim l_k^*"] &
\end{tikzcd}$$

Now by the continuity of λ_0 the two arrows on the left are isomorphisms, so

$$\varinjlim f_k^* = (\varinjlim \widehat{f}_k^*) \circ (\varinjlim \widehat{j}_k^*)^{-1} \circ \varinjlim j_k^*,$$

and this implies $(*)$.

Step 2. Next, we suppose that the systems $\{Y_k, f_k\}$, $\{Y_k, \widehat{f}_k\}$ have the same approximating sequence $\{Y_k\}$ for Y. Consider the enlarged sequence $\{W_k\}$, where $W_k = \widetilde{Y}_k$ for $k \in N$, and define $g_k, \widehat{g}_k \in \mathfrak{L}_0(X, W_k)$ by $g_k = l_k \circ f_k$, $\widehat{g}_k = l_k \circ \widehat{f}_k$, $k \in N$, where $l_k : Y_k \to W_k$ is the inclusion. We claim that

$(**)$ $\{W_k, g_k\}$, $\{W_k, \widehat{g}_k\}$ are approximating systems for f and

$$(\varinjlim g_k^*) \circ (\varinjlim (l_k j_k)^*)^{-1} = (\varinjlim \widehat{g}_k^*) \circ (\varinjlim (l_k j_k)^*)^{-1}.$$

Fix $k \in N$. In view of the homotopy relations $f_k \sim j_k f \sim \widehat{f}_k$ in $\mathfrak{L}$, we can find a homotopy $h_t^{(k)} : X \to Y_k$ in $\mathfrak{L}$ joining $f_k, \widehat{f}_k \in \mathfrak{L}(X, Y_k)$. Take a finite-dimensional homotopy $\widehat{h}_t^{(k)} : X \to E$ such that

$$\|h_t^{(k)}(x) - \widehat{h}_t^{(k)}(x)\| \le 1/k \qquad \text{for all } (x,t) \in X \times I.$$

Clearly, $\widehat{h}_t^{(k)}$ may be regarded as a homotopy $\widehat{h}_t^{(k)} : X \to W_k$ in $\mathfrak{L}_0$, and from the evident homotopy relations $g_k \approx \widehat{h}_0^{(k)}$, $\widehat{g}_k \approx \widehat{h}_1^{(k)}$ we get $g_k \approx \widehat{g}_k$. This implies $g_k^* = \widehat{g}_k^*$ for each $k \in N$, and thus $(**)$ follows.

Step 3. Finally, assume that $\{Y_k, f_k\}$, $\{\widehat{Y}_k, \widehat{f}_k\}$ are two arbitrary approximating systems for f. For each $k \in N$, set $W_k = (Y_k \cup \widehat{Y}_k)^{(k)}$ and define $g_k, \widehat{g}_k \in \mathfrak{L}_0(X, W_k)$ by $g_k = l_k \circ f_k$, $\widehat{g}_k = \widehat{l}_k \circ \widehat{f}_k$, where $l_k : Y_k \hookrightarrow W_k$, $\widehat{l}_k : \widehat{Y}_k \hookrightarrow W_k$ are the inclusions. In view of $(**)$, $\{W_k, g_k\}$, $\{W_k, \widehat{g}_k\}$ are both approximating systems for f, and because each of the pairs $\{W_k, g_k\}$,

$\{Y_k, f_k\}$ and $\{W_k, \widehat{g}_k\}$, $\{Y_k, \widehat{f}_k\}$ satisfies the assumptions of Step 1, we get

$$(\varinjlim f_k^*) \circ (\varinjlim j_k^*)^{-1} = (\varinjlim g_k^*) \circ (\varinjlim (l_k j_k)^*)^{-1},$$
$$(\varinjlim \widehat{f}_k^*) \circ (\varinjlim \widehat{j}_k^*)^{-1} = (\varinjlim \widehat{g}_k^*) \circ (\varinjlim (\widehat{l}_k \widehat{j}_k)^*)^{-1}.$$

Now our assertion follows from (**) of Step 2, and the proof is complete.□

(2.7) DEFINITION. Given a compact field $f \in \mathfrak{L}(X, Y)$, let $\{Y_k, f_k\}$ be an approximating sequence for f and let $j_k : Y \hookrightarrow Y_k$ be the inclusion. We define the *induced homomorphism* $f^* : \lambda(Y) \to \lambda(X)$ by

$$f^* = (\varinjlim f_k^*) \circ (\varinjlim j_k^*)^{-1}.$$

In view of Proposition (2.6), the definition of f^* does not depend on the choice of the approximating system $\{Y_k, f_k\}$ for f

We now establish the main properties of f^*:

(2.8) PROPOSITION. *If the fields $f, g \in \mathfrak{L}(X, Y)$ are homotopic in $\mathfrak{L}$, then $f^* = g^*$.*

PROOF. Let $h_t : X \to Y$ be a homotopy in $\mathfrak{L}$ joining $f = h_0$ and $g = h_1$. For each $k \in N$, using an argument as in (2.5), choose a homotopy $h_t^{(k)} : X \to \widetilde{Y}_k$ in $\mathfrak{L}_0$ such that

$$\|h_t^{(k)}(x) - h_t(x)\| \leq 1/k \qquad \text{for all } (x, t) \in X \times I$$

and define $f_k, g_k \in \mathfrak{L}_0(X, \widetilde{Y}_k)$ by $f_k = h_0^{(k)}$, $g_k = h_1^{(k)}$. We note that $\{\widetilde{Y}_k, f_k\}$ and $\{\widetilde{Y}_k, g_k\}$ are approximating systems for f and g, respectively, and since $f_k \approx g_k$ in $\mathfrak{L}_0$ for each $k \in N$, we get $f_k^* = g_k^*$, and hence $f^* = g^*$. □

(2.9) PROPOSITION. *The induced homomorphism f^* has the following properties:*

(a) $1_X^* = 1_{\lambda(X)}$,

(b) $(g \circ f)^* = f^* \circ g^*$

PROOF. Property (a) is evident. The proof of (b) is carried out in two steps.

STEP 1 (Special case: $g \in \mathfrak{L}_0$). Given $f \in \mathfrak{L}(X, Y)$ and $g \in \mathfrak{L}_0(Y, Z)$, let $h = g \circ f$; we shall prove that

$$h^* = f^* \circ g^*.$$

Let $\{Y^{(k)}, f_k\}$ be a standard approximating system for f. The Tietze theorem gives a finite-dimensional extension $\widehat{g} : Y^{(1)} \to E$ of g over $Y^{(1)}$. For each $k \in N$ let $W_k = \widehat{g}(Y^{(k)}) \cup Z$ and note that both $\{W_k\}$ and $\{\widetilde{W}_k\}$ are approximating sequences for Z

In the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f_k"'] & Y \arrow[r, "g"] \arrow[d, "i_k"] \arrow[dr, "g_k"] & Z \arrow[d, "j_k"] \\ & Y^{(k)} \arrow[r, "\widehat{g}_k"] & \widetilde{W}_k \end{tikzcd}$$

define $\widehat{g}_k \in \mathfrak{L}_0(Y^{(k)}, \widetilde{W}_k)$ by $\widehat{g}_k(y) = \widehat{g}(y)$ for $y \in Y^{(k)}$ and let $g_k = \widehat{g}_k \circ i_k$, $h_k = \widehat{g}_k \circ f_k$, where $i_k : Y \hookrightarrow Y^{(k)}$ and $j_k : Z \hookrightarrow \widetilde{W}_k$ are the inclusions. Since for each $k \in N$ both f_k and g_k are finite-dimensional, we have $g_k^* = i_k^* \circ \widehat{g}_k^*$, $h_k^* = f_k^* \circ \widehat{g}_k^*$, and therefore

$$\varinjlim h_k^* = (\varinjlim f_k^*) \circ (\varinjlim \widehat{g}_k^*) = (\varinjlim f_k^*) \circ (\varinjlim i_k^*)^{-1} \circ \varinjlim g_k^*,$$

and thus

$$(\varinjlim h_k^*) \circ (\varinjlim j_k^*)^{-1} = f^* \circ (\varinjlim g_k^*) \circ (\varinjlim j_k^*)^{-1}.$$

We also remark that $\{\widetilde{W}_k, h_k\}$ and $\{\widetilde{W}_k, g_k\}$ are approximating systems for h and g, respectively. This together with the last formula implies that $h^* = f^* \circ g^*$, and the proof is complete.

STEP 2 (General case). Let $f \in \mathfrak{L}(X,Y)$ and $g \in \mathfrak{L}(Y,Z)$ be arbitrary fields and $h = g \circ f$. We shall prove that $h^* = f^* \circ g^*$. Let $\{Z^{(k)}, h_k\}$ and $\{Z^{(k)}, g_k\}$ be standard approximating systems for h and g, respectively. From the inequalities

$$\|h_k(x) - h(x)\| \le 1/k \quad \text{for all } x \in X,$$
$$\|g_k \circ f(x) - h(x)\| \le 1/k \quad \text{for all } x \in X,$$

we infer that for every $k \in N$ the fields $g_k \circ f, h_k : X \to Z^{(k)}$ are homotopic in $\mathfrak{L}$. In view of (2.8) this implies that $h_k^* = (g_k \circ f)^*$, and therefore, because g_k is finite-dimensional, we get $(g_k \circ f)^* = f^* \circ g_k^*$ by Step 1, and thus

$$\varinjlim h_k^* = \varinjlim (f^* \circ g_k^*) = f^* \circ \varinjlim g_k^*.$$

This implies $h^* = f^* \circ g^*$ and completes the proof. □

Now define a function λ from the Leray–Schauder category $\mathfrak{L}$ to the category **Ab** by putting $\lambda(X) = \lambda_0(X)$, $\lambda(f) = f^*$ Observe that if the field f is in $\mathfrak{L}_0(X,Y)$, then it is clear (by taking $\{Y_k, f_k\}$ with $Y_k = Y$, $f_k = f$) that $\lambda(f) = \lambda_0(f)$, i.e., λ extends λ_0 over $\mathfrak{L}$.

We can now summarize the preceding discussion in the following

(2.10) THEOREM. *Let $\lambda_0 : \mathfrak{L}_0 \to \mathbf{Ab}$ be a continuous homotopy invariant cofunctor. Then λ_0 admits a unique extension over $\mathfrak{L}$ to a homotopy invariant cofunctor $\lambda : \mathfrak{L} \to \mathbf{Ab}$.* □

3. The Čech Cohomology Groups $H^{\infty-n}(X)$

This section is concerned with setting up some algebraic invariants of objects X of the category $\mathfrak{L}$, called the *finite-codimensional cohomology groups* $H^{\infty-n}(X)$. Throughout this section, and in fact throughout the rest of this paragraph, the following notation will be used. Let G be a fixed abelian group. We denote by $H^* = \{H^q, \delta^n\}$ (respectively $H_* = \{H_q, \partial^n\}$) the Čech cohomology (respectively the reduced singular homology) with coefficients in G; when no confusion can arise, the graded Čech cohomology (respectively reduced singular homology) of a space X over G is denoted simply by $H^*(X)$ (respectively $H_*(X)$).

A lemma on Alexander Pontrjagin duality

The main results of this paragraph are based on the Alexander Pontrjagin duality theory in $\boldsymbol{R}^n$ We now describe explicitly some special results of the theory that will be needed.

Given a compact subset $X \subset \boldsymbol{R}^k$, we let D_k denote the Alexander–Pontrjagin isomorphism

$$D_k : H^{k-n}(X) \to H_{n-1}(\boldsymbol{R}^k - X)$$

determined by the standard orientation of $\boldsymbol{R}^k$ (see Spanier's book [1966], p. 296). Let $X_0 = X \cap \boldsymbol{R}^{k-1}$, $i : \boldsymbol{R}^{k-1} - X_0 \hookrightarrow \boldsymbol{R}^k - X$ be the inclusion and Δ the Mayer–Vietoris homomorphism of the triad $(X; X \cap \boldsymbol{R}^k_+, X \cap \boldsymbol{R}^k_-)$. From the definition of D_k, it follows that the following diagram is sign-commutative:

$$\begin{tikzcd}
H^{k-1-n}(X_0) \arrow[r, "\Delta"] \arrow[d, "D_{k-1}"'] & H^{k-n}(X) \arrow[d, "D_k"] \\
H_{n-1}(\boldsymbol{R}^{k-1} - X_0) \arrow[r, "i_*"] & H_{n-1}(\boldsymbol{R}^k - X)
\end{tikzcd}$$

that is, $D_k \circ \Delta = a_{kn} i_* \circ D_{k-1}$, where $a_{kn} = \pm 1$.

We now let b_{kn}, $k \geq n$, be defined inductively on k by

$$b_{nn} = 1, \qquad b_{kn} = a_{kn} b_{k-1,n}.$$

Define the isomorphism

$$\mathcal{D}_k : H^{k-n}(X) \to H_{n-1}(\boldsymbol{R}^k - X)$$

by $\mathcal{D}_k = b_{kn} D_k$.

From the above property of D_k and the definition of $\mathcal{D}_k$ we obtain

(3.1) LEMMA. *For $X \subset \boldsymbol{R}^{k+1}$ compact, let $X_0 = X \cap \boldsymbol{R}^k$, and let Δ be the Mayer-Vietoris homomorphism of the triad $(X; X \cap \boldsymbol{R}^{k+1}_+, X \cap \boldsymbol{R}^{k+1}_-)$. Then the following diagram is commutative:*

$$\begin{tikzcd} H^{k-n}(X_0) \arrow[r, "\Delta"] \arrow[d, "\mathcal{D}_k"'] & H^{k-n+1}(X) \arrow[d, "\mathcal{D}_{k+1}"] \\ H_{n-1}(R^k - X_0) \arrow[r, "i_*"] & H_{n-1}(R^{k+1} - X) \end{tikzcd}$$

where $i : R^k - X_0 \hookrightarrow R^{k+1} - X$ *denotes the inclusion.* □

Orientation in E and the cohomology groups $H^{\infty - n}(X)$

Denote by R^∞ the linear space of all real sequences $x = (x_1, x_2, \ldots)$ with finitely many nonzero terms each, normed by $\|x\| = (\sum x_i^2)^{1/2}$, and set

$$\begin{aligned} R^k &= \{x \in R^\infty \mid x_i = 0 \text{ for } i \geq k+1\}, \\ R^k_+ &= \{x \in R^k \mid x_k \geq 0\}, \\ R^k_- &= \{x \in R^k \mid x_k \leq 0\}. \end{aligned}$$

Call two linear isomorphisms $l_1, l_2 : L_\alpha \to R^{d(\alpha)}$ equivalent if $l_1 \circ l_2^{-1} \in \mathrm{GL}_+(d(\alpha))$, i.e., the determinant of the corresponding matrix is positive. This equivalence relation decomposes the set of linear isomorphisms from L_α to $R^{d(\alpha)}$ into two equivalence classes, called orientations of L_α.

For each $\alpha \in \mathcal{L}$, choose an orientation O_α of L_α; we call the family $O = \{O_\alpha\}$ an *orientation* of E. Given a relation $\alpha \preceq \beta$ in $\mathcal{L}$ with $d(\beta) = d(\alpha)+1$ (for brevity such relations will be called *elementary*), and $l_\alpha \in O_\alpha$, there exists $l_\beta \in O_\beta$ such that $l_\beta | L_\alpha = l_\alpha$, and we let

$$L_\beta^+ = l_\beta^{-1}(R_+^{d(\beta)}), \qquad L_\beta^- = l_\beta^{-1}(R_-^{d(\beta)});$$

we note that the definition of L_β^+ and L_β^- depends only on the orientation of L_α and L_β, and so these orientations determine the triad $(L_\beta; L_\beta^+, L_\beta^-)$ with $L_\alpha = L_\beta^+ \cap L_\beta^-$. This implies that given an object X in $\mathfrak{L}$ and an elementary relation $\alpha \preceq \beta$ with $\alpha, \beta \in \mathcal{L}_X$, the orientations of L_α and L_β determine the triad $(X_\beta; X_\beta^+, X_\beta^-)$, where

$$X_\beta^+ = X \cap L_\beta^+, \qquad X_\beta^- = X \cap L_\beta^-, \qquad X_\alpha = X_\beta^+ \cap X_\beta^-.$$

Moreover, if $f \in \mathfrak{L}_\alpha(X, Y)$, then $f_\beta : X_\beta \to Y_\beta$ maps the triad $(X_\beta; X_\beta^+, X_\beta^-)$ into $(Y_\beta, Y_\beta^+, Y_\beta^-)$.

Let $n \geq 1$ be an integer, X an object in $\mathfrak{L}$, and $U = E - X$. First we fix an orientation $O = \{O_\alpha\}$ in the space E and choose an $l_\alpha \in O_\alpha$ for each $\alpha \in \mathcal{L}$. For any $\alpha \in \mathcal{L}_X$ with $d(\alpha) > n$, define

$$\mathcal{D}_\alpha : H^{d(\alpha)-n}(X_\alpha) \to H_{n-1}(U_\alpha)$$

to be the Alexander–Pontrjagin isomorphism transferred from $R^{d(\alpha)}$ to L_α

via l_α, i.e., the map that completes the diagram

$$\begin{tikzcd} H^{d(\alpha)-n}(X_\alpha) \arrow[d, dashed, "\mathcal{D}_\alpha"'] & H^{d(\alpha)-n}(l_\alpha(X_\alpha)) \arrow[l, "(l_\alpha)^*"'] \arrow[d, "\mathcal{D}_{d(\alpha)}"] \\ H_{n-1}(U_\alpha) \arrow[r, "(l_\alpha)_*"] & H_{n-1}(l_\alpha(U_\alpha)) \end{tikzcd}$$

From the definition of $\mathcal{D}_\alpha$ and Lemma (3.1) we obtain

(3.2) LEMMA. *For an object X in $\mathfrak{L}$, $U = E - X$, and an elementary relation $\alpha \preceq \beta$ in $\mathcal{L}_X$ (i.e., with $d(\beta) = d(\alpha) + 1$), let $i_{\alpha\beta} : U_\alpha \hookrightarrow U_\beta$ be the inclusion, and let*

$$\Delta_{\alpha\beta} = \Delta_{\alpha\beta}^{(n)} : H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta)$$

be the Mayer–Vietoris homomorphism of the triad $(X_\beta; X_\beta^+, X_\beta^-)$. Then the following diagram is commutative:

$$\begin{tikzcd} H^{d(\alpha)-n}(X_\alpha) \arrow[r, "\Delta_{\alpha\beta}"] \arrow[d, "\mathcal{D}_\alpha"'] & H^{d(\beta)-n}(X_\beta) \arrow[d, "\mathcal{D}_\beta"] \\ H_{n-1}(U_\alpha) \arrow[r, "(i_{\alpha\beta})_*"] & H_{n-1}(U_\beta) \end{tikzcd}$$ □

Now for any relation $\alpha \preceq \beta$ in $\mathcal{L}_X$ we define a homomorphism

$$\Delta_{\alpha\beta} : H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta)$$

as follows: if $\alpha = \beta$, we let $\Delta_{\alpha\beta}$ be the identity; if $\alpha \preceq \beta$ is elementary, we let $\Delta_{\alpha\beta}$ be the Mayer–Vietoris homomorphism defined above.

In order to define $\Delta_{\alpha\beta}^{(n)}$ for an arbitrary relation $\alpha \preceq \beta$ in $\mathcal{L}_X$, we need

(3.3) LEMMA. *Let X be an object in $\mathfrak{L}$, and $\alpha \preceq \beta$ a relation in $\mathcal{L}_X$ such that $d(\beta) = d(\alpha) + 2$. Then for any chains $\alpha \preceq \gamma \preceq \beta$, $\alpha \preceq \widehat{\gamma} \preceq \beta$ of elementary relations joining α and β, we have*

$$(*) \qquad \Delta_{\gamma\beta} \circ \Delta_{\alpha\gamma} = \Delta_{\widehat{\gamma}\beta} \circ \Delta_{\alpha\widehat{\gamma}}.$$

PROOF. This follows at once from Lemma (3.2). A direct proof of (3.3) is given at the end of this section. □

(3.4) DEFINITION. Let $\alpha \preceq \beta$ be an arbitrary relation in $\mathcal{L}_X$, and let $\alpha = \alpha_0 \preceq \alpha_1 \preceq \cdots \preceq \alpha_{k+1} = \beta$ be a chain of elementary relations in $\mathcal{L}_X$ joining α and β. We define

$$\Delta_{\alpha\beta} = \Delta_{\alpha_k\beta} \circ \cdots \circ \Delta_{\alpha_1\alpha_2} \circ \Delta_{\alpha\alpha_1}$$

as the composition of the corresponding Mayer–Vietoris homomorphisms.

It follows from Lemma (3.3) that the definition does not depend on the choice of the chain joining α and β.

For an object X in $\mathfrak{L}$, consider the abelian groups $H^{d(\alpha)-n}(X_\alpha)$ together with the homomorphisms $\Delta_{\alpha\beta}$ for all $\alpha \preceq \beta$ in $\mathcal{L}_X$. It follows from Lemma (3.2) that $\{H^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}_{\alpha\in\mathcal{L}_X}$ is a direct system of abelian groups, which we call the $(\infty - n)$-*cohomology system* of X corresponding to the orientation O of E.

(3.5) DEFINITION. For an object X in $\mathfrak{L}$, we define an abelian group

$$H^{\infty-n}(X) = \varinjlim_{\alpha\in\mathcal{L}_X} \{H^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}$$

to be the direct limit over $\mathcal{L}_X$ of the $(\infty - n)$-cohomology system of X.

(3.6) THEOREM (Alexander–Pontrjagin duality in E). *For any object X in $\mathfrak{L}$, we have an isomorphism*

$$H^{\infty-n}(X) \cong H_{n-1}(E - X).$$

PROOF. In view of Lemma (3.2) the $(\infty - n)$-cohomology system of X is isomorphic to the direct system $\{H_{n-1}(U_\alpha), (i_{\alpha\beta})_*\}$ of singular homology groups, where $U = E - X$.

Consider the direct family of homomorphisms

$$(i_\alpha)_* : H_{n-1}(U_\alpha) \to H_{n-1}(U),$$

where $i_\alpha : U_\alpha \hookrightarrow U$ are inclusions. We are going to show that

$$i_* = \varinjlim_{\alpha} (i_\alpha)_* : \varinjlim\{H_{n-1}(U_\alpha), (i_{\alpha\beta})_*\} \to H_{n-1}(U)$$

is an isomorphism.

(a) i_* *is epic*: Let $c \in Z_{n-1}(U)$ and denote by $|c|$ the compact support of c. Using (6.2.3) take a finite-dimensional ε-approximation $f_\varepsilon : |c| \to U$ of the inclusion map $i : |c| \hookrightarrow U$, where $\varepsilon < \operatorname{dist}(|c|, \partial U)$. Because the maps $i, f_\varepsilon : |c| \to U$ are homotopic, the cycles $C_{n-1}(f_\varepsilon)(c)$ and c represent the same homology class in $H_{n-1}(U)$. Since the support of the former cycle lies in some U_α, this proves that i_* is an epimorphism.

(b) i_* *is monic*: Suppose that a cycle $b \in Z_{n-1}(U_\alpha)$ is the boundary of some singular chain c in U. Let $f_\varepsilon : |c| \to U$ be a finite-dimensional ε-approximation of $j : |c| \hookrightarrow U$ with $f_\varepsilon(|c|) \subset L_\beta$, where $L_\beta \supset L_\alpha$ and $\varepsilon < \operatorname{dist}(|c|, \partial U)$. Because $j, f_\varepsilon : |c| \to U$ are homotopic and $C_n(f_\varepsilon)(c)$ is a chain in U_β, we infer that the cycle b represents in $H_{n-1}(U_\beta)$ the same homology class as the cycle

$$C_{n-1}(f_\varepsilon)(b) = C_{n-1}(f_\varepsilon)(\partial c) = \partial[C_n(f_\varepsilon)(c)],$$

and thus this homology class is zero. This implies that i_* is a monomorphism, and the proof of (3.6) is complete. □

Direct proof of Lemma (3.3)

We shall use the following notation:

$$\begin{aligned} Q^{k+1} &= \{x \in \boldsymbol{R}^{k+2} \mid x_{k+1} = 0\}, \\ Q_+^{k+1} &= \{x \in Q^{k+1} \mid x_{k+2} \geq 0\}, \\ Q_-^{k+1} &= \{x \in Q^{k+1} \mid x_{k+2} \leq 0\}, \\ P_+^{k+2} &= \{x \in \boldsymbol{R}^{k+2} \mid x_{k+1} \geq 0\}, \\ P_-^{k+2} &= \{x \in \boldsymbol{R}^{k+2} \mid x_{k+1} \leq 0\}. \end{aligned}$$

Letting $k = d(\alpha)$, we have $d(\gamma) = d(\widehat{\gamma}) = k+1$ and $d(\beta) = k+2$. Define a linear isomorphism $\Psi \cdot \boldsymbol{R}^{k+2} \to \boldsymbol{R}^{k+2}$ by

$$(x_1, \ldots, x_k, x_{k+1}, x_{k+2}) \mapsto (x_1, \ldots, x_k, x_{k+2}, x_{k+1})$$

and choose linear isomorphisms $l_\alpha : L_\alpha \to \boldsymbol{R}^k$, $l_\gamma : L_\gamma \to \boldsymbol{R}^{k+1}$, $l_{\widehat{\gamma}} : L_{\widehat{\gamma}} \to \boldsymbol{R}^{k+1}$ such that $l_\alpha \in O_\alpha$, $l_\gamma \in O_\gamma$, $l_{\widehat{\gamma}} \in O_{\widehat{\gamma}}$, and

$$l_\alpha(x) = l_\gamma(x) = l_{\widehat{\gamma}}(x) \qquad \text{for all } x \in L_\alpha.$$

There is a unique isomorphism $l : L_\beta \to \boldsymbol{R}^{k+2}$ such that

$$l(x) = \begin{cases} l_\gamma(x) & \text{for all } x \in L_\gamma, \\ \Psi \circ l_{\widehat{\gamma}}(x) & \text{for all } x \in L_{\widehat{\gamma}}. \end{cases}$$

Consider the following triads:

$$\begin{aligned} T_1 &= X \cap l^{-1}(\boldsymbol{R}^{k+1}; \boldsymbol{R}_+^{k+1}, \boldsymbol{R}_-^{k+1}), \\ T_2 &= X \cap l^{-1}(\boldsymbol{R}^{k+2}; \boldsymbol{R}_+^{k+2}, \boldsymbol{R}_-^{k+2}), \\ T_3 &= X \cap l^{-1}(Q^{k+1}; Q_+^{k+1}, Q_-^{k+1}), \\ T_4 &= X \cap l^{-1}(\boldsymbol{R}^{k+2}; P_+^{k+2}, P_-^{k+2}). \end{aligned}$$

By straightforward computation, (T_1, T_2), (T_3, T_4) are consecutive pairs of triads satisfying the assumption of Lemma (1.10), and therefore

$$\Delta(T_2)\Delta(T_1) = -\Delta(T_4)\Delta(T_3). \tag{1}$$

There are two cases to consider: either $l \in O_\beta$ or not. We show that in both cases we obtain the desired conclusion (*) of Lemma (3.3).

If $l \in O_\beta$, then $\Psi \circ l \notin O_\beta$, and we have

$$\begin{aligned} \Delta_{\alpha\gamma} &= \Delta(T_1), & \Delta_{\gamma\beta} &= \Delta(T_2), \\ \Delta_{\alpha\widehat{\gamma}} &= \Delta(T_3), & \Delta_{\widehat{\gamma}\beta} &= -\Delta(T_4). \end{aligned}$$

Thus, in view of (1), we obtain (*). If $l \notin O_\beta$, then $\Psi \circ l \in O_\beta$, and we have

$$\begin{aligned} \Delta_{\alpha\gamma} &= \Delta(T_1), & \Delta_{\gamma\beta} &= -\Delta(T_2), \\ \Delta_{\alpha\widehat{\gamma}} &= \Delta(T_3), & \Delta_{\widehat{\gamma}\beta} &= \Delta(T_4), \end{aligned}$$

and (*) follows again, completing the proof of Lemma (3.3). □

4. The Functor $H^{\infty-n} : (\mathfrak{L}, \sim) \to \mathbf{Ab}$

In this section, given any $n \geq 1$ we construct an h-cofunctor $H_0^{\infty-n}$ from $(\mathfrak{L}_0, \approx)$ to the category of abelian groups. Then, using the continuity of Čech cohomology and an algebraic lemma on interchanging double direct limits, we show that the functor $H_0^{\infty-n}$ is continuous. Finally, applying Theorem (2.10), we obtain the unique extension $H^{\infty-n}$ of $H_0^{\infty-n}$ over $(\mathfrak{L}, \sim)$. The section ends with the Alexander–Pontrjagin invariance theorem.

The induced homomorphism f^*

Having defined the Čech cohomology groups $H^{\infty-n}(X)$, we now proceed to define, for a finite-dimensional field f, the homomorphism f^* induced by f.

Let $f \in \mathfrak{L}_{\alpha_0}(X, Y)$ be an α_0-field, where $\alpha_0 \in \mathcal{L}_X$. Given a relation $\alpha \preceq \beta$ in $\mathcal{L}_X$ with $\alpha_0 \preceq \alpha$, consider the diagram

$$(\mathcal{D}) \qquad \begin{array}{ccc} H^{d(\alpha)-n}(Y_\alpha) & \xrightarrow{f_\alpha^*} & H^{d(\alpha)-n}(X_\alpha) \\ \Big\downarrow{\scriptstyle \Delta_{\alpha\beta}} & & \Big\downarrow{\scriptstyle \Delta_{\alpha\beta}} \\ H^{d(\beta)-n}(Y_\beta) & \xrightarrow{f_\beta^*} & H^{d(\beta)-n}(X_\beta) \end{array}$$

and observe that if $d(\beta) = d(\alpha) + 1$, then $f_\beta(X_\beta^+) \subset Y_\beta^+$, $f_\beta(X_\beta^-) \subset Y_\beta^-$, and hence the above diagram commutes by the functoriality of the Mayer–Vietoris homomorphism. Therefore, in view of the definition of $\Delta_{\alpha\beta}$, we see that $(\mathcal{D})$ in fact commutes for any relation $\alpha \preceq \beta$ in $\mathcal{L}_X$. Consequently, f induces the map

$$\{f_\alpha^*\} : \{H^{d(\alpha)-n}(Y_\alpha), \Delta_{\alpha\beta}\} \to \{H^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}$$

between the corresponding $(\infty - n)$-cohomology systems.

For $f \in \mathfrak{L}_0(X, Y)$, we now define the induced homomorphism

$$f^* = \varinjlim_{\alpha \in \mathcal{L}_Y} f_\alpha^* : H^{\infty-n}(Y) \to H^{\infty-n}(X)$$

to be the direct limit over $\mathcal{L}_Y$ of the family $\{f_\alpha^*\}$.

Let $H^{\infty-n}$ be the function assigning to an object X of $\mathfrak{L}_0$ the cohomology group $H^{\infty-n}(X)$ and to a field $f \in \mathfrak{L}_0(X, Y)$ the induced homomorphism $f^* : H^{\infty-n}(Y) \to H^{\infty-n}(X)$.

(4.1) THEOREM. *$H_0^{\infty-n}$ is an h-cofunctor from the category $(\mathfrak{L}_0, \approx)$ to the category of abelian groups.*

PROOF. This follows readily from the definitions involved with the aid of the Mayer–Vietoris homomorphism. □

Continuity of the functor $H_0^{\infty-n}$

Let Y be an object in $\mathfrak{L}$, and $\{Y_k\}$ an approximating system for Y. Denote by

$$j_k : Y \hookrightarrow Y_k, \qquad i_{kl} : Y_l \hookrightarrow Y_k \qquad (k \leq l)$$

the inclusions and consider the direct system $\{H^{d(\alpha)-n}(Y_k), i_{kl}^*\}$ over N together with the direct family of homomorphisms $j_k^* : H^{\infty-n}(Y_k) \to H^{\infty-n}(Y)$.

(4.2) THEOREM. *The map*

$$\varinjlim_k j_k^* : \varinjlim_k \{H^{\infty-n}(Y_k), i_{kl}^*\} \to H^{\infty-n}(Y)$$

*is an isomorphism; in other words, the functor $H_0^{\infty-n} : (\mathfrak{L}_0, \approx) \to$ **Ab** is continuous.*

PROOF. For any $\alpha \in \mathcal{L}_Y$ and $k \in N$, let $Y_\alpha^k = Y_k \cap L_\alpha$ and denote by

$$j_{k\alpha} : Y_\alpha \hookrightarrow Y_\alpha^k, \qquad i_{kl}^\alpha : Y_\alpha^l \hookrightarrow Y_\alpha^k \qquad (k \leq l)$$

the corresponding inclusions. Now for any pair of relations $k \leq l$ and $\alpha \preceq \beta$ in N and $\mathcal{L}_Y$, respectively, consider the diagram

$$\begin{tikzcd}
H^{d(\alpha)-n}(Y_\alpha^k) \arrow[r, "(i_{kl}^\alpha)^*"] \arrow[d, "\Delta_{\alpha\beta}^k"'] & H^{d(\alpha)-n}(Y_\alpha^l) \arrow[d, "\Delta_{\alpha\beta}^l"] \\
H^{d(\beta)-n}(Y_\beta^k) \arrow[r, "(i_{kl}^\beta)^*"] & H^{d(\beta)-n}(Y_\beta^l)
\end{tikzcd}$$

In view of Definition (3.4), the above diagram commutes, and consequently, the groups $H^{d(\alpha)-n}(Y_\alpha^k)$ together with the homomorphisms $(i_{kl}^\alpha)^*$ and $\Delta_{\alpha\beta}^k$ determine a double direct system of abelian groups over $N \times \mathcal{L}_Y$. Furthermore, by taking into account the commutativity relations between the various inclusions involved, we see that the family $\{j_{k\alpha}^*\}$ of homomorphisms $j_{k\alpha}^* : H^{d(\alpha)-n}(Y_\alpha^k) \to H^{d(\alpha)-n}(Y_k)$ maps the double direct system $\{H^{d(\alpha)-n}(Y_\alpha^k)\}$ to the direct system $\{H^{d(\alpha)-n}(Y_\alpha)\}$.

Now, for each $\alpha \in \mathcal{L}_Y$, by continuity of Čech cohomology, the map $\varinjlim_k j_{k\alpha}^*$ is an isomorphism, and therefore so is the map

$$\varinjlim_\alpha \varinjlim_k j_{k\alpha}^*.$$

In view of the lemma on interchanging double direct limits (cf. Appendix, Theorem (D.8)), the map

$$\varinjlim_k j_k^* = \varinjlim_k \varinjlim_\alpha j_{k\alpha}^*$$

is also an isomorphism, and the proof is complete. □

In view of Theorems (4.2) and (2.10), we summarize the preceding discussion in the following

(4.3) THEOREM. *The functor $H_0^{\infty-n}$ extends uniquely from $(\mathfrak{L}_0, \approx)$ to an h-functor $H^{\infty-n} : (\mathfrak{L}, \sim) \to \mathbf{Ab}$.* □

(4.4) COROLLARY. *If the objects X and Y are equivalent or homotopy equivalent in $(\mathfrak{L}, \sim)$, then $H^{\infty-n}(X) \cong H^{\infty-n}(Y)$ for each $n \geq 1$.* □

(4.5) THEOREM (Alexander-Pontrjagin invariance in E). *Let X and Y be equivalent or homotopy equivalent objects in $(\mathfrak{L}, \sim)$. Then the singular homology groups*

$$H_{n-1}(E - X; G) \quad \text{and} \quad H_{n-1}(E - Y; G)$$

are isomorphic for any $n \geq 1$ and any group G of coefficients.

PROOF. This is an immediate consequence of (3.4) and (4.4). □

(4.6) COROLLARY (Leray- Alexandroff invariance theorem). *Let X and Y be equivalent or homotopy equivalent objects of the Leray–Schauder category $(\mathfrak{L}, \sim)$. Then the complements $E - X$ and $E - Y$ have the same number of components.* □

5. Cohomology Theory on $\mathfrak{L}$

Having defined the absolute cohomology functors $H^{\infty-n}$, we now turn to the case of relative cohomology. Throughout the section we let $F = E \oplus \boldsymbol{R}$ be the direct product of E and the real line $\boldsymbol{R}$, and we regard E as a 1-codimensional linear subspace of F; we let $y^+ = (0, 1)$, where $0 \in E$, $1 \in \boldsymbol{R}$.

The relative cohomology functor $H^{\infty-n}$

We begin by fixing an orientation $\{O_\alpha\}$ in the space E. Denote by $\mathcal{L}_E = \{\alpha, \beta, \gamma, \ldots\}$ and $\mathcal{L}_F = \{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \ldots\}$ the directed sets of finite-dimensional linear subspaces of E and F, respectively, and let $\mathcal{L}_0 = \{L_{\widehat{\beta}} \in \mathcal{L}_F \mid y^+ \in L_{\widehat{\beta}}\}$; clearly, $\mathcal{L}_0$ is cofinal in $\mathcal{L}_F$. For $\alpha \in \mathcal{L}_E$, we denote by α' the element of $\mathcal{L}_0$ given by $L_{\alpha'} = L_\alpha \oplus \boldsymbol{R}y^+$.

Starting with the orientation $\{O_\alpha\}$ in E we define an orientation $\{\overline{O}_{\widehat{\alpha}}\}$ in F by the following rule:

(i) If $\widehat{\alpha} \in \mathcal{L}_E$, we let $\overline{O}_{\widehat{\alpha}} = O_\alpha$.
(ii) If $\widehat{\alpha} \notin \mathcal{L}_E$ and $\widehat{\alpha} \notin \mathcal{L}_0$, we define $\overline{O}_{\widehat{\alpha}}$ arbitrarily.
(iii) If $\widehat{\alpha} \notin \mathcal{L}_E$ and $\widehat{\alpha} \in \mathcal{L}_0$, we first take a $\beta \in \mathcal{L}_E$ such that $\beta' = \widehat{\alpha}$; then we choose a representative $l = (l_1, \ldots, l_k) : L_\beta \to \boldsymbol{R}^k$ in O_β, where $k = d(\beta)$, and define a linear map $\bar{l} : L_{\widehat{\alpha}} \to \boldsymbol{R}^{k+1}$ by

$$\bar{l}(x) = \begin{cases} (0, l_1(x), \ldots, l_k(x)) & \text{for } x \in L_\beta, \\ (1, 0, \ldots, 0) & \text{for } x = y^+. \end{cases}$$

We let $\overline{O}_{\tilde{\alpha}}$ be the orientation of $L_{\tilde{\alpha}}$ determined by $\bar{l}$. Thus, we have defined an orientation $\{\overline{O}_{\tilde{\alpha}}\}$ in F, which we call an *extension* of $\{O_{\tilde{\alpha}}\}$ from E over F. From now on, we assume that such an orientation $\{\overline{O}_{\tilde{\alpha}}\}$ in F is fixed.

Next, consider the categories $\mathfrak{L}_E = \mathfrak{L}(E)$, $\mathfrak{L}_F = \mathfrak{L}(F)$, and observe that $\mathfrak{L}_E$ and $\mathfrak{L}_0(E)$ are h-subcategories of $\mathfrak{L}_F$ and $\mathfrak{L}_0(F)$, respectively. We will denote by $\mathfrak{L}_E^2$ and $\mathfrak{L}_F^2$ the corresponding h-categories of pairs.

Let $C : \mathfrak{L}_E \to \mathfrak{L}_F$ be the *cone functor* given by

$$CA = \begin{cases} \{x \in F \mid x = ta + (1-t)y^+,\ (a,t) \in A \times I\} & \text{for } A \subset E,\ A \neq \emptyset, \\ \emptyset & \text{for } A = \emptyset. \end{cases}$$

and

$$Cf(x) = tf(a) + (1-t)y^+, \qquad x \in CA,$$

for any $f : A \to B$ in $\mathfrak{L}_E$. The reader will easily verify that:

(i) C is well defined.

(ii) C sends $\mathfrak{L}_0(E)$ to $\mathfrak{L}_0(F)$.

(iii) C is homotopy invariant with respect to $\sim$.

(iv) $C|\mathfrak{L}_0(E)$ is homotopy invariant from $(\mathfrak{L}_0(E), \approx)$ to $(\mathfrak{L}_0(F), \approx)$.

Now, given (X, A) in $\mathfrak{L}_E^2$, let $\varrho(X, A) = X \cup CA$, and for $f : (X, A) \to (Y, B)$ in $\mathfrak{L}_E$ let $\varrho(f) = \tilde{f} : X \cup CA \to Y \cup CB$ be defined by

$$\tilde{f}(x) = \begin{cases} Cf(x) & \text{for all } x \in CA, \\ f(x) & \text{for all } x \in X. \end{cases}$$

(5.1) PROPOSITION. *The assignments $(X, A) \mapsto X \cup CA$, $f \mapsto \tilde{f}$ determine an h-functor ϱ from $\mathfrak{L}_E^2$ to $\mathfrak{L}_F$ such that $\varrho(\mathfrak{L}_0^2(E)) \subset \mathfrak{L}_0(F)$.* □

Now, for (X, A) in $\mathfrak{L}_E^2$, we shall define the relative groups $H^{\infty - n}(X, A)$. To this end, for $\alpha \in \mathcal{L}_E$ such that $A_\alpha = A \cap L_\alpha \neq \emptyset$, let

$$\begin{aligned} e_\alpha &: (X_\alpha, A_\alpha) \to (X_\alpha \cup CA_\alpha, CA_\alpha), \\ j_\alpha &: X_\alpha \cup CA_\alpha \to (X_\alpha \cup CA_\alpha, CA_\alpha), \\ i_\alpha &: CA_\alpha \to X_\alpha \cup CA_\alpha \end{aligned}$$

denote the corresponding inclusions. Because e_α is an excision, the induced map

$$e_\alpha^* : H^n(X_\alpha \cup CA_\alpha, CA_\alpha) \to H^n(X_\alpha, A_\alpha)$$

is an isomorphism. Writing the exact sequences

$$\cdots \to H^n(X_\alpha \cup CA_\alpha, CA_\alpha) \xrightarrow{j_\alpha^*} H^n(X_\alpha \cup CA_\alpha) \xrightarrow{i_\alpha^*} H^n(CA_\alpha) \to \cdots$$

of the pair $(X_\alpha \cup CA_\alpha, CA_\alpha)$ and taking into account that CA_α is contractible, we see that i_α^* is epic and j_α^* is an isomorphism for $n \geq 1$.

Let $\eta_\alpha : H^n(X_\alpha, A_\alpha) \to H^n(X_\alpha \cup CA_\alpha)$ be given by $\eta_\alpha = j_\alpha^* \circ (e_\alpha^*)^{-1}$; clearly, η_α is an isomorphism for $n \geq 1$, and we note that the sequence

$$H^n(X_\alpha, A_\alpha) \xrightarrow{\eta_\alpha} H^n(X_\alpha \cup CA_\alpha) \xrightarrow{i_\alpha^*} H^n(CA_\alpha)$$

is exact.

Let $\alpha \preceq \beta$ be an elementary relation in $\mathcal{L}_E$ and suppose that A_α is nonempty. Then $(A_\beta, A_\beta^+, A_\beta^-)$ is a proper subtriad of $(X_\beta; X_\beta^+, X_\beta^-)$, and we let

$$\Delta_{\alpha\beta} = \Delta_{\alpha\beta}^{(n)} : H^{d(\alpha)-n}(X_\alpha, A_\alpha) \to H^{d(\beta)-n}(X_\beta, A_\beta)$$

be the corresponding relative Mayer–Vietoris homomorphism. Note that $(X \cup CA)_{\alpha'} = X_\alpha \cup CA_\alpha$. Let $\Delta_{\alpha'\beta'}$ denote the Mayer–Vietoris homomorphisms of the triads $((X \cup CA)_{\beta'}; (X \cup CA)_{\beta'}^+, (X \cup CA)_{\beta'}^-)$ and $((CA)_{\beta'}; (CA)_{\beta'}^+, (CA)_{\beta'}^-)$. Clearly, we have the following

(5.2) LEMMA. *The following diagram commutes:*

$$\begin{CD}
H^{d(\alpha)-n}(X_\alpha, A_\alpha) @>{\eta_\alpha}>> H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) @>{i_\alpha^*}>> H^{d(\alpha)-n}(CA_\alpha) \\
@V{\Delta_{\alpha\beta}}VV @V{\Delta_{\alpha'\beta'}}VV @VV{\Delta_{\alpha'\beta'}}V \\
H^{d(\beta)-n}(X_\beta, A_\beta) @>{\eta_\beta}>> H^{d(\beta)-n}(X_\beta \cup CA_\beta) @>{i_\beta^*}>> H^{d(\beta)-n}(CA_\beta)
\end{CD}$$ □

Assume now that $\alpha \preceq \beta$ is an arbitrary relation in $\mathcal{L}_A$ and let $\alpha = \alpha_0 \preceq \alpha_1 \preceq \cdots \preceq \alpha_{k+1} = \beta$ be a chain of elementary relations joining α and β. We define

$$\Delta_{\alpha\beta} = \Delta_{\alpha_k\beta} \circ \cdots \circ \Delta_{\alpha\alpha_1}$$

as the composition of the corresponding Mayer–Vietoris homomorphisms. In view of (3.3) and (5.2) and since η_α, η_β are isomorphisms, the above definition does not depend on the choice of the chain. Furthermore, the groups $H^{d(\alpha)-n}(X_\alpha, A_\alpha)$ together with the homomorphisms $\Delta_{\alpha\beta}$ form a direct system of abelian groups over $\mathcal{L}_A$, which we will call the $(\infty - n)$-*cohomology system* of the pair (X, A) coresponding to the orientation $\{O_\alpha\}$ in E.

(5.3) DEFINITION. For $n \in N$ we define the relative cohomology group

$$H^{\infty-n}(X, A) = \varinjlim_{\alpha \in \mathcal{L}_A} \{H^{d(\alpha)-n}(X_\alpha, A_\alpha), \Delta_{\alpha\beta}\}$$

as the direct limit of the $(\infty - n)$-cohomology system of the pair (X, A).

Evidently, this definition extends that of the absolute group $H^{\infty-n}(X)$ given in Section 3.

Now we apply the construction of the previous section to the space F and denote by $\overline{H}^{\infty-n}: \mathfrak{L}_F \to \mathbf{Ab}$ the functor corresponding to H^* and the orientation $\{\overline{O}_{\tilde{\alpha}}\}$ in F.

In view of Lemma (5.2), we see that the family $\{\eta_\alpha\}$ of isomorphisms is in fact a direct family of maps from $\{H^{d(\alpha)-n}(X_\alpha, A_\alpha), \Delta^{(n)}_{\alpha\beta}\}$ to $\{H^{d(\alpha')-n-1}((X \cup CA)_{\alpha'}), \Delta^{(n+1)}_{\alpha'\beta'}\}$, and consequently the direct limit map

$$\eta = \varinjlim \eta_\alpha : H^{\infty-n}(X, A) \to \overline{H}^{\infty-(n+1)}(X \cup CA)$$

is an isomorphism. Moreover, the sequence

$$H^{\infty-n}(X, A) \underset{\cong}{\xrightarrow{\eta}} \overline{H}^{\infty-(n+1)}(X \cup CA) \xrightarrow{\overline{H}(i)} \overline{H}^{\infty-(n+1)}(CA),$$

in which $i : CA \hookrightarrow X \cup CA$ denotes the inclusion, is exact.

(5.4) DEFINITION. For $f : (X, A) \to (Y, B)$ in $\mathfrak{L}^2_E$ we define the induced map $f^* = H^{\infty-n}(f)$ by imposing commutativity of the following diagram:

$$\begin{array}{ccc} H^{\infty-n}(Y, B) & \underset{\cong}{\xrightarrow{\eta}} & \overline{H}^{\infty-n-1}(Y \cup CB) \\ \downarrow{\scriptstyle H^{\infty-n}(f)} & & \downarrow{\scriptstyle \overline{H}^{\infty-n-1}(\tilde{f})} \\ H^{\infty-n}(X, A) & \underset{\cong}{\xrightarrow{\eta}} & \overline{H}^{\infty-n-1}(X \cup CA) \end{array}$$

Since both arrows denoted by η are isomorphisms, the induced map f^* is well defined.

(5.5) PROPOSITION. *The assignments $(X, A) \mapsto H^{\infty-n}(X, A)$ and $f \mapsto f^*$ define an h-cofunctor $H^{\infty-n}$ from the category $\mathfrak{L}^2_E$ to the category of abelian groups. Moreover, η is a natural equivalence from the functor $H^{\infty-n}$ to $\overline{H}^{\infty-(n+1)} \circ \varrho$.*

PROOF. This follows from the definitions involved and Theorem (4.3). ☐

The homomorphism δ^n_α

We now establish some results that will be used in defining the coboundary transformation $\delta^{\infty-n}$. Recall that for an object A in $\mathfrak{L}_E$ we write $\mathcal{L}_A = \{\alpha \in \mathcal{L}_E \mid A_\alpha \neq \emptyset\}$.

(5.6) DEFINITION. For (X, A) in $\mathfrak{L}^2_E$ and $\alpha \in \mathcal{L}_A$ let

$$\delta^n_\alpha : H^{d(\alpha)-n-1}(A_\alpha) \to H^{d(\alpha)-n}(X_\alpha, A_\alpha)$$

be given by $\delta^n_\alpha = (-1)^{d(\alpha)}\delta_{(X_\alpha, A_\alpha)}$, where $\delta_{(X_\alpha, A_\alpha)}$ is the coboundary homomorphism of the pair (X_α, A_α).

(5.7) LEMMA. *If (X, A) is in $\mathfrak{L}_E^2$, then for every relation $\alpha \preceq \beta$ in $\mathcal{L}_A$ the following diagram commutes:*

$$\begin{tikzcd}
H^{d(\alpha)-n-1}(A_\alpha) \arrow[r, "\delta_\alpha^n"] \arrow[d, "\Delta_{\alpha\beta}^{(n+1)}"'] & H^{d(\alpha)-n}(X_\alpha, A_\alpha) \arrow[d, "\Delta_{\alpha\beta}^{(n)}"] \\
H^{d(\beta)-n-1}(A_\beta) \arrow[r, "\delta_\beta^n"] & H^{d(\beta)-n}(X_\beta, A_\beta)
\end{tikzcd}$$

PROOF. Assume first that $\alpha \preceq \beta$ is elementary. Let

$$\widetilde{\delta}_\alpha^n : H^{d(\alpha)-n-1}(A_\alpha) \to H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) = H^{d(\alpha')-n-1}((X \cup CA)_{\alpha'})$$

be the Mayer–Vietoris homomorphism of the triad $(X_\alpha \cup CA_\alpha; CA_\alpha, X_\alpha)$. Evidently, we have

$$\widetilde{\delta}_\alpha^n = (-1)^{d(\alpha)} \eta_\alpha \circ \delta_\alpha^n.$$

Next, by observing that the consecutive pairs of triads

$$(X_\alpha \cup CA_\alpha; CA_\alpha, X_\alpha), \quad ((X \cup CA)_{\beta'}; (X \cup CA)_{\beta'}^+, (X \cup CA)_{\beta'}^-),$$
$$(A_\beta; A_\beta^+, A_\beta^-), \quad (X_\beta \cup CA_\beta; CA_\beta, X_\beta)$$

satisfy the assumptions of Lemma (1.10), we have

$$\Delta_{\alpha'\beta'}^{(n)} \circ \widetilde{\delta}_\alpha^n = -\widetilde{\delta}_\beta^n \circ \Delta_{\alpha\beta}^{(n+1)}.$$

Now we consider the diagram

$$\begin{tikzcd}
H^{d(\alpha)-n-1}(A_\alpha) \arrow[r, "\delta_\alpha^n"] \arrow[d, "\Delta_{\alpha\beta}^{(n+1)}"'] & H^{d(\alpha)-n}(X_\alpha, A_\alpha) \arrow[r, "\eta_\alpha"] \arrow[d, "\Delta_{\alpha\beta}^{(n)}"] & H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) \arrow[d, "\Delta_{\alpha'\beta'}^{(n)}"] \\
H^{d(\beta)-n-1}(A_\beta) \arrow[r, "\delta_\beta^n"] & H^{d(\beta)-n}(X_\beta, A_\beta) \arrow[r, "\eta_\beta"] & H^{d(\beta)-n}(X_\beta \cup CA_\beta)
\end{tikzcd}$$

The composition of the top row homomorphisms equals $(-1)^{d(\alpha)}\widetilde{\delta}_\alpha^n$, and the composition of the bottom row is $(-1)^{d(\beta)}\widetilde{\delta}_\beta^n$. Since the right square is commutative by (5.2), we have

$$\begin{aligned}
\eta_\beta \circ \Delta_{\alpha\beta}^{(n)} \circ \delta_\alpha^n &= \Delta_{\alpha'\beta'}^{(n)} \circ \eta_\alpha \circ \delta_\alpha^n = (-1)^{d(\alpha)} \Delta_{\alpha'\beta'}^{(n)} \circ \widetilde{\delta}_\alpha^n \\
&= (-1)^{d(\beta)} \widetilde{\delta}_\beta^n \circ \Delta_{\alpha\beta}^{(n+1)} = \eta_\beta \circ \delta_\beta^n \circ \Delta_{\alpha\beta}^{(n+1)}.
\end{aligned}$$

From this, since η_β is an isomorphism, we get $\Delta_{\alpha\beta}^{(n)} \circ \delta_\alpha^n = \delta_\beta^n \circ \Delta_{\alpha\beta}^{(n+1)}$, and the proof is complete. □

The next two results follow at once from the definition of δ_α^n.

(5.8) PROPOSITION. *Let (X, A) be in $\mathfrak{L}_E^2$. Then, for every $\alpha \in \mathcal{L}_A$, the sequence*

$$\cdots \to H^{d(\alpha)-n}(X_\alpha) \to H^{d(\alpha)-n}(A_\alpha) \xrightarrow{\delta_\alpha^{n+1}} H^{d(\alpha)-(n+1)}(X_\alpha, A_\alpha) \to \cdots$$

is exact, where all unmarked maps are induced by inclusions. □

(5.9) PROPOSITION. *Let (X, A), (Y, B) be in $\mathfrak{L}_E^2$, and $f : (X, A) \to (Y, B)$ an α_0-field. Then, for any $\alpha \in \mathcal{L}_A$ such that $\alpha_0 \preceq \alpha$, the following diagram commutes:*

$$\begin{tikzcd}
H^{d(\alpha)-n-1}(B_\alpha) \arrow[r, "\delta_\alpha^n"] \arrow[d, "(f_\alpha|A_\alpha)^*"'] & H^{d(\alpha)-n}(Y_\alpha, B_\alpha) \arrow[d, "f_\alpha^*"] \\
H^{d(\alpha)-n-1}(A_\alpha) \arrow[r, "\delta_\alpha^n"] & H^{d(\alpha)-n}(X_\alpha, A_\alpha)
\end{tikzcd}$$

□

The coboundary transformation $\delta^{\infty-n}$

Let (X, A) be in $\mathfrak{L}_E^2$. From (5.7) it follows that the family $\{\delta_\alpha^n\}$ is a direct family of homomorphisms, and we define the coboundary homomorphism

$$\delta_{(X,A)}^{\infty-n} : H^{\infty-n-1}(A) \to H^{\infty-n}(X, A)$$

by

$$\delta_{(X,A)}^{\infty-n} = \varinjlim_{\alpha} \delta_\alpha^n.$$

We let the homomorphism

$$\widetilde{\delta}_{(X,A)}^{\infty-n} : H^{\infty-n-1}(A) \to \overline{H}^{\infty-n-1}(X \cup CA)$$

be given by $\widetilde{\delta}_{(X,A)}^{\infty-n} = \eta \circ \delta_{(X,A)}^{\infty-n}$.

We let $\vartheta : \mathfrak{L}^2(E) \to \mathfrak{L}^2(E)$ denote the functor defined by

$$\vartheta(X, A) = (A, \emptyset) = A \quad \text{for } (X, A) \text{ in } \mathfrak{L}^2(E),$$
$$\vartheta(f) = f|A : A \to B \quad \text{for } f : (X, A) \to (Y, B) \text{ in } \mathfrak{L}^2(E).$$

(5.10) PROPOSITION. *The family $\delta^{\infty-n} = \{\delta_{(X,A)}^{\infty-n}\}$ indexed by (X, A) in $\mathfrak{L}_E^2$ is a natural transformation from $H^{\infty-n-1} \circ \vartheta$ to $H^{\infty-n}$*

PROOF. Given $f : (X, A) \to (Y, B)$ in $\mathfrak{L}^2(E)$, consider the diagram

$$\begin{tikzcd}
H^{\infty-n-1}(B) \arrow[r, "\delta^{\infty-n}"] \arrow[d, "(f|A)^*"'] & H^{\infty-n}(Y, B) \arrow[r, "\eta"] \arrow[d, "f^*"] & \overline{H}^{\infty-n-1}(Y \cup CB) \arrow[d, "\bar{f}^*"] \\
H^{\infty-n-1}(A) \arrow[r, "\delta^{\infty-n}"] & H^{\infty-n}(X, A) \arrow[r, "\eta"] & \overline{H}^{\infty-n-1}(X \cup CA)
\end{tikzcd}$$

Assume first that we have proved $\widetilde{f}^* \circ \widetilde{\delta}^{\infty-n} = \widetilde{\delta}^{\infty-n} \circ (f|A)^*$ Then, because the right-hand square is commutative, we have

$$\eta \circ f^* \circ \delta^{\infty-n} = \widetilde{f}^* \circ \eta \circ \delta^{\infty-n} = \widetilde{\delta}^{\infty-n} \circ (f|A)^* = \eta \circ \delta^{\infty-n} \circ (f \mid A)^*,$$

and since η is an isomorphism, this implies $f^* \circ \delta^{\infty-n} = \delta^{\infty-n} \circ (f \mid A)^*$

Thus in order to prove our assertion it is enough to show that for a field $f : (X, A) \to (Y, B)$ in $\mathfrak{L}^2_E$ the following diagram commutes:

$$\begin{tikzcd}
H^{\infty-n-1}(B) \arrow[r, "\widetilde{\delta}^{\infty-n}"] \arrow[d, "(f|A)^*"'] & \overline{H}^{\infty-n-1}(Y \cup CB) \arrow[d, "\widetilde{f}^*"] \\
H^{\infty-n-1}(A) \arrow[r, "\widetilde{\delta}^{\infty-n}"] & \overline{H}^{\infty-n-1}(X \cup CA)
\end{tikzcd}$$

To this end, assume first that f is finite-dimensional. In this special case, we pass to the direct limit in the commutative diagrams of (5.7) and (5.9), and then observe that the desired conclusion follows from $\widetilde{\delta}^{\infty-n}_{(X,A)} = \eta \circ \delta^{\infty-n}_{(X,A)}$.

Consider now the general case and take an approximating system $f^{(k)} : (X, A) \to (Y_k, B_k)$ for f. The definition and the proof of the existence of such a system are similar to those in the absolute case and are omitted. It follows from (5.1) that the sequence $\widetilde{f}^{(k)} : X \cup CA \to Y_k \cup CB_k$ forms an approximating system for $\widetilde{f} : X \cup CA \to Y \cup CB$.

Consider the inclusions

$$j_k : Y \cup CB \hookrightarrow Y_k \cup CB_k, \qquad j'_k : B \hookrightarrow B_k,$$
$$i_{kl} : Y_l \cup CB_l \hookrightarrow Y_k \cup CB_k, \qquad i'_{kl} : B_l \hookrightarrow B_k \quad (k \leq l).$$

By the special case of our assertion, the following diagram commutes for each pair $k \leq l$:

$$\begin{tikzcd}
H^{\infty-n-1}(A) \arrow[rrr, "\widetilde{\delta}^{\infty-n}_{(X,A)}"] & & & \overline{H}^{\infty-n-1}(X \cup CA) \\
& H^{\infty-n-1}(B_l) \arrow[r] \arrow[ul] & \overline{H}^{\infty-n-1}(Y_l \cup CB_l) \arrow[ur] & \\
H^{\infty-n-1}(B_k) \arrow[rrr] \arrow[uu, "(f^k_A)^*"] \arrow[ur] \arrow[dr] \arrow[dd, "(j'_k)^*"'] & & & \overline{H}^{\infty-n-1}(Y_k \cup CB_k) \arrow[uu, "(\widetilde{f}^{(k)})^*"'] \arrow[ul] \arrow[dl] \arrow[dd, "j_k^*"] \\
& H^{\infty-n-1}(B_l) \arrow[r] \arrow[dl] & \overline{H}^{\infty-n-1}(Y_l \cup CB_l) \arrow[dr] & \\
H^{\infty-n-1}(B) \arrow[rrr, "\widetilde{\delta}^{\infty-n}_{(Y,B)}"] & & & \overline{H}^{\infty-n-1}(Y \cup CB)
\end{tikzcd}$$

Applying the direct limit functor to the corresponding commutative diagram in the category of direct systems of abelian groups, we obtain the following commutative diagram:

$$\begin{tikzcd}
H^{\infty-n-1}(A) \arrow[r, "\tilde{\delta}^{\infty-n}_{(X,A)}"] & \overline{H}^{\infty-n-1}(X \cup CA) \\
\varinjlim_k H^{\infty-n-1}(B_k) \arrow[r] \arrow[u, "\varinjlim_k (f_A^{(k)})^*"] \arrow[d, "\varinjlim_k (j_k')^*"'] & \varinjlim_k \overline{H}^{\infty-n-1}(Y_k \cup CB_k) \arrow[u, "\varinjlim_k (\tilde{f}^{(k)})^*"'] \arrow[d, "\varinjlim_k j_k^*"] \\
H^{\infty-n-1}(B) \arrow[r, "\tilde{\delta}^{\infty-n}_{(Y,B)}"] & \overline{H}^{\infty-n-1}(Y \cup CB)
\end{tikzcd}$$

By Theorem (4.2) the homomorphisms $\varinjlim (j_k')^*$ and $\varinjlim (j_k)^*$ are invertible. This, in view of the definition of the induced map, implies our assertion, and thus the proof is complete. □

We summarize the preceding discussion in the following

(5.11) THEOREM. *$H^{\infty-*} = \{H^{\infty-n}, \delta^{\infty-n}\}$ is a cohomology theory on $\mathfrak{L}^2$. Moreover, $H^{\infty-1}(S) \approx H^0(point)$, i.e., the coefficients of the theory $H^{\infty-*}$ coincide with those of the theory H^**

PROOF. (i) The exactness axiom follows from (5.8) and the definition of $H^{\infty-n}$ by passing to the limit with α.

(ii) The homotopy axiom has already been established.

(iii) To show that the excision axiom is fulfilled, let $(X; A, B)$ be a triad in $\mathfrak{L}$, i.e., $A \cup B = X$; if $k : (A, A \cap B) \to (X, B)$ is the inclusion, then so is $\tilde{k} : A \cup C(A \cap B) \to X \cup CB$. Since $(\tilde{k}_\alpha)^*$ is an isomorphism for each α, it follows that so also is $\tilde{k}^* = \varinjlim_\alpha \tilde{k}_\alpha^*$. From this, by considering the commutative diagram

$$\begin{tikzcd}
H^{\infty-n}(X, B) \arrow[r, "\eta", "\cong"'] \arrow[d, "H^{\infty-n}(k)"'] & \overline{H}^{\infty-n-1}(X \cup CB) \arrow[d, "\tilde{k}^*"] \\
H^{\infty-n}(A, A \cap B) \arrow[r, "\eta", "\cong"'] & \overline{H}^{\infty-n-1}(A \cup C(A \cap B))
\end{tikzcd}$$

we infer that $H^{\infty-n}(k)$ is an isomorphism, and the proof of strong excision is complete.

(iv) To show that the dimension axiom is satisfied take the $(\infty - 1)$-cohomology system $\{H^{d(\alpha)-1}(S_\alpha); \Delta_{\alpha\beta}\}$ of the unit sphere S in E; note that if $\alpha \preceq \beta$ is elementary, then $\Delta_{\alpha\beta}$ coincides with the suspension isomorphism. Consequently, for sufficiently large α, we have $H^{\infty-1}(S) \cong H^{d(\alpha)-1}(S_\alpha)$, and our assertion follows. □

(5.12) COROLLARY. *If (X, A) and (Y, B) are two equivalent (or more generally, h-equivalent) pairs in $\mathfrak{L}_E^2$ then for every $n \geq 1$ we have an isomorphism $H^{\infty-n}(X, A) \cong H^{\infty-n}(Y, B)$.* □

6. Miscellaneous Results and Examples

A. The functor $H^{\infty-1}$ and the Leray Schauder degree

Let $A_{r,R} = \{x \in E \mid r \leq \|x\| \leq R\}$, $0 < r < R$. Clearly, $H^{\infty-1}(A_{r,R}; Z) \cong Z$. We identify the above groups with Z by isomorphisms compatible with those induced by the inclusions $A_{r,R} \subset A_{r_1,R_1}$ for $r_1 \leq r < R \leq R_1$; we let 1 denote the generator of Z.

(A.1) Let X be in $\mathfrak{L}_E$, $U_1, U_2, \ldots$ be the (finite or transfinite) family of all bounded components of $E - X$, $x_i \in U_i$, and let $f_i : X \to A_{r,R}$ for some $R > r > 0$ be given by $f_i(x) = x - x_i$ for $x \in X$. Show:

(a) f_i^* does not depend on x_i.

(b) $H^{\infty-1}(X; Z)$ is a free abelian group with generators $\{a_i = f_i^*(1) \mid i = 1, 2, \ldots\}$.

(c) If $g : X \to A_{r,R}$ is a compact field, then $g^*(1) = \sum_{j=1}^k n_j a_{i_j}$.

(A.2) Let U be an open bounded subset of E. Show: To each compact field $f : (\overline{U}, \partial U) \to (E, E - \{0\})$ corresponds an integer $\deg(f)$ with the following properties:

(i) (*Normalization*) If $x_0 \in U$ and $f : (\overline{U}, \partial U) \to (E, E - \{0\})$ is given by $f(x) = x - x_0$, then $\deg(f) = 1$.

(ii) (*Homotopy*) If the fields $f, g \in \mathfrak{C}((\overline{U}, \partial U), (E, E - \{0\}))$ are homotopic, then $\deg(f) = \deg(g)$.

(iii) (*Additivity*) If V_1, V_2 are two open disjoint subsets of U such that $Z(f) \subset U_1 \cup U_2$, then $\deg(f) = \deg(f|V_1) + \deg(f|V_2)$.

(A.3) Show: The integer $\deg(f)$ coincides with the Leray-Schauder degree.

B. Compact fields and α-fields

We use the terminology and notation of Section 1. Given an object X in $\mathfrak{L}$, we set $\mathcal{L}_X = \{\alpha \in \mathcal{L} \mid X_\alpha \neq \emptyset\}$. By $\pi : \mathfrak{L}_E \to \mathbf{Ens}$ we denote the cofunctor assigning to X in $\mathfrak{L}$ the set $\pi(X, E - \{0\})$ of homotopy classes of compact fields $f : X \to E - \{0\}$. We use the following notation:

$$\begin{aligned} \mathfrak{C}(X) &= \mathfrak{C}(X, E - \{0\}), & \pi(X) &= \pi(X, E - \{0\}), \\ \mathfrak{C}_\alpha(X) &= \mathfrak{C}_\alpha(X, E - \{0\}), & \pi_\alpha(X) &= \pi_\alpha(X, E - \{0\}), \\ \mathfrak{C}(X_\alpha) &= \mathfrak{C}(X_\alpha, L_\alpha - \{0\}), & \pi(X_\alpha) &= \pi(X_\alpha, L_\alpha - \{0\}). \end{aligned}$$

(B.1) Let f' be a map in $\mathfrak{C}(X_\alpha)$. Show: There exists an α-field $f \in \mathfrak{C}_\alpha(X)$ such that $f_\alpha = f|X_\alpha = f'$.

(B.2) Let f, g be two α-fields in $\mathfrak{C}_\alpha(X)$, and assume that $h_t' : X_\alpha \to L_\alpha - \{0\}$ is a homotopy joining f_α and g_α. Show: There exists an α-homotopy $h_t \;\; X \to E - \{0\}$ in $\mathfrak{C}_\alpha(X)$ joining f and g, such that $h_t|X_\alpha = h_t'$ for $t \in I$.

(B.3) For X in $\mathfrak{L}$ and $\alpha \in \mathcal{L}_X$, let $\tau_\alpha : \pi_\alpha(X) \to \pi(X_\alpha)$ be given by $[f]_\alpha \mapsto [f_\alpha]$. Show: The map τ_α is bijective.

(B.4) Let $f, g : X \to E - \{0\}$ be two fields such that $f \sim g$ in $\mathfrak{C}(X)$. Show: There exist an α and α-fields $\hat{f}, \hat{g} \in \mathfrak{C}_\alpha(X)$ satisfying (i) $\hat{f} \sim_\alpha \hat{g}$ in $\mathfrak{C}_\alpha(X)$; (ii) $\hat{f} \sim f, \hat{g} \sim g$ in $\mathfrak{C}(X)$.

(B.5) Let $f \in \mathfrak{C}_\alpha(X)$ and $g \in \mathfrak{C}_\beta(X)$. Show: If $f \sim g$ in $\mathfrak{C}(X)$, then there exists a $\gamma \in \mathcal{L}_X$ with $\alpha \preceq \gamma$, $\beta \preceq \gamma$ such that $f \sim_\gamma g$ in $\mathfrak{C}_\gamma(X)$.

(B.6) (*Inessential fields*) Let X be an object in $\mathfrak{L}$. We say that a field f in $\mathfrak{C}(X)$ (respectively in $\mathfrak{C}_\alpha(X)$) is *inessential* if for any Y in $\mathfrak{L}$ containing X, there exists an $\hat{f}$ in $\mathfrak{C}(Y)$ (respectively in $\mathfrak{C}_\alpha(Y)$) such that $\hat{f}|X = f$. Show:

(i) Any $\mathfrak{C}(X)$ (respectively $\mathfrak{C}_\alpha(X)$) contains an inessential field (respectively α-field).

(ii) Any two inessential fields (respectively α-fields) $f', f'' : X \to E - \{0\}$ are homotopic (respectively α-homotopic) in $\mathfrak{C}(X)$ (respectively in $\mathfrak{C}_\alpha(X)$).

(B.7) Let A and B be two objects in $\mathfrak{L}$, and $f : A \to B$ a compact field. Show: If a field $\varphi : B \to E - \{0\}$ is inessential, then so also is the composite $\varphi \circ f : A \to E - \{0\}$.

(B.8) Denote by $\pi : \mathfrak{L} \to$ **Ens** the cofunctor assigning to an object X in $\mathfrak{L}$ the set $\pi(X)$ and to a compact field $f : X \to Y$ the induced map $\pi(f) : \pi(Y) \to \pi(X)$. Show: $\pi : \mathfrak{L} \to$ **Ens*** is an h-cofunctor from $\mathfrak{L}$ to the category **Ens*** of based sets.

[Let 0 denote the *zero homotopy class* in $\mathfrak{C}(X)$ consisting of all inessential fields, and consider 0 as the base point in $\pi(X)$. Prove that if $f : X \to Y$ is a field, then $\pi(f)(0) = 0$.]

(B.9) Prove: The cofunctor $\pi : \mathfrak{L} \to$ **Ens*** is continuous.

C. The homotopy systems $\{\pi_\alpha(X), i_{\alpha\beta}\}$ *and* $\{\pi(X_\alpha), j_{\alpha\beta}\}$

In this subsection X stands for a fixed object in $\mathfrak{L}$.

(C.1) For each $\alpha \in \mathcal{L}$ regard $\pi_\alpha(X)$ as a based set with the 0_α-homotopy class as the base element. For any relation $\alpha \preceq \beta$ in $\mathcal{L}_X$, let $i_{\alpha\beta} : \pi_\alpha(X) \to \pi_\beta(X)$ be given by $[f]_\alpha \mapsto [f]_\beta$. Prove: The family $\{\pi_\alpha(X), i_{\alpha\beta}\}$ is a direct system of based sets over $\mathcal{L}_X$.

(C.2) For each $\alpha \in \mathcal{L}_X$ let $i_\alpha : \pi_\alpha(X) \to \pi(X)$ be defined by $[f]_\alpha \mapsto [f]$. Show:

(i) The family $\{i_\alpha\}$ is a direct family in **Ens*** over $\mathcal{L}_X$.

(ii) The direct limit map

$$i^* = \varinjlim_\alpha i_\alpha : \varinjlim_\alpha \{\pi_\alpha(X), i_{\alpha\beta}\} \to \pi(X)$$

is invertible in **Ens***

(C.3) For each relation $\alpha \preceq \beta$ in $\mathcal{L}_X$, define $j_{\alpha\beta} : \pi(X_\alpha) \to \pi(X_\beta)$ in **Ens*** to be the composite

$$\pi(X_\alpha) \xrightarrow{(\tau_\alpha^\beta)^{-1}} \pi_\alpha(X_\beta) \xrightarrow{i_\alpha^\beta} \pi(X_\beta),$$

where the first map is the inverse of the bijective based map $\tau_\alpha^\beta : \pi_\alpha(X_\beta) \to \pi(X_\alpha)$ given by $[f]_\alpha \mapsto [f_\alpha]$, and the based map $i_\alpha^\beta : \pi_\alpha(X_\beta) \to \pi(X_\beta)$ is defined by $[f]_\alpha \mapsto [f]$. Prove: The family $\{\pi(X_\alpha), j_{\alpha\beta}\}$ is a direct system in **Ens*** over $\mathcal{L}_X$.

(C.4) For an object X in $\mathfrak{L}$, let $\tilde{\pi}(X) = \varinjlim_\alpha \{\pi(X_\alpha), j_{\alpha\beta}\}$ and for a field $f : X \to Y$ in $\mathfrak{L}_0$ define the based map $\tilde{\pi}(f) = \tilde{f} : \tilde{\pi}(Y) \to \tilde{\pi}(X)$ by $\tilde{f} = \varinjlim_\alpha f_\alpha^*$, where $\{f_\alpha : X_\alpha \to Y_\alpha \mid f(X_\alpha) \subset Y_\alpha \text{ for } \alpha \succeq \gamma\}$ and γ are determined by f. Prove: $\tilde{\pi}$ is an h-cofunctor from $(\mathfrak{L}_0, \approx)$ to **Ens***

(C.5) For X in $\mathfrak{L}$ and $\alpha \in \mathfrak{L}_X$, let $\tau_{\alpha X} : \pi_\alpha(X) \to \pi(X_\alpha)$ be given by $[f]_\alpha \mapsto [f_\alpha]$, and for any relation $\alpha \preceq \beta$ in $\mathfrak{L}_X$ consider the commutative diagram

$$\begin{tikzcd} \pi_\alpha(X) \arrow[r, "i_{\alpha\beta}"] \arrow[d, "\tau_{\alpha X}"'] & \pi_\beta(X) \arrow[d, "\tau_{\beta X}"] \\ \pi(X_\alpha) \arrow[r, "j_{\alpha\beta}"] & \pi(X_\beta) \end{tikzcd}$$

Define a family $\tau = \{\tau_X\}$ of based maps $\tau_X : \pi_0(X) \to \tilde{\pi}(X)$ by $\tau_X = \varinjlim_\alpha \tau_{\alpha X}$. Prove:

(i) τ_X is a bijective based map for each X in $\mathfrak{L}$.

(ii) $\tau : \pi_0 \to \tilde{\pi}$ is a natural equivalence of the cofunctors $\pi_0, \tilde{\pi} : \mathfrak{L}_0 \to \mathbf{Ens}^*$

(The above results are due to Gęba-Granas [1965a], [1965b].)

7. Notes and Comments

Finite-codimensional Čech cohomology

The main results of this paragraph are due to Gęba–Granas ([1965a,b], [1967a,b], [1969], [1972]). The $H^{\infty-*}$ theory generalizes the Leray–Schauder degree (cf. "Miscellaneous Results and Examples") and provides a convenient tool for the treatment of various infinite-dimensional problems. Gęba [1978] showed that $H^{\infty-*}$ cohomology can be used to give an "algebraic" proof of the basic bifurcation results in Banach spaces (Theorem (13.1.10)).

Some other uses of $H^{\infty-*}$ cohomology and related results may be found in Colvin [1982], Gel'man [1975], and Dawidowicz [1994]. For some refinements and extensions of the $H^{\infty-*}$ theory the reader is referred to Szulkin [1992] and Abbondandolo [1997]. For finite-codimensional cohomology and duality results on Banach manifolds, see Eells [1961], Mukherjea [1970], and also Eells's survey [1966] and Namioka [1965].

Historical comments

In a letter sent to J. Schauder in 1935, J. Leray asserted the possibility of extending to infinite dimensions Betti groups etc. (cf. Schauder [1936]); the details, however, never appeared in print. The first attempts of extending to the infinite-dimensional case the topological invariants other than the Brouwer degree were made by L. Lusternik.

In the early 1930s, L. Lusternik observed that many problems of the "calculus of variations in the large" can be reduced to the study of the topological properties of "locally linear spaces" (1) (Banach manifolds). Furthermore, soon after the discovery, in 1935, of cohomology groups, he

(1) These spaces, presently known as Banach manifolds, were introduced in Lavrentiev–Lusternik [1935]; the notion was also familiar to Schauder [1936].

became aware that these groups provide an effective tool for the study of "$(\infty - k)$-dimensional cycles" in locally linear spaces. Given a locally linear space M, Lusternik [1943a] defines its *length* $\mathrm{Long}(M)$ ([1]) and proves that $\mathrm{Long}(M) \leq \mathrm{Cat}(M) - 1$, where $\mathrm{Cat}(M)$ is the Lusternik–Schnirelmann category of M.

Consider now the (locally linear) space M_{ab} of rectifiable arcs joining two points a and b of S^2. Using cohomology, Lusternik [1943a], [1943b] outlines a method for showing that $\mathrm{Long}(M_{ab}) = \infty$, and as application proves the existence of infinitely many geodesics joining two given points of S^2. For details concerning the method (which could, in fact, also be applied to other similarly defined spaces), see Lusternik's tract [1947].

Around 1954, the first author of this book learned from his Ph.D. supervisor L. Lusternik the following:

PROBLEM (Lusternik). Is it possible to construct the "$(\infty - k)$-dimensional homology groups" in a Banach space within the framework of the Leray–Schauder theory?

This problem (especially on the technical side) turned out not to be evident. We now describe the main consecutive steps which have led, over a number of years, to a positive solution of the Lusternik problem.

L. Lusternik, 1950

In the 1930s and 1940s, K. Borsuk discovered that a number of geometric results concerning the topology of $\boldsymbol{R}^n$ can be developed by using only simple notions of homotopy theory. To this circle of ideas belonged Borsuk's

([1]) For finite-dimensional manifolds, the notion of length was introduced by Froloff–Elsholz, Mat. Sb. 42 (1935), 637–643.

approach to the disconnection theory in $\boldsymbol{R}^n$ based on describing the structure of the cohomotopy group $\pi^{n-1}(X)$ of a compactum $X \subset \boldsymbol{R}^n$ Granas [1959], [1962] showed that some of these ideas and results can be extended to the framework of compact fields in Banach spaces.

The techniques developed in those papers (and based on the homotopy extension theorem for compact fields) provided tools for extending Borsuk's cohomotopy groups to infinite dimensions.

Let E be a normed linear space, X a closed bounded subset of E, and let $\pi(X) = \pi(X, E - \{0\})$ denote the set of homotopy classes $[f]$ of nonvanishing compact fields $f : X \to E - \{0\}$. We first consider two special finite-dimensional cases.

(a) *The case* $\dim E = 2$. In this case, $E = \boldsymbol{C}$, and $\pi(X)$ can be identified with the set $\pi^1(X)$ of homotopy classes of maps $f : X \to \boldsymbol{C} - \{0\}$. Observe that $\pi^1(X)$ is an abelian group with multiplication $[f][g] = [f \cdot g]$.

In Eilenberg's 1935 thesis, the following result was established:

THEOREM (Eilenberg). *Let $X \subset \boldsymbol{C}$ be compact, $U_1, U_2, \ldots$ the sequence (finite or infinite) of all bounded components of $\boldsymbol{C} - X$, and $z_i \in U_i$. Then for any $f : X \to \boldsymbol{C} - \{0\}$ we have*

$$f(z) \simeq (z - z_{i_1})^{k_1} \ldots (z - z_{i_m})^{k_m}$$

for some integers $k_1, \ldots, k_m$ that do not depend on the choice of z_i in U_i. In other words, the homotopy classes $[(z-z_1)|X]$, $[(z-z_2)|X], \ldots$ are generators of the free abelian group $\pi^1(X)$.

(b) *The case* $\dim E = n$. In this case, $E = \boldsymbol{R}^n$, and $\pi(X)$ coincides with the set $\pi^{n-1}(X)$ of homotopy classes $[f]$ of maps $f : X \to \boldsymbol{R}^n - \{0\}$. Given $f, g : X \to \boldsymbol{R}^n - \{0\}$ such that for each $x \in X$, either $f(x) = 1$ or $g(x) = 1$ (where 1 denotes the point $(1, 0, \ldots, 0)$ of $\boldsymbol{R}^n$), the product $f(x) \cdot g(x)$ is well defined if we assume the obvious rule $1 \cdot p = p = p \cdot 1$ (for each $p \in \boldsymbol{R}^n$); thus, in this particular case, f and g are "multipliable". By a theorem of Borsuk [1936], for each pair $f, g : X \to \boldsymbol{R}^n - \{0\}$, there exists a pair of multipliable maps $\widehat{f}$, $\widehat{g}$ such that $[f] = [\widehat{f}]$ and $[g] = [\widehat{g}]$. We set $[f]\ [g] = [\widehat{f} \cdot \widehat{g}]$; this definition does not depend on the choice of $\widehat{f}$, $\widehat{g}$ and in this case, as was shown by Borsuk, $\pi^{n-1}(X)$ becomes an abelian group.

The following extension of the Eilenberg theorem due to Borsuk [1950] describes the structure of the cohomotopy group $\pi^{n-1}(X)$.

THEOREM (Borsuk). *Let $X \subset \boldsymbol{R}^n$ be compact, $U_1, U_2, \ldots$ the sequence (finite or infinite) of all bounded components of $\boldsymbol{R}^n - X$, and $x_i \in U_i$. Then the homotopy classes $[(x - x_1)|X]$, $[(x - x_2)|X], \ldots$ are generators of the free abelian (cohomotopy) group $\pi^{n-1}(X)$*

(c) *The case* $\dim E = \infty$. In this case $\pi(X)$ is the set $\pi^{\infty-1}(X)$ of homotopy classes of compact fields $f : X \to E - \{0\}$.

The following result (Granas [1962]) extends Borsuk's theorem to the framework of compact fields:

THEOREM (Granas). *Let $X \subset E$ be closed and bounded, $U_1, U_2, \dots$ the sequence (finite or transfinite) of all bounded components of $E - X$, and $x_i \in U_i$. Then*

(i) *$\pi^{\infty-1}(X)$ admits the structure of an abelian group,*

(ii) *the homotopy classes $[(x-x_1)|X]$, $[(x-x_2)|X]$, $\dots$ are the generators of $\pi^{\infty-1}(X)$.*

The last theorem suggested the study of "the groups $\pi^{\infty-n}(X)$" and provided a starting point for constructing the first full-fledged infinite-dimensional cohomology. In his 1962 thesis, K. Gęba constructed in fact such a theory by extending to Banach spaces the theory of cohomotopy groups of Borsuk as well as the Spanier–Whitehead duality.

To describe the main results of this theory (Gęba [1962], [1964]), we introduce some notation. Let $E = E^\infty$ be an infinite-dimensional normed linear space together with a sequence $\{E^{\infty-n} \oplus E^n\}$ of direct sum decompositions of E such that

(i) $E^0 \subset E^1 \subset E^2 \subset \cdots$,

(ii) $E^\infty \supset E^{\infty-1} \supset E^{\infty-2} \supset \cdots$,

(iii) $\operatorname{codim} E^{\infty-n} = \dim E^n = n$.

Let $O = \{O_\alpha\}$ be a fixed orientation in E. For each $n \geq 1$ choose an isomorphism $l_n : E^n \to R^n$ representing the orientation of E^n such that $l_n(x) = l_{n+1}(x)$ for $x \in E^n$, and let $U^{\infty-n} = E - E^{n-1}$, $V^{\infty-n} = E - l_n^{-1}(R^n_+)$.

THEOREM (Gęba). *Let $\mathfrak{L} = \mathfrak{L}_E$ be the Leray–Schauder category associated with E.*

1° *There exists a sequence $\{\pi^{\infty-n}\}_{n=1}^{\infty}$ of contravariant functors $\pi^{\infty-n} : \mathfrak{L}^2 \to$ Ab such that $\pi^{\infty-n}(X, A)$ is the set of homotopy classes of compact fields $f \in \mathfrak{C}((X, A), (U^{\infty-n}, V^{\infty-n}))$.*

2° *There exists a family of natural transformations $\delta^{\infty-n} : \pi^{\infty-n}(A) \to \pi^{\infty-n+1}(X, A)$ such that the following properties hold:*

(i) (Homotopy) *If the fields $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then $\pi^{\infty-n}(f_0) = \pi^{\infty-n}(f_1)$.*

(ii) (Exactness) *For each (X, A) in $\mathfrak{L}^2$, the sequence*

$$\to \pi^{\infty-n}(X, A) \to \pi^{\infty}(X) \to \pi^{\infty-n}(A) \xrightarrow{\delta^{\infty-n}} \pi^{\infty-n+1}(X, A) \to \cdots$$

is exact, where all unmarked maps are induced by inclusions.

(iii) (Strong excision) *The inclusion* $e : (A, A \cap B) \to (A \cup B, B)$ *induces an isomorphism*

$$\pi^{\infty-n}(e) : \pi^{\infty-n}(A \cup B, B) \to \pi^{\infty-n}(A, A \cap B)$$

for all $n = 1, 2, \ldots$.

3° *The graded group* $\{\pi^{\infty-n}(S)\}_{n=1}^{\infty}$, *where* $S = \{x \in E \mid \|x\| = 1\}$, *is* $\{\Sigma_{m+n-1}(S^m)\}_{n=1}^{\infty}$, *where* $\Sigma_{m+n-1}(S^m)$ *is the stable homotopy group of* S^m.

The technique used in the proof of this theorem suggested how to proceed with the construction of the $H^{\infty-*}$ theory; in the joint work of Gęba–Granas between 1967 and 1969, the solution of the Lusternik problem was completed.

Generalized degree

Let K denote a closed ball in E with center 0, let $S = \partial K$ be its boundary, and $U^{\infty-k} = E - E^{k-1}$

For each $n = 1, 2, \ldots$ we define the class $\mathfrak{C}^{(n)}$ of compact fields by the condition $f \in \mathfrak{C}^{(n)}$ if and only if $f : (K, \partial K) \to (E^{\infty-n}, E^{\infty-n} - \{0\})$. If $f \in \mathfrak{C}^{(n)}$, we let $\hat{f} = f|S : S \to E^{\infty-n} - \{0\}$ and $j : E^{\infty-n} - \{0\} \to U^{\infty-n}$ be the inclusion.

DEFINITION. Let f be in $\mathfrak{C}^{(n)}$ We define the *generalized degree* DEG(f) of f as the element of the Gęba cohomotopy group $\pi^{\infty-n}(S)$ given by

$$\mathrm{DEG}(f) = [j\hat{f}],$$

where $[j\hat{f}]$ is the homotopy class of the field $j\hat{f} : S \to U^{\infty-n}$.

THEOREM. *Let f be in $\mathfrak{C}^{(n)}$. If* DEG(f) *is a nontrivial element of the group* $\pi^{\infty-n}$, *then* $Z(f) \neq \emptyset$.

Since the group $\pi^{\infty-n}(S) \cong \Sigma^{\infty-n}(S)$ is isomorphic to the stable homotopy group of spheres $\Sigma_{m+n-1}(S^m)$, the above theorem can only be of interest when the latter is nontrivial (this is known to be the case, for example, for $n = 1, 2, 3, 4, 7, 8$, but not for $n = 5, 6$).

For some applications of the generalized degree to nonlinear problems see Nirenberg [1970].

General infinite-dimensional cohomology theories

Let $\mathfrak{L} = \mathfrak{L}_E$ be the Leray–Schauder category. By a *cohomology theory* $h^{\infty-*}$ on $\mathfrak{L}$ is meant a sequence $\{h^{\infty-n}\}_{n=1}^{\infty}$ of contravariant functors $h^{\infty-n} : \mathfrak{L}^2 \to$ Ab together with a sequence of natural transformations $\delta^{\infty-n} : h^{\infty-n}(A) \to$

$h^{\infty-n+1}(X, A)$ satisfying the homotopy, exactness, and strong excision axioms; the graded group $\{h^{\infty-n-1}(S)\}$, where S is the unit sphere in E, is the *group of coefficients* of the theory. A systematic and unified treatment of such theories, based on the important work of G.W. Whitehead [1962], can be found in Gęba–Granas [1973].

We list some of the central results of the general cohomology theory.

(I) THEOREM. *To any cohomology theory h^* on the category of polyhedra corresponds a cohomology theory $h^{\infty-*}$ on $\mathfrak{L}$ with the same group of coefficients; moreover, the assignment $h^* \mapsto h^{\infty-*}$ is natural with respect to the theories.*

Thus, in particular, to any spectrum [1] $\mathbb{A}$ in the sense of G.W. Whitehead corresponds a theory $\{h^{\infty-*}, \mathbb{A}\}$ called the *cohomology theory on $\mathfrak{L}$ with coefficients in $\mathbb{A}$*. If $\mathbb{A} = \mathbb{K}(\pi)$ is the Eilenberg–MacLane spectrum, then we get the "ordinary Čech cohomology" $H^{\infty-*}(\cdot; \pi)$ on $\mathfrak{L}$ (which was described in §18) with coefficients in π. If $\mathbb{A}$ is the sphere spectrum $\mathbb{S}$, then the corresponding theory, denoted by $\Sigma^{\infty-*}$, is called the *stable cohomotopy* on $\mathfrak{L}$. The Hopf–Hurewicz map $h : \mathbb{S} \to \mathbb{K}(Z)$ between the spectra induces a natural transformation h^* from $\Sigma^{\infty-*}$ to $H^{\infty-*}(\cdot; Z)$.

(II) THEOREM. *There exists a natural equivalence between the functors $\pi^{\infty-n}$ and $\Sigma^{\infty-n}$ from $\mathfrak{L}^2$ to* **Ab**.

This theorem implies that the stable cohomotopy theory $\Sigma^{\infty-*}$ can be in fact identified with the cohomotopy theory $\pi^{\infty-*}$

(III) THEOREM (Duality theorem in E). *Let $\mathbb{A}$ be either the spectrum of spheres $\mathbb{S}$ or the Eilenberg–MacLane spectrum $\mathbb{K}(\pi)$ and let $h^{\infty-n}(\cdot; \mathbb{A})$ (respectively $h_n(\cdot; \mathbb{A})$) be the cohomology (respectively homology) with coefficients in $\mathbb{A}$. Then*

(i) *for each pair (X, Y) in $\mathfrak{L}$ and $V = E - Y$, $U = E - X$ there exists a duality map*

$$D : h^{\infty-n}(X, Y; \mathbb{A}) \to h_n(V, U; \mathbb{A}).$$

---

[1] Let $\mathbf{CW}_*$ be the category of based CW-complexes. By a *spectrum* $\mathbb{A}$ is meant a sequence $\{A_n\}$ of objects of $\mathbf{CW}_*$ together with a sequence of maps $\alpha_n : \Sigma A_n \to A_{n+1}$ in $\mathbf{CW}_*$, where Σ is reduced suspension. The simplest example is provided by the spectrum of spheres $\mathbb{S} = \{S^n, \alpha_n\}$, where $\alpha_n : \Sigma S^n \to S^{n+1}$ is the natural identification. To any spectrum $\mathbb{A}$ there corresponds a homology $h_*(\cdot; \mathbb{A})$ and cohomology $h^*(\cdot; \mathbb{A})$ on $\mathbf{CW}_*$ with coefficients in $\mathbb{A}$. If $\mathbb{A} = \mathbb{K}(\pi)$ is the Eilenberg–MacLane spectrum corresponding to an abelian group π, then $h_*(\cdot; \mathbb{K}(\pi))$, $h^*(\cdot; \mathbb{K}(\pi))$ are naturally isomorphic to the singular homology $H_*(\cdot; \pi)$ and singular cohomology $H^*(\cdot; \pi)$ with coefficients in π. The homology and cohomology theories with coefficients in $\mathbb{S}$ are isomorphic to the stable homotopy and cohomotopy theories, respectively. For details see G.W. Whitehead [1962] and Switzer's book [1975].

(ii) *D maps the cohomology sequences of a triple (X, Y, Z) into the homology sequences of the complementary triple $(W, V, U) = (E - Z, E - Y, E - X)$, i.e., the following diagram commutes:*

$$\begin{array}{ccccccccc}
\cdots \to & h^{\infty-n}(X,Y;\mathbb{A}) & \to & h^{\infty-n}(X,Z;\mathbb{A}) & \to & h^{\infty-n}(Y,Z;\mathbb{A}) & \to & h^{\infty-n+1}(X,Y;\mathbb{A}) & \to \cdots \\
& \downarrow D & & \downarrow D & & \downarrow D & & \downarrow D & \\
\cdots \to & h_n(V,U;\mathbb{A}) & \to & h_n(W,U;\mathbb{A}) & \to & h_n(W,V;\mathbb{A}) & \to & h_{n-1}(V,U;\mathbb{A}) & \to \cdots
\end{array}$$

(iii) *D is natural with respect to the Hopf–Hurewicz map of spectra $h : \mathbb{S} \to \mathbb{K}(Z)$, i.e., the following diagram commutes:*

$$\begin{array}{ccc}
\Sigma^{\infty-n}(X,Y) & \xrightarrow{h^*} & H^{\infty-n}(X,Y;Z) \\
D\downarrow & & \downarrow D \\
\Sigma_n(V,U) & \xrightarrow{h_*} & H_n(V,U;Z)
\end{array}$$

The duality combined with the Hurewicz theorem in $\mathbb{S}$-theory yields the following:

(IV) THE HOPF THEOREM. *For any pair in $\mathfrak{L}$ the first nonvanishing stable cohomotopy group is isomorphic to the first nonvanishing cohomology group over Z. More precisely, we have*

(i) *$\pi^{\infty-q}(X,Y) \cong \Sigma^{\infty-q}(X,Y) = 0 \Leftrightarrow H^{\infty-q}(X,Y;Z) = 0$ for any $0 \le q < n$,*

(ii) *if $\pi^{\infty-q}(X,Y) \cong \Sigma^{\infty-n}(X,Y) = 0$ for $0 \le q < n$, then the Hopf map $h^* : \Sigma^{\infty-n}(X,Y) \to H^{\infty-n}(X,Y;Z)$ is an isomorphism.*

Codimension

Let Y be a closed bounded subset of E, and let $U \subset E$ be open. Call U an *extension object* of Y provided given any closed subset A of Y, any compact field $f \in \mathfrak{C}(A, U)$ admits an extension $f \in \mathfrak{C}(Y, U)$. We let

$$\mathfrak{L}(U^{\infty-k}) = \{Y \in \mathfrak{L} \mid U^{\infty-k} = E - E^{k-1} \text{ is an extension object for } Y\}.$$

Let $X \subset E$ be closed and bounded, and let $H^{\infty-*}(\ ;G)$ be the cohomology theory on $\mathfrak{L}$ with coefficients in an abelian group G. We say that:

(a) the *codimension of X with respect to E* is equal to n (we write $\operatorname{Codim} X = n$) if:
   (i) $X \in \mathfrak{L}(U^{\infty-k})$ for each $k \in [n-1]$,
   (ii) $X \notin \mathfrak{L}(U^{\infty-k})$ for $k = n$;

(b) the *cohomological codimension of X with respect to E* is equal to n (we write $\operatorname{Codim}_G X = n$) provided n is the smallest integer such that $H^{\infty-n}(X, A; G) \neq 0$ for some closed subset A of X.

We note that:

(i) if X and Y are two equivalent objects of the category $\mathfrak{L}$, then $\operatorname{Codim} X = \operatorname{Codim} Y$ and $\operatorname{Codim}_G X = \operatorname{Codim}_G Y$,

(ii) in the finite-dimensional case the definition of codimension coincides with a theorem of P Alexandroff characterizing the dimension of compacta by maps into S^n.

The main theorems of the previous subsection yield the following result:

THEOREM. *Let X be an object of the Leray-Schauder category $\mathfrak{L}$.*

(a) $\operatorname{Codim} X = \operatorname{Codim}_{\mathbb{Z}} X$.

(b) *If* $\operatorname{Codim} X \geq 2$, *then* $b_0(E-X) = 0$ (¹), *i.e., X does not disconnect the space E.*

(c) (Phragmén-Brouwer theorem) *If $(Y; Y_1, Y_2)$ is an additive triad in $\mathfrak{L}$ and* $\operatorname{Codim}(Y_1 \cap Y_2) \geq 2$, *then*

$$b_0(E - Y) = b_0(E - Y_1) + b_0(E - Y_2).$$

We remark that (a) is analogous to the well known "fundamental theorem in dimension theory" due to Alexandroff.

L. Lusternik [1940a,b], in the context of his studies on variational problems in "locally linear spaces" (= Banach manifolds), introduced the notion of codimension. Let M be a Banach manifold. A submanifold $N \subset M$ is said to be of *codimension* k (or dimension $\infty - k$) if N can be locally determined by a system of equations $\Phi_i(x) = 0$, $i \in [k]$, with linearly independent differentiable functionals $\Phi_1, \dots, \Phi_k$.

Fix $n \geq 1$ and consider the differential equation

$$(*) \qquad y'' + \varphi(x, y, y') = \lambda y, \quad y(a) = y(b) = 0, \quad \int_a^b y^2(x)\, dx = 1,$$

where φ is continuous and $\varphi(x, y, z) = -\varphi(x, -y, -z)$.

If $\varphi \equiv 0$, there exists a λ for which $(*)$ has a solution that vanishes exactly n times inside $[a, b]$. In 1941, L. Lusternik gave a procedure that allows extending this result to the case $\varphi \not\equiv 0$ and established the following

THEOREM (Lusternik [1941]). *Under suitable growth conditions on φ:*

(i) *there exists a countable sequence $\lambda_1, \lambda_2, \dots$ of eigenvalues of $(*)$,*

(ii) *to each eigenvalue λ_k there corresponds an eigenfunction y_k that vanishes exactly n times inside $[a, b]$.*

Lusternik's proof of this theorem was based on using the continuation method in conjunction with the following important property of codimension: *If M is a connected Banach manifold and $N \subset M$ a submanifold with* $\operatorname{Codim} N \geq 2$, *then N does not disconnect M.*

(¹) Here $b_0(E - X)$ denotes the number of bounded components of $E - X$.

§19. Vietoris Fractions and Coincidence Theory

Given two maps $f, s : \Gamma \to X$ between spaces Γ and X, by a *coincidence point* for the pair f, s is meant a point $y_0 \in \Gamma$ for which $f(y_0) = s(y_0)$. In this paragraph, using Čech homology and the well known Vietoris theorem as the main tools, we establish a general coincidence theorem which includes the Lefschetz fixed point theorem for compact maps of ANRs given in §15. The paragraph ends with applications to the fixed point theory of set-valued maps.

1. Preliminary Remarks

Throughout the entire paragraph only metrizable spaces are considered, and we always use Čech homology with compact supports over the rationals Q; when no confusion can arise, the Q will be omitted in the notation, so that the graded homology of X over Q is denoted simply by $H_*(X) = \{H_n(X)\}$. In this section we gather the relevant facts of the Čech theory that will be needed.

Given two spaces X and Y, we say that a map $s : X \to Y$ is *inverse acyclic* if the fiber $s^{-1}(y)$ is acyclic ([1]) for each y. Note that an inverse acyclic map is necessarily surjective.

The main results of this paragraph rely heavily on the following theorem:

(1.1) THEOREM (Vietoris). *Let $s : X \to Y$ be an inverse acyclic map between compact metric spaces X and Y. Then the induced homomorphism $s_* : H_*(X) \to H_*(Y)$ is an isomorphism.* □

Assume that Γ is a compact metric space, P a finite polyhedron, and let $f : \Gamma \to P$, $s : \Gamma \to P$ be continuous maps with s inverse acyclic. We first note that $s_*^{-1} : H_*(P) \to H_*(\Gamma)$ is well defined, in view of the Vietoris theorem, and therefore $f_* s_*^{-1} : H_*(P) \to H_*(P)$. We define the *Lefschetz number* $\lambda[(f, s)]$ of the pair (f, s) by

$$\lambda[(f, s)] = \lambda(f_* s_*^{-1}),$$

where $\lambda(f_* s_*^{-1})$ is the ordinary Lefschetz number of the endomorphism $f_* s_*^{-1}$; clearly the latter is defined, because the homology of P is of finite type.

With this terminology, we are now in a position to state the basic coincidence theorem:

([1]) Recall that X is *acyclic* if: (i) X is nonempty, (ii) $H_q(X) = 0$ for $q \geq 1$, (iii) the reduced 0th homology group $\widetilde{H}_0(X)$ is 0.

(1.2) THEOREM (Eilenberg–Montgomery). *Let Γ be a compact metric space, P a finite polyhedron, and let $f : \Gamma \to P$, $s : \Gamma \to P$ be continuous maps with s inverse acyclic. If $\lambda[(f,s)] \neq 0$, then the maps f and s have a coincidence.* □

The original proof of the Eilenberg–Montgomery theorem is quite elementary and closely analogous to the proof of the Lefschetz–Hopf theorem given in §9; it uses, however, the notions of Vietoris cycles and the chain approximation technique, which are not discussed in this book.

Observe that if we take in (1.2) $\Gamma = P$ and $s = \mathrm{id}$, then the integer $\lambda[(f,s)]$ coincides with the Lefschetz number $\lambda(f)$; thus Theorem (1.2) includes the Lefschetz–Hopf fixed point theorem for polyhedra.

It is our objective in this paragraph to extend the Eilenberg–Montgomery theorem to wider classes of maps and spaces. We now describe the approach and the main steps of our discussion:

(a) Using the concept of a Vietoris map, we first enlarge the category of metrizable spaces by adding new morphisms (Vietoris fractions).

(b) We extend the Čech homology functor over this larger category of Vietoris fractions.

(c) Assuming the Eilenberg–Montgomery coincidence theorem for polyhedra to be known, we derive the main coincidence results by using essentially the same algebraic and factorization techniques that were previously employed in the Lefschetz fixed point theory for compact maps.

2. Category of Fractions

We begin by collecting some simple notions of category theory that will be needed.

(2.1) DEFINITION. Let $\mathbf{T}$ be a category and $\mathscr{V}$ a family of morphisms in $\mathbf{T}$ such that

(F.1) the identities of $\mathbf{T}$ are all in $\mathscr{V}$,

(F.2) if $s : X \to Y$ and $t : Y \to Z$ are in $\mathscr{V}$, then so is ts,

(F.3) any diagram in $\mathbf{T}$

$$\begin{tikzcd} \circ \arrow[r, dashed] \arrow[d, dashed] & \circ \arrow[d, Rightarrow, "s"] \\ \circ \arrow[r, "f"] & \circ \end{tikzcd}$$

with s in $\mathscr{V}$ can be completed to a commutative diagram in $\mathbf{T}$

$$\begin{tikzcd} \circ \arrow[r, "f'"] \arrow[d, Rightarrow, "s'"'] & \circ \arrow[d, Rightarrow, "s"] \\ \circ \arrow[r, "f"] & \circ \end{tikzcd}$$

where s' is in $\mathscr{V}$ and s', f' is a pull-back of the pair s, f (for the definition of pull-back, see Appendix).

When (F.1)–(F.3) are satisfied, we say that the family $\mathscr{V}$ admits a *calculus of fractions.*

Given objects X and Y in $\mathbf{T}$, let $\mathscr{D}(X,Y)$ denote the set of pairs (f,s) of morphisms

$$X \overset{s}{\Longleftarrow} \circ \xrightarrow{f} Y,$$

where the morphism s "in the wrong direction" belongs to $\mathscr{V}$. Given two pairs $(f,s),(g,t)$ in $\mathscr{D}(X,Y)$, we write $(f,s) \sim (g,t)$ if there exists a commutative diagram

$$(*) \qquad \begin{tikzcd} & \circ \arrow[dl, Rightarrow, "s"'] \arrow[dr, "f"] \arrow[dd, "u"] & \\ X & & Y \\ & \circ \arrow[ul, Rightarrow, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

with an invertible morphism u in $\mathbf{T}$.

It is clear that $\sim$ is an equivalence relation in $\mathscr{D}(X,Y)$; the equivalence class of (f,s) is denoted by $f|s$. We let $\mathscr{F}(X,Y)$ denote the corresponding set of equivalence classes; its elements are called *fractions* from X to Y.

(2.2) DEFINITION. We now define a new category $\mathscr{V}(\mathbf{T})$ as follows:

(i) the objects of $\mathscr{V}(\mathbf{T})$ coincide with those of $\mathbf{T}$,

(ii) given X and Y in $\mathscr{V}(\mathbf{T})$, the set of morphisms from X to Y in $\mathscr{V}(\mathbf{T})$ is defined as the set $\mathscr{F}(X,Y)$ of fractions from X to Y,

(iii) the composition law

$$\mathscr{F}(X,Y) \times \mathscr{F}(Y,Z) \to \mathscr{F}(X,Z)$$

is given by $[g|t][f|s] = gf'|st'$, where $t' \in \mathscr{V}$ and t', f' is a pull-back of the pair t, f making the following diagram commutative:

$$\begin{tikzcd} \circ \arrow[r, "f'"] \arrow[d, Rightarrow, "t'"'] & \circ \arrow[r, "g"] \arrow[d, Rightarrow, "t"] & Z \\ \circ \arrow[r, "f"] \arrow[d, Rightarrow, "s"'] & Y & \\ X & & \end{tikzcd}$$

Conditions (F.1)–(F.3) imply that the composition law is well defined and makes $\mathscr{V}(\mathbf{T})$ a category called the *category of fractions* of $\mathbf{T}$ by $\mathscr{V}$.

3. Vietoris Maps and Fractions

Let now $\mathbf{T}$ denote the category of metrizable spaces. In this section we introduce a family $\mathscr{V}$ of morphisms in $\mathbf{T}$ (called Vietoris maps) that admits a calculus of fractions and consider the corresponding category of fractions.

(3.1) DEFINITION. Given X and Y in $\mathbf{T}$, we say that $s : X \to Y$ is a *Vietoris map* (we write $s : X \Rightarrow Y$) if (i) s is proper and (ii) s is inverse acyclic; we denote by $\mathscr{V}$ the family of Vietoris maps.

The following theorem asserting that the family $\mathscr{V}$ admits a calculus of fractions is of fundamental importance.

(3.2) THEOREM. *The family $\mathscr{V}$ has the following properties*:
(F.1) *the identities of $\mathbf{T}$ are in $\mathscr{V}$*,
(F.2) *if $s : X \to Y$ and $t : Y \to Z$ are in $\mathscr{V}$, then so is $ts : X \to Z$*,
(F.3) *the pull-back of a Vietoris map is a Vietoris map.*

PROOF. (F.1) is obvious.

(F.2): Assuming that s and t are in $\mathscr{V}$, we note that ts is proper. To prove that $(ts)^{-1}(z)$ is acyclic for each $z \in Z$, note that since s is inverse acyclic and sends $s^{-1}(t^{-1}(z)) = (ts)^{-1}(z)$ to $t^{-1}(z)$, the restriction map $\widehat{s} = s|s^{-1}(t^{-1}(z))$ induces an isomorphism $\widehat{s}_* : H_*((ts)^{-1}(z)) \to H_*(t^{-1}(z))$ by (1.1). Since $t^{-1}(z)$ is acyclic, the required conclusion follows.

(F.3): We have to complete in a suitable manner the diagram

$$\begin{tikzcd} \circ \arrow[r, dashed] \arrow[d, dashed] & \Gamma' \arrow[d, Rightarrow, "s"] \\ \Gamma \arrow[r, "f"] & Y \end{tikzcd}$$

Let

$$\Gamma \times_Y \Gamma' = \{(x, y) \in \Gamma \times \Gamma' \mid f(x) = s(y)\}$$

be the *fiber product* of f, s, and let the pair of maps $f' : \Gamma \times_Y \Gamma' \to \Gamma'$ and $s' : \Gamma \times_Y \Gamma' \to \Gamma$ given by $f'(x, y) = y$, $s'(x, y) = x$ for $(x, y) \in \Gamma \times_Y \Gamma'$ be the *pull-back* of the pair (f, s). It can be easily seen that the diagram

$$\begin{tikzcd} \Gamma \times_Y \Gamma' \arrow[r, "f'"] \arrow[d, Rightarrow, "s'"'] & \Gamma' \arrow[d, Rightarrow, "s"] \\ \Gamma \arrow[r, "f"] & Y \end{tikzcd}$$

commutes and also that s' is proper and inverse acyclic. □

Using the terminology and notation of Section 2, we let $\mathscr{V}(\mathbf{T})$ denote the category of Vietoris fractions of $\mathbf{T}$ by $\mathscr{V}$, and given two spaces X and Y we denote by $\mathscr{F}(X, Y)$ the set of Vietoris fractions from X to Y

To compose two Vietoris fractions $\varphi = f|s \in \mathscr{F}(X, Y)$ and $\psi = g|t \in \mathscr{F}(Y, Z)$ we write a commutative diagram:

$$\begin{tikzcd} \Gamma \times_Y \Gamma' \arrow[r, "f'"] \arrow[d, Rightarrow, "t'"'] & \Gamma' \arrow[r, "g"] \arrow[d, Rightarrow, "t"] & Z \\ \Gamma \arrow[r, "f"] \arrow[d, Rightarrow, "s"'] & Y & \\ X & & \end{tikzcd}$$

in which $\Gamma \times_{Y} \Gamma'$ is the fiber product of f, t, and f', t' is the pull-back of f, t. Now the composite $\varphi = (g|t)(f|s)$ is equal to $gf'|st'$.

In what follows we regard the category $\mathbf{T}$ of metrizable spaces as a subcategory of the Vietoris fractions $\mathscr{V}(\mathbf{T})$.

In order to define a homomorphism of homology groups induced by a fraction, we need to extend Theorem (1.1) to abitrary Vietoris maps.

(3.3) THEOREM (Vietoris mapping theorem). *Let $s : X \to Y$ be a Vietoris map between spaces X and Y in $\mathbf{T}$. Then the induced map $s_* : H_*(X) \to H_*(Y)$ is an isomorphism.*

PROOF. We recall the definition of the Čech homology functor with compact supports. Given X in $\mathbf{T}$, let $\mathcal{D}_X = \{X_\alpha\}$ denote the set of all compact subspaces of X partially ordered by inclusion. Assigning to each such X_α its graded Čech homology $H_*(X_\alpha)$ and to each pair $X_\alpha \subset X_\beta$ the homomorphism $(i_{\alpha\beta})_* : H_*(X_\alpha) \to H_*(X_\beta)$ induced by inclusion, we obtain the direct system $\{H_*(X_\alpha), (i_{\alpha\beta})_*\}$ of graded vector spaces.

Given $s : X \to Y$ in $\mathbf{T}$ and letting $\mathcal{D}_Y = \{Y_\lambda\}$ denote the partially ordered set of all compact subspaces of Y, we see that the assignment $X_\alpha \mapsto s(X_\alpha)$ determines an order preserving map $\mathcal{D}_X \to \mathcal{D}_Y$. This implies that the family $\{s_\alpha\}$, where $s_\alpha : X_\alpha \to s(X_\alpha)$ is defined by s, determines a map of direct systems

$$\{(s_\alpha)_*\} : \{H_*(X_\alpha), (i_{\alpha\beta})_*\} \to \{H_*(Y_\lambda), (j_{\lambda\mu})_*\}.$$

The functor H from $\mathbf{T}$ to the category **GrVect** of graded vector spaces is defined by taking direct limits:

$$H(X) = \varinjlim_{\alpha \in \mathcal{D}_X} \{H_*(X_\alpha), (i_{\alpha\beta})_*\} \quad \text{for } X \text{ in } \mathbf{T},$$

$$H(s) = s_* = \varinjlim_{\alpha \in \mathcal{D}_X} (s_\alpha)_* \quad \text{for } s : X \to Y \text{ in } \mathbf{T}.$$

We now start the proof and assume that $s : X \to Y$ is in $\mathscr{V}$ We will show that $s_* = \varinjlim (s_\alpha)_*$ is an isomorphism. To this end, for each $X_\alpha \in \mathcal{D}_X$

let $\widehat{X}_\alpha = s^{-1}s(X_\alpha)$ and observe that s sends each $\widehat{X}_\alpha$ onto $s(X_\alpha)$; moreover, since the restriction map $\widehat{s}_\alpha = s|\widehat{X}_\alpha$ from $\widehat{X}_\alpha$ onto $s(X_\alpha)$ is inverse acyclic, the induced map $(\widehat{s}_\alpha)_* : H_*(\widehat{X}_\alpha) \to H_*(s(X_\alpha))$ is an isomorphism by the Vietoris theorem (1.1).

Next we note that the set $\widehat{\mathcal{D}} = \{\widehat{X}_\alpha \mid X_\alpha \in \mathcal{D}_X\}$ is cofinal in $\mathcal{D}_X$ and this gives $s_* = \varinjlim (s_\alpha)_* = \varinjlim (\widehat{s}_\alpha)_*$. Now because each $(\widehat{s}_\alpha)_*$ is an isomorphism, the conclusion follows. □

4. Induced Homomorphisms and the Lefschetz Number

(4.1) DEFINITION. Let X, Y be in $\mathbf{T}$, and let $\varphi = f|s \in \mathscr{F}(X,Y)$ be a fraction. The homomorphism induced by φ is defined as the composite $H_*(X) \xrightarrow{s_*^{-1}} H_*(\Gamma) \xrightarrow{f_*} H_*(Y)$, and we write $f_*s_*^{-1} = H_*(\varphi) = \varphi_*$.

It is easily seen, by considering the commutative diagram $(*)$ appearing in the definition of the equivalence relation $\sim$ in $\mathcal{D}(X,Y)$, that the definition of $\varphi_* = (f|s)_* : H_*(X) \to H_*(Y)$ does not depend on the choice of a representative $f|s$ of φ, and thus the above definition makes sense; we also note that if $\varphi = f|\operatorname{id}_X = f$, then $f_* = \varphi_*$.

We now define a function $\overline{H}_* : \mathscr{V}(\mathbf{T}) \to \mathbf{GrVect}$ by

$$\overline{H}_*(X) = H_*(X) \qquad \text{for } X \in \mathscr{V}(\mathbf{T}),$$
$$\overline{H}_*(\varphi) = \varphi_* \qquad \text{for } \varphi \in \mathscr{F}(X,Y) \in \mathscr{V}(\mathbf{T}).$$

(4.2) PROPOSITION. *The function $\overline{H}_* : \mathscr{V}(\mathbf{T}) \to \mathbf{GrVect}$ is a functor extending the Čech homology functor $H_* : \mathbf{T} \to \mathbf{GrVect}$.*

PROOF. Let $\varphi = f|s \in \mathscr{F}(X,Y)$ and $\psi = g|t \in \mathscr{F}(Y,Z)$ be two fractions. Clearly to establish our assertion we need only show that $(\psi\varphi)_* = \psi_*\varphi_*$. To this end consider a commutative diagram

$$\begin{tikzcd}
\Gamma \times_Y \Gamma' \arrow[r, "f'"] \arrow[d, Rightarrow, "t'"'] & \Gamma' \arrow[r, "g"] \arrow[d, Rightarrow, "t"] & Z \\
\Gamma \arrow[r, "f"] \arrow[d, Rightarrow, "s"'] & Y & \\
X & &
\end{tikzcd}$$

By definition of the homomorphism induced by a fraction we have

$$\begin{aligned}(\psi\varphi)_* &= [(g|t)(f|s)]_* = (gf'|st')_* = (gf')_*(st')_*^{-1} = (g_*f'_*)(t')_*^{-1}s_*^{-1} \\ &= [g_*t_*^{-1}] \circ [f_*s_*^{-1}] = [(g|t)_*] \circ [(f|s)_*] = \psi_*\varphi_*.\end{aligned}$$ □

Using the functor H we now define for the Vietoris fractions some notions analogous to those introduced for ordinary maps in §15.

(4.3) DEFINITION. Let X, Y be in $\mathbf{T}$, and let $\varphi = f|s \in \mathscr{F}(X,Y)$ be a fraction. We say that $\varphi = f|s \in \mathscr{F}(X,X)$ is a *Lefschetz fraction* if the induced map $\varphi_* : H_*(X) \to H_*(X)$ is a Leray endomorphism; for such a φ we define its *generalized Lefschetz number* by $\Lambda(\varphi) = \Lambda(\varphi_*)$.

With this terminology, we have the following simple general result that is frequently used:

(4.4) LEMMA. *Let $\alpha \in \mathscr{F}(K,X)$ and $\beta \in \mathscr{F}(X,K)$ be two fractions. Then $\beta\alpha \in \mathscr{F}(K,K)$ is a Lefschetz fraction if and only if so is $\alpha\beta \in \mathscr{F}(X,X)$. In other words, given commutative diagrams*

$$\begin{tikzcd} & K \arrow[dl, "\alpha"'] \arrow[d, "\varphi"] \\ X \arrow[r, "\beta"'] & K \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} & X \arrow[dl, "\beta"'] \arrow[d, "\Phi"] \\ K \arrow[r, "\alpha"'] & X \end{tikzcd}$$

in $\mathscr{V}(\mathbf{T})$, if one of φ, Φ is a Lefschetz fraction, then so also is the other, and $\Lambda(\varphi) = \Lambda(\Phi)$.

PROOF. Applying the functor $H_* : \mathscr{V}(\mathbf{T}) \to \mathbf{GrVect}$ to the above diagrams we obtain the commutative diagrams of graded vector spaces

$$\begin{tikzcd} & H_*(K) \arrow[dl, "\alpha_*"'] \arrow[d, "\varphi_*"] \\ H_*(X) \arrow[r, "\beta_*"'] & H_*(K) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & H_*(X) \arrow[dl, "\beta_*"'] \arrow[d, "\Phi_*"] \\ H_*(K) \arrow[r, "\alpha_*"'] & H_*(X) \end{tikzcd}$$

Now the desired assertions follow at once from the definitions involved and (15.2.2)(A). □

5. Coincidence Spaces

(5.1) DEFINITION. Let Γ, X be in $\mathbf{T}$, and let $f : \Gamma \to X$, $s : \Gamma \to X$ be two maps. Given $\varepsilon > 0$, a point $y \in \Gamma$ is called an *ε-coincidence point* for the pair (f, s) if $d(f(y), s(y)) < \varepsilon$, where d is a metric in X.

(5.2) LEMMA. *Let $f, s : \Gamma \to X$ be two maps such that s is proper and f is compact. If for each $\varepsilon > 0$ there exists an ε-coincidence point for (f, s), then the pair has a coincidence point.*

PROOF. The proof is strictly analogous to that of (6.3.1) and is left to the reader. □

Let $\varphi = f|s \in \mathscr{F}(X,X)$ be a Vietoris fraction, and $(f,s),(g,t)$ two representatives of φ. We let

$$\operatorname{Coin}(f,s) = \{y \in \Gamma \mid f(y) = s(y)\}, \qquad \operatorname{Coin}(g,t) = \{y \in \Gamma' \mid g(y) = t(y)\},$$

and observe that if $tu = s$ and $ug = f$ for some homeomorphism $u : \Gamma \to \Gamma'$, then u maps homeomorphically $\operatorname{Coin}(g,t)$ onto $\operatorname{Coin}(f,s)$. Thus, up to homeomorphism, we can assign to each Vietoris fraction $\varphi \in \mathscr{F}(X,X)$ its coincidence set $\operatorname{Coin}(f|s)$.

We say that a fraction $\varphi = f|s \in \mathscr{F}(X,X)$ *has a coincidence* if the set $\operatorname{Coin}\varphi$ is nonempty.

(5.3) LEMMA. *Assume that each of the diagrams*

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[d, "\varphi"] \\ Y \arrow[r, "\beta"'] & X \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} & Y \arrow[dl, "\beta"'] \arrow[d, "\Phi"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

is commutative in the category of fractions $\mathscr{V}(\mathbf{T})$. Then

$$\varphi \textit{ has a coincidence} \Leftrightarrow \Phi \textit{ has a coincidence.}$$

PROOF. Consider the following two commutative diagrams, which appear in the definition of $\varphi = \beta\alpha$ and $\Phi = \alpha\beta$:

$$\begin{tikzcd} \Gamma \times_Y \Gamma' \arrow[r, "f'"] \arrow[d, Rightarrow, "t'"'] & \Gamma' \arrow[r, "g"] \arrow[d, Rightarrow, "t"] & X \\ \Gamma \arrow[r, "f"] \arrow[d, Rightarrow, "s"'] & Y & \\ X & & \end{tikzcd} \qquad \begin{tikzcd} \Gamma' \times_X \Gamma \arrow[r, "g'"] \arrow[d, Rightarrow, "s'"'] & \Gamma \arrow[r, "f"] \arrow[d, Rightarrow, "s"] & Y \\ \Gamma' \arrow[r, "g"] \arrow[d, Rightarrow, "t"'] & X & \\ Y & & \end{tikzcd}$$

The reader can now verify that if $(x,y) \in \Gamma \times_Y \Gamma'$ is a coincidence point for (gf', st'), then $(y,x) \in \Gamma' \times_X \Gamma$ and (y,x) is a coincidence point for (fg', ts'); this clearly yields the desired conclusion. □

We call a fraction $\varphi = f|s \in \mathscr{F}(X,Y)$ *compact* if the map f is compact.

We remark that if $\alpha = f|s \in \mathscr{F}(X,Y)$ and $\beta = g|t \in \mathscr{F}(Y,Z)$ are two Vietoris fractions, then their composite $\beta\alpha \in \mathscr{F}(X,Z)$ is compact whenever either α or β is compact. This clearly follows from the definitions involved.

(5.4) DEFINITION. We say that X is a *coincidence space* if:

(i) any compact $\varphi = f|s \in \mathscr{F}(X,X)$ is a Lefschetz fraction,

(ii) $\Lambda(\varphi) \neq 0$ implies that $\varphi = f|s$ has a coincidence.

We describe two simple ways of forming new coincidence spaces from old ones.

(5.5) LEMMA. *If X is a coincidence space, then so also is every retract A of X.*

PROOF. Let $r : X \to A$ be the retraction and $i : A \hookrightarrow X$ the inclusion. Given a compact Vietoris fraction $\varphi = f|s \in \mathscr{F}(A, A)$, consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "\varphi"'] & X \arrow[d, "i\varphi r"] \arrow[dl, "\varphi r"'] \\ A \arrow[r, hook, "i"] & X \end{tikzcd}$$

and note that the fraction $\psi = i\varphi r \in \mathscr{F}(X, X)$ is also compact. Now the desired assertion follows from (4.4) and (5.3). □

If K is compact, the existence of a coincidence for a fraction $\varphi : K \to K$ can sometimes be determined by factorization.

(5.6) LEMMA. *Let K be a compact metric space and assume that a fraction $\varphi \in \mathscr{F}(K, K)$ factors through a coincidence space X, i.e., there are maps α and β making the diagram*

$$\begin{tikzcd} & K \arrow[dl, "\alpha"'] \arrow[d, "\varphi"] \\ X \arrow[r, "\beta"'] & K \end{tikzcd}$$

commutative. Then φ is Lefschetz, and $\Lambda(\varphi) \neq 0$ implies that φ has a coincidence.

PROOF. This is an obvious consequence of (4.4) and (5.3). □

(5.7) LEMMA. *Let K be a compact metric space such that every fraction $\varphi = f|s \in \mathscr{F}(K, K)$ factors through a coincidence space. Then K is a coincidence space.* □

6. Some General Coincidence Theorems

Using the Eilenberg–Montgomery coincidence theorem for polyhedra we first establish a preliminary result.

(6.1) THEOREM. *Every open subset of a normed linear space is a coincidence space.*

PROOF. Let U be open in a normed linear space E, and let

$$\varphi = \{U \overset{s}{\Leftarrow} \Gamma \overset{f}{\to} U\} \in \mathscr{F}(U,U)$$

be a compact fraction. We first show that φ is a Lefschetz fraction. By applying the approximation theorem (12.3.1) to f, choose a sufficiently small $\varepsilon > 0$ and an ε-approximation $f_\varepsilon : \Gamma \to U$ of f such that $f_\varepsilon(\Gamma) \subset P_\varepsilon \subset U$, where P_ε is a finite polyhedron and f_ε is homotopic to f. To prove that φ is Lefschetz, consider the fraction

$$\varphi_\varepsilon = \{U \overset{s}{\Leftarrow} \Gamma \overset{f_\varepsilon}{\to} U\} \in \mathscr{F}(U,U)$$

and the commutative diagram

$$\begin{tikzcd} P_\varepsilon \arrow[r, hook] \arrow[d, "\varphi'_\varepsilon"'] & U \arrow[d, "\varphi_\varepsilon"] \arrow[dl, "\tilde{\varphi}_\varepsilon"'] \\ P_\varepsilon \arrow[r, hook] & U \end{tikzcd}$$

where the fractions φ'_ε and $\tilde{\varphi}_\varepsilon$ are determined by φ_ε. Since the homology $H_*(P_\varepsilon)$ is of finite type, the fraction φ'_ε is Lefschetz, and by (4.4), so also is the fraction φ_ε, and $\lambda(\varphi'_\varepsilon) = \Lambda(\varphi_\varepsilon)$. Because $f \simeq f_\varepsilon$ we infer that $\Lambda(\varphi)$ is defined, and moreover,

$$\Lambda(\varphi) = \Lambda(\varphi_\varepsilon) = \lambda(\varphi'_\varepsilon).$$

Assuming now that $\Lambda(\varphi) \neq 0$, we have $\lambda(\varphi'_\varepsilon) \neq 0$, and hence by the Eilenberg–Montgomery theorem (1.2), the fraction $\varphi'_\varepsilon : P_\varepsilon \to P_\varepsilon$ has a coincidence. From this, in view of the commutativity of the above diagram and (5.3), we infer that also φ_ε has a coincidence, and thus $f_\varepsilon(x_0) = s(x_0)$ for some x_0; this shows that x_0 is an ε-coincidence point for the maps f and s. Since f is compact, s is proper, and $\varepsilon > 0$ was arbitrary, it follows by (5.2) that f and s have a coincidence. □

Recall that a space Y is called a NES(*compact metric*) *space* if for any pair (X, A) of compacta (1) and any $f : A \to Y$ there exists a nbd $W \supset A$ and an extension $F : W \to Y$ of f. Our aim now is to show that any metric NES(compact metric) space is a coincidence space.

Using the terminology of (4.3) and the factorization technique, we first establish the following preliminary result:

(6.2) THEOREM.

(a) *Let K be a compactum and assume that a fraction $\varphi : K \to K$ can be factored as*

$$K \overset{f}{\to} X \overset{\beta}{\to} K,$$

(1) Recall that a *compactum* is a compact metric space.

where X is a metric NES(*compact metric*) *space, β is a fraction, and f is an ordinary single-valued map. Then φ is a Lefschetz fraction, and $\Lambda(\varphi) \neq 0$ implies that φ has a coincidence.*

(a)* *Let X be a metric* NES(*compact metric*) *space and suppose a fraction $\Phi : X \to X$ can be factored as*

$$X \xrightarrow{\beta} K \xrightarrow{f} X,$$

where K is a compactum, β is a fraction, and f is an ordinary map. Then Φ is Lefschetz, and $\Lambda(\Phi) \neq 0$ implies that Φ has a coincidence.

PROOF. In view of (4.4) and (5.3), it is clearly enough to establish (a). To this end consider in $\mathscr{V}(\mathbf{T})$ the diagram

$$\begin{tikzcd}
H & U \arrow[l, hook'] \arrow[dd, dashed, "g"'] & & \widehat{K} \arrow[ll, hook', "j"'] \arrow[dl, shift left, "h^{-1}"] \\
& & K \arrow[ur, shift left, "h"] \arrow[dl, "f"'] \arrow[d, "\varphi"] & \\
& X \arrow[r, "\beta"] & K &
\end{tikzcd}$$

where $h : K \to \widehat{K}$ is a homeomorphism of K onto a subset $\widehat{K}$ of a Hilbert space H. Since X is a NES(compact metric), there is an extension $g : U \to X$ of the map $fh^{-1} : \widehat{K} \to X$ over an open nbd U of $\widehat{K}$ in H, i.e., $gj = fh^{-1}$, where $j : \widehat{K} \to U$ is the inclusion. Consider now the composite

$$K \xrightarrow{jh} U \xrightarrow{\beta g} K.$$

We have $(\beta g)(jh) = \beta(gj)h = \beta(fh^{-1})h = \beta f = \varphi$, which shows that φ factors through an open set $U \subset H$. Now because U is a coincidence space by (6.1), we conclude by (5.6) that the fraction φ is Lefschetz, and $\Lambda(\varphi) \neq 0$ implies that φ has a coincidence. □

As an obvious consequence of (6.2)(a)* we obtain at once the desired extension of the Eilenberg–Montgomery theorem:

(6.3) THEOREM. *Any metric* NES(*compact metric*) *space, and in particular, any ANR, is a coincidence space.*

(6.4) COROLLARY (Eilenberg–Montgomery). *Let Γ be a compact metric space, X a compact ANR, and $f : \Gamma \to X$, $s : \Gamma \to X$ two maps, with s inverse acyclic. If $\lambda[(f, s)] = \lambda(f_* s_*^{-1}) \neq 0$, then the maps f and s have a coincidence.* □

7. Fixed Points for Compact and Acyclic Set-Valued Maps

We now return to the study of fixed points for set-valued maps. The results of Section 6 are applied to obtain some fixed point theorems for set-valued maps analogous to those in §15.

In the remainder of this paragraph the set-valued maps are denoted by capital letters; for convenience, we write $S : X \to Y$ instead of $S : X \to 2^Y$.

(7.1) DEFINITION. Let X and Y be two spaces, and let $S : X \to Y$ be a set-valued map. Then:

(i) S said to be *acyclic* if S is upper semicontinuous (u.s.c.) (1) and has acyclic values.

(ii) S is called *compact* if the image $S(X) = \bigcup\{Sx \mid x \in X\}$ is contained in a compact subset of Y.

Let X be a space and $S : X \to X$ be an acyclic map. Let $\Gamma = \{(x,y) \in X \times X \mid y \in S(x)\}$ be the graph of S, and let $f : \Gamma \to X$ and $s : \Gamma \to X$ denote the natural projections given by $(x,y) \mapsto y$ and $(x,y) \mapsto x$, respectively. Because $s^{-1}(x)$ is homeomorphic to $S(x)$ for every $x \in X$ and s is proper, we see that the projection $s : \Gamma \to X$ is a Vietoris map, and hence in view of the Vietoris mapping theorem, we may form the endomorphism

$$S_* = f_* s_*^{-1} : H_*(X) \to H_*(X).$$

(7.2) DEFINITION. We say that $S : X \to X$ is a *Lefschetz map* if $S_* : H_*(X) \to H_*(X)$ is a Leray endomorphism. For such a map S the integer $\Lambda(S) = \Lambda(S_*)$ is called the *Lefschetz number* of S.

Note that whenever X is of finite type, then any acyclic $S : X \to X$ is a Lefschetz map and $\Lambda(S)$ coincides with the ordinary Lefschetz number $\lambda(S)$ of S.

We now prove a general fixed point theorem for compact acyclic maps.

(7.3) THEOREM. *Let X be either a metric* NES(*compact metric*) *space or an ANR, and $S : X \to X$ a compact acyclic map. Then S is a Lefschetz map, and $\Lambda(S) \neq 0$ implies that S has a fixed point.*

PROOF. Clearly, it is enough to prove the assertion in the case where X is a metric NES(compact metric) space. To this end let Γ be the graph of S, and let

$$\varphi = \{X \overset{s}{\Leftarrow} \Gamma \overset{f}{\to} X\} \in \mathscr{F}(X,X)$$

be the fraction determined by S. We make the following observations:

---

(1) We recall that a set-valued map $S : X \to Y$ is *upper semicontinuous* if (i) $S(x)$ is compact for each $x \in X$ and (ii) for any open $V \subset Y$ the set $\{x \in X \mid S(x) \subset V\}$ is open.

(i) $S_* = \varphi_*$, where $\varphi_* = f_* s_*^{-1}$ is induced by φ.
(ii) f is compact, and therefore so is φ.
(iii) $\Lambda(S) = \Lambda(\varphi_*)$ is defined by (6.3).

Assume that $\Lambda(S) = \Lambda(\varphi_*) \neq 0$. Then, by (6.3), the maps $f, s : \Gamma \to X$ have a coincidence, i.e., for some pair (x_0, y_0) with $y_0 \in S(x_0)$ we have $f(x_0, y_0) = s(x_0, y_0)$; now because $f(x_0, y_0) = y_0$ and $s(x_0, y_0) = x_0$, this gives $x_0 \in S(x_0)$, and the conclusion follows. □

As obvious consequences, we obtain the following results:

(7.4) COROLLARY (Eilenberg–Montgomery). *Let X be a compact ANR and $S : X \to X$ an acyclic map such that $\lambda(S) \neq 0$. Then S has a fixed point.* □

(7.5) COROLLARY. *Assume that X is one of the following:*
(i) *an acyclic metric* NES(*compact metric*) *space,*
(ii) *an acyclic ANR,*
(iii) *a metric* ES(*compact metric*) *space,*
(iv) *an AR,*
(v) *a convex (not necessarily closed) subset of a locally convex metrizable space.*

Then any compact acyclic map S of X into itself has a fixed point. □

Observe that (7.5)(v) represents a generalization of fixed point results for compact Kakutani maps in (7.8.4).

Now, as an immediate corollary of (7.5), we shall establish some results of the Leray–Schauder type for compact acyclic maps in normed linear spaces. Let E be a normed linear space; given a bounded subset $A \subset E$ we let $\|A\| = \sup\{\|a\| \mid a \in A\}$.

(7.6) THEOREM (Nonlinear alternative). *Let E be a normed linear space, and let K_ϱ be the closed ball $\{x \in E \mid \|x\| \leq \varrho\}$, where $\varrho > 0$. Then each compact acyclic map $T : K_\varrho \to E$ has at least one of the following two properties:*
(a) *T has a fixed point,*
(b) *there are $x \in \partial K_\varrho$ and $\lambda \in (0,1)$ such that $x \in \lambda T x$.*

PROOF. Let $r : E \to K_\varrho$ be the standard retraction given by

$$(*) \qquad r(y) = \begin{cases} y & \text{for } \|y\| \leq \varrho, \\ \varrho y / \|y\| & \text{for } \|y\| > \varrho, \end{cases}$$

and consider the composite map $E \xrightarrow{r} K_\varrho \xrightarrow{T} E$. Since $T \circ r$ is compact and acyclic, we infer by (7.5)(v) that it has a fixed point, i.e., $x \in Tr(x)$

for some $x \in E$; this obviously implies that $y = r(x)$ is a fixed point for $rT : K_\varrho \to K_\varrho$, i.e., $y \in rTy$.

We now examine two possible cases: (i) $\|Ty\| \leq \varrho$ and (ii) $\|Ty\| > \varrho$. In case (i), $rTy = Ty$, and therefore $y \in Ty$, i.e., property (a) holds. In case (ii), there exists a $z \in Ty$ with $\|z\| > \varrho$ and $y = r(z)$, and hence, in view of (*), we get

$$y = \varrho \frac{z}{\|z\|} \in \partial K_\varrho \quad \text{and} \quad z = \frac{\|z\|}{\varrho} y \in Ty.$$

This gives $y \in \lambda Ty$ with $\lambda = \varrho/\|z\| < 1$. Thus, property (b) holds, and the proof is complete. □

Several fixed points for compact acyclic maps can be derived from (7.6) by imposing conditions that prevent the occurrence of the second property.

(7.7) THEOREM. *Let $T : K_\varrho \to E$ be a compact acyclic map and assume that for all $x \in \partial K_\varrho$ one of the following conditions is satisfied:*
(i) $\|Tx\| \leq \max\{\|x\|, \|x - Tx\|\}$,
(ii) $\|Tx\| \leq (\|x - Tx\|^2 + \|x\|^2)^{1/2}$.
Then T has a fixed point.

PROOF. The routine verification that property (b) in (7.6) cannot occur is left to the reader. □

Let $T : E \to E$ be an acyclic map. We say that T is *completely continuous* if it is compact on bounded subsets of E.

Theorem (7.6), applied to completely continuous acyclic maps, yields

(7.7) THEOREM (Leray–Schauder alternative). *Let $T : E \to E$ be a completely continuous acyclic map, and let*

$$\mathcal{E}(T) = \{x \in E \mid x \in \lambda Tx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\mathcal{E}(T)$ is unbounded, or T has a fixed point.

PROOF. Assume that $\mathcal{E}(T)$ is bounded, and let K_ϱ be a ball containing $\mathcal{E}(T)$ in its interior. Then $T|K_\varrho : K_\varrho \to E$ is a compact acyclic map, and since no $x \in \partial K_\varrho$ can satisfy the second property in (7.6), the map T has a fixed point, and the proof is complete. □

8. Miscellaneous Results and Examples

A. Degree mod 2 for set-valued acyclic maps

Throughout subsections A, B, C, only metric spaces are considered. By $H_*(X)$ we denote the Čech homology of X with Z_2 coefficients and with compact supports. Set-valued

maps are denoted by $\varphi, \psi, \ldots$, and single-valued maps by $f, g, \ldots$; for convenience we write $\varphi : X \to Y$ instead of $\varphi : X \to 2^Y$ With $\varphi : X \to Y$ we associate the diagram

$$X \xleftarrow{p} \Gamma_\varphi \xrightarrow{q} Y$$

in which $\Gamma = \Gamma_\varphi$ is the graph of φ, and p, q are the natural projections.

A map $\varphi : X \to Y$ is called *acyclic* if φ is u.s.c. and $\varphi(x)$ is acyclic for each $x \in X$; for such a φ, the sets $p^{-1}(x)$ are also acyclic, and by the Vietoris theorem, p induces an isomorphism $p_* : H_*(\Gamma) \cong H_*(X)$. The homomorphism $\varphi_* = q_* p_*^{-1} : H_*(X) \to H_*(Y)$ is said to be *induced* by φ.

Let $\varphi : S^n \to \boldsymbol{R}^{n+1} - \{0\}$ be an acyclic map. Since $H_n(S^n) \cong H_n(\boldsymbol{R}^{n+1} - \{0\}) \cong Z_2$, we may identify φ_* with an integer mod 2. This integer is called the *degree mod 2* of φ and is denoted by $d(\varphi)$; for any $y_0 \in \boldsymbol{R}^{n+1} - \varphi(S^n)$, we let $d(\varphi, y_0) = d(\psi)$, where $\psi(x) = \varphi(x) - y_0$.

(A.1) Show: If $\varphi : (K^{n+1}, S^n) \to (\boldsymbol{R}^{n+1}, \boldsymbol{R}^{n+1} - \{0\})$ is an acyclic map with $d(\varphi|S^n) \neq 0$, then $0 \in \varphi(x_0)$ for some $x_0 \in K^{n+1}$

[Supposing the contrary, consider the commutative diagram

$$\begin{tikzcd}
K^{n+1} & \Gamma_\varphi \arrow[l, "p_1"'] \arrow[dr, "q_1"] & \\
 & & \boldsymbol{R}^{n+1} - \{0\} \\
S^n \arrow[uu, "i"] & \Gamma_{\varphi|S^n} \arrow[l, "p_2"] \arrow[uu, "j"'] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

where i, j are inclusions and p_i, q_i are the natural projections of the respective graphs; to get a contradiction, apply the Vietoris theorem to show that $d(\varphi|S^n) = 0$.]

(A.2) Define the relation of acyclic homotopy between acyclic maps and show:
- (a) If the acyclic maps $\varphi, \psi : S^n \to \boldsymbol{R}^{n+1} - \{0\}$ are acyclically homotopic, then $d(\varphi) = d(\psi)$.
- (b) If $f : S^n \to \boldsymbol{R}^{n+1} - \{0\}$ is a selector of an acyclic map $\varphi : S^n \to \boldsymbol{R}^{n+1} - \{0\}$, then $d(\varphi)$ is equal to the ordinary degree mod 2 of the map f.

(A.3) (*Generalized Brouwer theorem*) Let $\varphi : K^n \to K^n$ be an acyclic map. Show: φ has a fixed point.

B. Antipodal theorem for acyclic maps

Throughout this subsection M is a compact space with $H_*(M) = H_*(S^n)$.

(B.1)* (*Generalized Borsuk theorem*) Let $h : M \to M$ be a fixed point free involution. Show: If a map f $M \to S^n$ satisfies $f(x) \neq f(h(x))$ for all $x \in M$, then $f_* : H_n(M) \to H_n(S^n)$ is nontrivial (Jaworowski [1955]).

(B.2) A set-valued map $\Phi : M \to M$ is called a *set-valued involution* if $y \in \Phi(x)$ implies $x \in \Phi(y)$ for all $(x, y) \in M \times M$. Let $\Phi : M \to M$ be an involution and $\varphi : M \to \boldsymbol{R}^{n+1} - \{0\}$ an acyclic map satisfying the following condition:

(∗) no ray starting at $0 \in \boldsymbol{R}^{n+1}$ intersects both $\varphi(x)$ and $\varphi(\Phi(x))$ for any $x \in M$.

Show: The homomorphism $\varphi_* : H_n(M) \to H_n(\boldsymbol{R}^{n+1} - \{0\})$ is nontrivial.

[Consider the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "s"'] & P^{n+1} \arrow[r, "r"] & S^n \\ M & \Gamma_\varphi \arrow[l, "p"'] \arrow[u, "q"'] & \end{tikzcd}$$

where $P^{n+1} = R^{n+1} - \{0\}$,

$$Z = \{(x, y, u, v) \mid u \in \varphi(x),\ v \in \varphi(y),\ x \in \Phi(y)\} \subset M \times M \times P^{n+1} \times P^{n+1},$$

p, q are natural projections and s, f, r are given by

$$s(x, y, u, v) = x, \qquad r(u) = u/\|u\|, \qquad f(x, y, u, v) = u - v.$$

Prove that Z is compact and establish successively the following assertions: (i) f is well defined; (ii) r_* and p_* are isomorphisms; (iii) s is the composition of the maps $(x, y, u, v) \mapsto (x, y, u) \mapsto (x, y) \mapsto x$, having counterimages homeomorphic to $\varphi(y)$, $\varphi(x)$, and $\Phi(x)$, respectively; (iv) s_* is an isomorphism, and thus $H_*(Z) \cong H_*(S^n)$. Letting $h : Z \to Z$ be the involution $(x, y, u, v) \mapsto (y, x, v, u)$, observe that the map $g = rf : Z \to S^n$ satisfies $g(z) \neq g(h(z))$ for each $z \in Z$. Using (B.1), conclude that $g_* \quad H_n(Z) \to H_n(S^n)$ is nontrivial, and hence so is φ_*.]

(B.3) Let $\varphi : (K^{n+1}, S^n) \to (R^{n+1}, R^{n+1} - \{0\})$ be an acyclic map such that no ray starting at the origin intersects both the sets $\varphi(x)$ and $\varphi(-x)$ for any $x \in S^n$ Show: $0 \in \varphi(x_0)$ for some $x_0 \in K^{n+1}$

(B.4) (*Generalized Borsuk–Ulam theorem*) Let $\varphi : S^n \to R^n$ be an acyclic map. Show: There exists a point $x_0 \in S^n$ such that $\varphi(x_0) \cap \varphi(-x_0) \neq \emptyset$.

C. Invariance of domain for acyclic maps

(C.1) Let $A \subset R^{n+1}$ be compact, and $a \in A$. Show: $a \in \operatorname{Int} A$ if and only if there exists an acyclic map $\varphi : K^{n+1} \to A$ such that $\varphi(S^n) \subset A - \{a\}$ and $d(\varphi|S^n, a) \neq 0$.

[For sufficiency, first show that $a \in \operatorname{Int} M \Leftrightarrow \operatorname{Ker}[H_n(A - \{a\}) \to H_n(A)] \neq 0$; then observe that the above kernel contains $\varphi_*(H_n(S^n))$ which is nonzero by assumption.]

(C.2) A set-valued map $\varphi : X \to Y$ is called an *ε-map* if $\varphi(x_1) \cap \varphi(x_2) \neq \emptyset$ implies $d(x_1, x_2) < \varepsilon$ for all $x_1, x_2 \in X$. Let $\varphi : K^{n+1} \to R^{n+1}$ be an acyclic 1-map and $\psi = \varphi|S^n$ Show: For any $y_0 \in \varphi(0)$ we have $d(\psi, y_0) \neq 0$.

[Letting $\widehat{\varphi}(x) = \varphi(x) - y_0$ and $\widehat{\psi} = \widehat{\varphi}|S^n$, note that $0 \in \widehat{\varphi}(0)$. Let

$$\begin{aligned} W &= \Gamma_{\widehat{\psi}} = \{(x, u) \mid x \in S^n,\ u \in \widehat{\varphi}(x)\}, \\ Z &= \{(x, y) \in K^{n+1} \times K^{n+1} \mid \varrho(x, y) = 1\}, \\ M &= \{(x, y, u, v) \mid (x, y) \in Z,\ u \in \widehat{\varphi}(x),\ v \in \widehat{\varphi}(y)\}, \\ Y &= \{(x, y, u, v) \in M \mid y = 0\}. \end{aligned}$$

Observing that $(x, u) \mapsto (x, 0, u, 0)$ defines a map $\iota : W \to Y$, consider the commutative

diagram

$$\begin{tikzcd}
 & & S^n \\
 & Y \arrow[ur, "h"] \arrow[dl, "t"'] \arrow[d, "j"] & W \arrow[l, "i"'] \arrow[u, "p"'] \arrow[d, "q"] \\
Z & M \arrow[l, "s"'] \arrow[r, "f"] & P^{n+1}
\end{tikzcd}$$

where p, q are the natural projections, j is the inclusion, and s, t, h, and f are defined by

$$s(x,y,u,v) = (x,y), \qquad h(x,0,u,v) = x,$$
$$t(x,0,u,v) = (x,0), \qquad f(x,y,u,v) = u - v.$$

Then establish successively the following assertions: (i) t and f are well defined; (ii) s is the composition of the two Vietoris maps $(x,y,u,v) \mapsto (x,y,u) \mapsto (x,y)$; (iii) s_*, t_*, h_* are isomorphisms; (iv) i_* and j_* are isomorphisms and hence $H_*(M) \cong H_*(S^n)$. Define $k : M \to M$ by $(x,y,u,v) \mapsto (y,x,v,u)$, and note that k is an involution such that $fk(z) = -f(z)$ for $z \in M$. Applying (B.2) with k taken for Φ and f for φ, find that $f_* \neq 0$, implying that $q_* \neq 0$, and thus $d(\psi, y_0) \neq 0$.]

(C.3) (*Lemma on ε-maps*) Let $\varepsilon > 0$ and $K_\varepsilon(x_0) = \{x \in R^{n+1} \mid \|x - x_0\| \leq \varepsilon\}$. Show: If $\varphi : K_\varepsilon(x_0) \to R^{n+1}$ is an acyclic ε-map, then every point $y_0 \in \varphi(x_0)$ is an interior point of $\varphi(K_\varepsilon(x_0))$.

(C.4) (*Invariance of domain*) Let $U \subset R^{n+1}$ be open and $\varphi : U \to R^{n+1}$ an acyclic map such that $\varphi(x_1) \cap \varphi(x_2) = \emptyset$ for any $x_1 \neq x_2$ in U. Show: $\varphi(U)$ is open in R^{n+1}

(For the results of subsections A, B, C, see Granas–Jaworowski [1959].)

9. Notes and Comments

Vietoris fractions and coincidence theory

The idea of using "Vietoris fractions" in the context of the fixed point theory of set-valued maps is due to Calvert [1970]. The present sharp formulation of the method was first conceived by Granas in an unpublished work ([1]) and further developed in joint publications with L. Górniewicz (cf. Górniewicz–Granas [1981], [1989], where the Vietoris fractions appear under the name of "morphisms"). An appropriate coverage of the theory of Vietoris fractions would require a separate volume. In this presentation, we are content to develop the theory to the point where it can serve as a convenient tool in the study of fixed points of set-valued maps.

Here we remark that as shown in Górniewicz–Granas [1981], the principal notions and main results of this paragraph extend to nonmetrizable spaces; using the Čech homology $H_*(\cdot, Q)$ with compact supports and the Vietoris–Begle theorem (cf. Begle [1950]), the main Lefschetz-type fixed point results

([1]) The main results of this work were presented in Granas's lectures at the University of Bonn in 1975.

of §17 can be extended to the framework of coincidence theory. Thus, for example, we have the following generalization (Górniewicz-Granas [1989]) of the Eilenberg-Montgomery coincidence theorem:

THEOREM. *Let Γ be a space, X an* ANES(*compact*) *space, and $f, s : \Gamma \to X$ two maps such that f is compact and s is proper and inverse acyclic. Then:*
 (i) *the Lefschetz number $\Lambda(f,s) = \Lambda(f_* s_*^{-1})$ is defined,*
 (ii) *if $\Lambda(f,s) \neq 0$, then the maps f and s have a coincidence.*

Fixed points for acyclic maps

The fixed point theory for acyclic maps was founded in 1946 by S. Eilenberg and D. Montgomery. Using the Vietoris mapping theorem (and also the chain approximation technique, which appears in the original proof of the above theorem by Vietoris [1927]), Eilenberg–Montgomery [1946] extended the Lefschetz-Hopf theorem to set-valued acyclic maps. It soon became clear that the same method can be used to extend to acyclic maps other known topological results. Thus, the Borsuk antipodal theorem, the invariance of domain, and the mod 2 degree were extended to acyclic maps (see Jaworowski [1956], [1957], Granas–Jaworowski [1959], and "Miscellaneous Results and Examples"). Theorems (1.2) and (6.4) are due to Eilenberg–Montgomery [1946]. Theorem (7.3) represents an extension to acyclic compact maps of the Lefschetz fixed point theorem for ANRs given in §15; the proof of this result in the special case where X is a topologically complete ANR was found by Powers [1970] and independently by Górniewicz–Granas [1970]. We remark that (7.3) yields in particular the generalized Schauder theorem for compact acyclic maps of ARs.

For other results on acyclic maps the reader may consult the survey of Borisovich et al. [1980].

Lefschetz theorem for maps having acyclic decompositions

Let $\mathscr{P}$ be the class of set-valued maps such that $S : X \to X$ is in $\mathscr{P}$ if and only if X is a compact ANR and for some sequence $\varphi = (S_k, \dots, S_1)$ of acyclic maps $S_i : X_i \to X_{i+1}$, $i \in [k]$ (where $X_1 = X_{k+1} = X$), $S = S_k \circ \cdots \circ S_1$; any such φ is called an *acyclic decomposition* of S. If $S : X \to X$ is in $\mathscr{P}$ and $\varphi = (S_k, \dots, S_1)$ is an acyclic decomposition of S, we let $\lambda(\varphi) = \lambda(\varphi_*)$, where $\varphi_* = (S_k)_* \circ \cdots \circ (S_1)_* : H_*(X) \to H_*(X)$, and call $\lambda(\varphi)$ the *Lefschetz number* of φ; clearly, different acyclic decompositions have in general different Lefschetz numbers.

The class $\mathscr{P}$ was introduced by M. Powers [1972] to whom the following extension of the Eilenberg–Montgomery theorem is due.

THEOREM (Powers). *Let $S : X \to X$ be a map in $\mathscr{P}$. If for some acyclic decomposition φ of S we have $\lambda(\varphi) \neq 0$, then S has a fixed point.*

Set-valued maps determined by fractions

Let X be a space and $\varphi = f|s \in \mathscr{F}(X,X)$ a fraction. We say that a set-valued map $S : X \to X$ is *determined by* φ if $S(x) = f(s^{-1}(x))$ for $x \in X$ (Górniewicz–Granas [1981]; see also Jankowski [1975]).

We say that $S : X \to X$ determined by $f|s$ is a *Lefschetz map* if $f_* s_*^{-1} : H_*(X) \to H_*(X)$ is a Leray endomorphism; for such an S we define its *Lefschetz number* $\Lambda(S)$ by $\Lambda(S) = \Lambda(f_* s_*^{-1})$.

Because any set-valued map determined by a fraction is upper semicontinuous, the general coincidence result stated at the beginning of this section implies (cf. Górniewicz–Granas [1981], [1991]) the following

THEOREM. *Let X be an ANR, or more generally an* ANES(*compact*) *space, and assume that $S : X \to X$ is a set-valued map determined by a compact fraction. Then S is a Lefschetz map, and $\Lambda(S) \neq 0$ implies that S has a fixed point.*

Because any composition of acyclic maps is determined by a fraction, this theorem implies the above theorem of Powers and a number of other Lefschetz-type results for compact set-valued maps that admit acyclic decompositions.

For details and more general or related results, see Powers [1972], Górniewicz [1976], Dzedzej [1985], Kryszewski [1994], and Górniewicz's book [1999], where further references can be found.

The local index for acyclic maps of polyhedra

Let $\mathscr{A}^*$ be the class of all triples (X, φ, U), where $\varphi : X \to X$ is an acyclic self-map of a compact polyhedron X, $U \subset X$ is open, and $\mathrm{Fix}(\varphi|\partial U) = \emptyset$. The following result was established by H. Siegberg and G. Skordev.

THEOREM. *There exists on $\mathcal{A}^*$ a local index $(X, \varphi, U) \mapsto i(X, \varphi, U)$ with values in Q satisfying the properties*: (i) *strong normalization*; (ii) *homotopy*; (iii) *additivity*; (iv) *excision*; *and* (v) *commutativity.*

This index is developed through the use of chain approximations (not necessarily induced by simplicial approximations). The same technique permits one in fact to extend the above index to maps in the class $\mathscr{P}$ and to prove the mod p property of the index. For details and more general or related results the reader is referred to Siegberg–Skordev [1982], Skordev [1984], von Haeseler–Skordev [1992], and Kryszewski [1996].

Homotopy method

We now briefly comment on the study of fixed points for set-valued maps that is based on homotopy considerations and relies only on using elementary tools of geometric topology.

In 1958, Granas asked the following question: *Given a compactum X in $\boldsymbol{R}^m$ and an acyclic map $\varphi : X \to \boldsymbol{S}^n$, does there exist an acyclic homotopy joining φ to a single-valued map?*

Brahana–Fort–Horstmann [1965] gave an affirmative answer to this question, assuming that the maps and the homotopies under consideration are cellular ([1]). A similar result (as shown by Kucharski [1988]) also holds for the class of strongly acyclic maps: *Any strongly acyclic map $\varphi : X \to \boldsymbol{S}^n$ is homotopic via a strongly acyclic homotopy to a single-valued map.*

We remark that the first (respectively second) result implies that the Brouwer degree for maps $\boldsymbol{S}^n \to \boldsymbol{S}^n$ can be extended to cellular (respectively strongly acyclic) maps.

We mention two other noteworthy results obtained by the homotopy method:

(i) (*The index for $\boldsymbol{Z}$-acyclic maps of ENRs*) Using the properties of strongly acyclic maps, Bielawski [1987] established the following result: *Let $\widehat{\mathscr{F}}$ be the class of set-valued maps defined by the condition: $\varphi \in \widehat{\mathscr{F}}$ if and only if $\varphi : U \to X$ is compactly fixed and $\boldsymbol{Z}$-acyclic (i.e., each $\varphi(x)$ has $\boldsymbol{Z}$-acyclic Čech cohomology), where X is an ENR and $U \subset X$ is open. Then there exists a unique function $I : \widehat{\mathscr{F}} \to \boldsymbol{Z}$ satisfying the same properties as the index for single-valued maps of ENRs.*

(ii) (*The index for R_δ-maps of compact ANRs*) Let X be a compact ANR, $U \subset X$ open, and let $\mathscr{C}_{\partial U}(\overline{U}, X)$ denote the set of all R_δ-maps $\varphi : \overline{U} \to X$ with no fixed point on ∂U. A homotopy $\{\chi_t\}$ between members of $\mathscr{C}_{\partial U}(\overline{U}, X)$ is understood to consist of R_δ-maps with no fixed point on ∂U. We denote by $\mathscr{C}_{\partial U}[\overline{U}, X]$ the corresponding set of homotopy classes, and by $C_{\partial U}[\overline{U}, X]$ the set of homotopy classes of single-valued maps $f : \overline{U} \to X$ with $\mathrm{Fix}(f|\partial U) = \emptyset$. Górniewicz–Granas–Kryszewski [1991] constructed a bijection between $\mathscr{C}_{\partial U}[\overline{U}, X]$ and $C_{\partial U}[\overline{U}, X]$. This implies the existence of the fixed point index for R_δ-maps of compact ANRs with the usual properties.

We remark that from the above index theory for R_δ-maps it follows that the Hilbert cube I^∞ is a fixed point space for R_δ-maps. This in turn (by using the factorization technique) implies a broad generalization of the Schauder theorem: *If X is an AR, then any compact R_δ-map of X into itself has a fixed point.*

For details and related results the reader may consult Kryszewski [1996] and also the book of Górniewicz [1999].

([1]) Let $\varphi : S^n \to S^n$ be an u.s.c. set-valued map. Then: (i) φ is *cellular* if each $\varphi(x)$ is the intersection of a descending sequence of topological n-cells; (ii) φ is *strongly acyclic* if the homotopy groups $\pi_k(S^n - \varphi(x))$ vanish for all $x \in S^n$ and $k = 0, 1, \dots$ (cf. Kucharski [1988]); (iii) φ is an *R_δ-map* if each $\varphi(x)$ is an R_δ-set, i.e., an intersection of a descending sequence of compact ARs.

§20. Further Results and Supplements

This paragraph is concerned with three topics not closely related to one another but each having points in common with the Leray–Schauder theory. The "Notes and Comments" end with a concise overview of a few important areas of the fixed point theory that were not treated in the book.

1. Degree for Equivariant Maps in $\boldsymbol{R}^n$

Topological degree in $\boldsymbol{R}^n$ has been extended to various classes of equivariant maps. Our aim in this section is to outline one such extension, in a simple setting which does not require knowledge of representation theory and equivariant topology.

Throughout this section, we let U denote an open bounded subset of $\boldsymbol{R}^n$. For the convenience of the reader, we first recall some definitions and relevant results.

Let $f : U \to \boldsymbol{R}^k$ be a smooth map (¹). We say that $x_0 \in U$ is a *regular point* of f if the differential $Df(x_0) \in \mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^k)$ is surjective; otherwise, x_0 is called a *critical point*. The images of critical points are *critical values*. An element $y \in \boldsymbol{R}^k$ is a *regular value* if it is not a critical value.

With this terminology, we can state the following special case of the Sard theorem:

(1.1) THEOREM. *If $f : U \to \boldsymbol{R}^n$ is a smooth map, then the set of regular values of f is dense in $\boldsymbol{R}^n$.* □

Let $K \subset \boldsymbol{R}^n$ be compact, $\{U_1, \dots, U_k\}$ an open covering of K, and $U = \bigcup_{i=1}^k U_i$. By a *smooth partition of unity subordinate to* $\{U_i\}_{i=1}^k$ is meant a family $\{\lambda_i\}_{i=1}^k$ of smooth functions $\lambda_i : U \to \boldsymbol{I}$ such that:

(i) $\operatorname{supp} \lambda_i \subset U_i$ for $i \in [k]$,

(ii) $\sum_{i=1}^k \lambda_i(y) = 1$ for all $y \in U$.

Using the well-known fact that such a partition of unity always exists, we now show that if $K \subset \boldsymbol{R}^n$ is compact, then any continuous map $f : K \to \boldsymbol{R}^n$ can be uniformly approximated on K by smooth maps.

(1.2) THEOREM. *Let $K \subset \boldsymbol{R}^n$ be compact and $f : K \to \boldsymbol{R}^n$ continuous. Then for each $\varepsilon > 0$ there exists a smooth map $g : K \to \boldsymbol{R}^n$ such that $\|f - g\| < \varepsilon$, where $\| \ \|$ is the supremum norm.*

PROOF. Let $\varepsilon > 0$ be given. Since K is compact, uniform continuity of f gives a $\delta > 0$ such that $\|f(x) - f(y)\| < \varepsilon$ whenever $\|x - y\| < 2\delta$.

(¹) Given $X \subset \boldsymbol{R}^n$, a map $f : X \to \boldsymbol{R}^k$ is called *smooth* if there exists an open $V \supset X$ and a C^∞ map $g : V \to \boldsymbol{R}^k$ such that $g|X = f$.

This implies that there exists a finite subset $\{x_1, \dots, x_k\}$ of K and an open covering $\{U_i\}_{i=1}^k$ of K by balls $U_i = B(x_i, \delta)$ such that $\|f(x) - f(y)\| < \varepsilon$ whenever $x, y \in U_i$, $i \in [k]$.

Take now a smooth partition of unity $\{\lambda_i\}$ subordinate to $\{U_i\}$ and define a smooth map $g : U \to \boldsymbol{R}^n$ by

$$g(x) = \sum_{i=1}^{k} \lambda_i(x) f(x_i), \qquad x \in U.$$

Then for any $x \in K$,

$$\|f(x) - g(x)\| \le \sum_{i=1}^{k} \lambda_i(x) \|f(x) - f(x_i)\| \le \sum_{i=1}^{k} \lambda_i(x) \varepsilon = \varepsilon. \qquad \square$$

Admissible maps and homotopies

A map $f : \overline{U} \to \boldsymbol{R}^n$ is called *admissible* if $f|U : U \to \boldsymbol{R}^n$ is smooth and $f(x) \neq 0$ for $x \in \partial U$. We let $\mathscr{A}(\overline{U})$ denote the set of all admissible maps. A map $f \in \mathscr{A}(\overline{U})$ is called *generic* if 0 is a regular value of $f|U$. By an *admissible homotopy* is meant a homotopy $h : \overline{U} \times \boldsymbol{I} \to \boldsymbol{R}^n$ such that $h|U \times \boldsymbol{I}$ is smooth and $h(x,t) \neq 0$ for $(x,t) \in \partial U \times \boldsymbol{I}$; the set of admissible homotopies is denoted by $\mathscr{A}\mathscr{H}(\overline{U})$.

(1.3) DEFINITION. Two maps $f, g \in \mathscr{A}(\overline{U})$ are said to be *homotopic in* $\mathscr{A}(\overline{U})$ (written $f \sim g$ in $\mathscr{A}(\overline{U})$) if there exists an admissible homotopy $h \in \mathscr{A}\mathscr{H}(\overline{U})$ joining f and g (i.e., such that $h(x,0) = f(x)$, $h(x,1) = g(x)$ for $x \in \overline{U}$).

(1.4) THEOREM. *For each $f \in \mathscr{A}(\overline{U})$, there exists a generic map $g \in \mathscr{A}(\overline{U})$ homotopic to f in $\mathscr{A}(\overline{U})$.*

PROOF. Given $f \in \mathscr{A}(\overline{U})$, choose $\alpha > 0$ with $\alpha < \inf\{\|f(x)\| \mid x \in \partial U\}$. By Theorem (1.1) there exists a regular value $y_0 \in \boldsymbol{R}^n$ of f with $\|y_0\| < \alpha$. Set

$$g(x) = f(x) - y_0, \qquad x \in \overline{U},$$

and

$$h(x,t) = (1-t) f(x) + t g(x), \qquad (x,t) \in \overline{U} \times \boldsymbol{I}.$$

Clearly, g is generic, and (since $\|h(x,t) - f(x)\| \le t\|y_0\| < \alpha$) h is the desired admissible homotopy joining f and g. $\square$

Let $f : \overline{U} \to \boldsymbol{R}^n$ be generic. By the inverse function theorem, the set $f^{-1}(0)$ is finite:

$$f^{-1}(0) = \{x_1, \dots, x_k\}.$$

We let

$$\sigma(f,U) = \sum_{i=1}^{k} \operatorname{sgn}\det Df(x_i)$$

and observe that $\sigma(f,U) = d(f,U)$.

Equivariant maps and homotopies

From now on, we deal with a fixed Euclidean space $\boldsymbol{R}^n$ equipped with a splitting $\boldsymbol{R}^n = \boldsymbol{R}^p \oplus \boldsymbol{R}^q$, where p, q are fixed natural numbers with $p+q = n$. For $x \in \boldsymbol{R}^n$, we write $x = (u,v)$ with $u \in \boldsymbol{R}^p$, $v \in \boldsymbol{R}^q$ and we let

$$T(u,v) = (u,-v) \qquad \text{for } (u,v) \in \boldsymbol{R}^n$$

Clearly, the map $T : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a linear isomorphism and an *involution*, i.e., $T^2 = \mathrm{id}_{\boldsymbol{R}^n}$.

A set $X \subset \boldsymbol{R}^n$ is called *T-invariant* if $T(X) \subset X$; clearly if X is T-invariant, then $T(X) = X$ and $T|X : X \to X$ is an involution on X.

Let $X \subset \boldsymbol{R}^n$ be T-invariant. A map $f : X \to \boldsymbol{R}^n$ is called *equivariant* if

$$f(T(x)) = T(f(x)) \qquad \text{for } x \in X.$$

If we write $f = (f_1, f_2)$, where $f_1 : X \to \boldsymbol{R}^p$, $f_2 : X \to \boldsymbol{R}^q$, then equivariance of f means that $f_1(u,-v) = f_1(u,v)$ and $f_2(u,-v) = -f_2(u,v)$ for all $(u,v) \in X$.

For convenience, call $U \subset \boldsymbol{R}^n$ *allowable* if it is open, bounded, and T-invariant; then $\overline{U}$ is also T-invariant. Let U be a fixed allowable subset of $\boldsymbol{R}^n$. A map $f : \overline{U} \to \boldsymbol{R}^n$ is called *T-admissible* if $f \in \mathscr{A}(\overline{U})$ and f is equivariant; we denote by $\mathscr{E}\mathscr{A}(\overline{U})$ the set of all T-admissible maps. A homotopy $h : \overline{U} \times \boldsymbol{I} \to \boldsymbol{R}^n$ is called *T-admissible* if $h \in \mathscr{A}\mathscr{H}(\overline{U})$ and each h_t is equivariant, i.e., $h(Tx,t) = T(h(x,t))$ for $(x,t) \in \overline{U} \times \boldsymbol{I}$. The set of all T-admissible homotopies is denoted by $\mathscr{E}\mathscr{A}\mathscr{H}(\overline{U})$.

(1.5) Definition. Two maps $f, g \in \mathscr{E}\mathscr{A}(\overline{U})$ are called *homotopic in* $\mathscr{E}\mathscr{A}(\overline{U})$ (written $f \sim g$ in $\mathscr{E}\mathscr{A}(\overline{U})$) if there is a T-admissible homotopy $h \in \mathscr{E}\mathscr{A}\mathscr{H}(\overline{U})$ joining f and g.

Clearly, homotopy is an equivalence relation in $\mathscr{E}\mathscr{A}(\overline{U})$; we let $[f]$ be the homotopy class of $f \in \mathscr{E}\mathscr{A}(\overline{U})$ and denote by $\mathscr{E}\mathscr{A}[\overline{U}]$ the set of all such classes.

Generic equivariant maps

Our aim now is to show that under suitable conditions every homotopy class in $\mathscr{E}\mathscr{A}[\overline{U}]$ contains a generic equivariant map. We first establish some lemmas.

(1.6) LEMMA (Homotopy extension lemma). *Let (U, U_0) be a pair of open bounded T-invariant subsets of $\boldsymbol{R}^n$, and let $f_0, g_0 \in \mathscr{E}\mathscr{A}(\overline{U}_0)$ be homotopic in $\mathscr{E}\mathscr{A}(\overline{U}_0)$. Assume that f_0 extends to a map $f \in \mathscr{E}\mathscr{A}(\overline{U})$. Then there exist $g \in \mathscr{E}\mathscr{A}(\overline{U})$ and an open T-invariant set $V \subset U_0$ such that:*

(i) $f \sim g$ *in* $\mathscr{E}\mathscr{A}(\overline{U})$,
(ii) $g(x) = f(x)$ *for all* $x \in \overline{U} - U_0$,
(iii) $g_0^{-1}(0) \cap U_0 = g^{-1}(0) \cap U_0 \subset V$,
(iv) $g(x) = g_0(x)$ *for all* $x \in V$

PROOF. Let $k \in \mathscr{E}\mathscr{A}\mathscr{H}(\overline{U}_0)$ be a T-admissible homotopy joining f_0 and g_0. Choose an open T-invariant (1) subset V of U_0 such that $\overline{V} \subset U_0$ and $k(x,t) \neq 0$ for all $(x,t) \in (\overline{U}_0 - V) \times \boldsymbol{I}$ and then choose an open T-invariant subset V_1 of U_0 such that $\overline{V} \subset V_1 \subset \overline{V}_1 \subset U_0$. Let $\lambda : \boldsymbol{R}^n \to \boldsymbol{I}$ be a smooth T-invariant function such that

$$\lambda(x) = \begin{cases} 1 & \text{for } x \in \overline{V}, \\ 0 & \text{for } x \in \boldsymbol{R}^n - V_1. \end{cases}$$

Letting now

$$h(x,t) = \begin{cases} f(x) & \text{for } x \in \overline{U} - \overline{V}_1, \\ k(x, \lambda(x)t) & \text{for } x \in U_0, \end{cases} \qquad g(x) = h(x,1) \quad \text{for } x \in \overline{U},$$

we obtain a map g and a T-admissible homotopy $h : \overline{U} \times \boldsymbol{I} \to \boldsymbol{R}^n$ joining f and g and satisfying the assertion of the lemma. □

To state the next lemma, we need some terminology.

Let K be a compact T-invariant subset of $\boldsymbol{R}^n$, and k a positive integer. By a (T,k)-*simple covering* of K is meant a family $\{U_i\}_{i=1}^k$ of open subsets of $\boldsymbol{R}^n$ such that:

(i) $U_i \cap T(U_i) = \emptyset$ for each $i \in [k]$,
(ii) $K \subset \bigcup_{i=1}^k (U_i \cup T(U_i))$.

A compact $K \subset \boldsymbol{R}^n$ is said to be (T,k)-*simple* whenever it has a (T,k)-simple covering.

In what follows, given $f \in \mathscr{E}\mathscr{A}(\overline{U})$, we let

$$R(f) = \{x \in f^{-1}(0) \mid Df(x) \text{ is an isomorphism}\}$$

denote the set of *regular zeros* of f

In the remainder of this section we let U denote an allowable subset of $\boldsymbol{R}^n$ such that T acts freely on $\overline{U}$ (i.e., $Tx \neq x$ for all $x \in \overline{U}$, in other words, $\overline{U} \cap \boldsymbol{R}^p = \emptyset$).

---

(1) Observe that if $\widehat{V} \subset U_0$ is open, then $V = \widehat{V} \cup T\widehat{V} \subset U_0$ is also open and moreover T-invariant; similarly, given a smooth Urysohn function $\widehat{\lambda} . \boldsymbol{R}^n \to \boldsymbol{I}$, the function $\lambda : \boldsymbol{R}^n \to \boldsymbol{I}$ defined by $\lambda = \frac{1}{2}(\widehat{\lambda} + \widehat{\lambda} \circ T)$ is a smooth T-invariant Urysohn function.

(1.7) LEMMA. *Let $f \in \mathscr{E}\mathscr{A}(\overline{U})$, where $\overline{U}$ is (T,s)-simple for some $s \geq 2$. Then there exists a $g \in \mathscr{E}\mathscr{A}(\overline{U})$ such that:*
(i) *$f \sim g$ in $\mathscr{E}\mathscr{A}(\overline{U})$,*
(ii) *the compact set $g^{-1}(0) - R(g)$ is $(T, s-1)$-simple.*

PROOF. Assume that $\{U_i\}_{i=1}^{s}$ is a (T,s)-simple cover of $\overline{U}$. Set

$$K = \overline{U} - \bigcup_{i=2}^{s}(U_i \cup T(U_i))$$

and observe that:
(i) K is a compact subset of $U_1 \cup T(U_1)$,
(ii) $K = K_1 \cup T(K_1)$, where $K_1 = K \cap U_1$.
Take a regular value y_0 of $f|U \cap U_1$ such that $\|y_0\| < \inf\{\|f(x)\| \mid x \in \partial U\}$ and choose a smooth function $\lambda : \boldsymbol{R}^n \to \boldsymbol{I}$ with $\lambda(x) = 1$ for $x \in K_1$ and $\lambda(x) = 0$ for $x \in \boldsymbol{R}^n - U_1$.

The reader can now verify that $g : \overline{U} \to \boldsymbol{R}^n$ given by

$$g(x) = \begin{cases} f(x) - \lambda(x)y_0 & \text{for } x \in U_1 \cap \overline{U}, \\ f(x) - \lambda(Tx)Ty_0 & \text{for } x \in T(U_1) \cap \overline{U}, \\ f(x) & \text{for } x \in \overline{U} - (U_1 \cup T(U_1)), \end{cases}$$

has the desired properties. □

(1.8) LEMMA. *Let $f \in \mathscr{E}\mathscr{A}(\overline{U})$ and assume that the set $f^{-1}(0) - R(f)$ is (T,s)-simple for some $s \geq 1$. Then there exists a $g \in \mathscr{E}\mathscr{A}(\overline{U})$ such that:*
(i) *$f \sim g$ in $\mathscr{E}\mathscr{A}(\overline{U})$,*
(ii) *the compact set $g^{-1}(0) - R(g)$ is $(T, s-1)$-simple.*

PROOF. Since by assumption, $f^{-1}(0) - R(f)$ is (T,s)-simple, there exists an open T-invariant set $U_0 \subset U$ such that:
(i) $f^{-1}(0) - R(f) \subset U_0$,
(ii) $R(f) \subset U - \overline{U}_0$,
(iii) $\overline{U}_0$ is (T,s)-simple.
Set now $f_0 = f|\overline{U}_0 : \overline{U}_0 \to \boldsymbol{R}^n$ and observe that $f_0 \in \mathscr{E}\mathscr{A}(\overline{U}_0)$. By Lemma (1.7), there exists a homotopy $h \in \mathscr{E}\mathscr{A}\mathscr{H}(\overline{U}_0)$ joining f_0 to some g_0 such that the compact set $g_0^{-1}(0) - R(g_0)$ is $(T, s-1)$-simple. By the homotopy extension lemma, there exists a $g \in \mathscr{E}\mathscr{A}(\overline{U})$ with $f \sim g$ in $\mathscr{E}\mathscr{A}(\overline{U})$ such that the compact set $g^{-1}(0) - R(g) = g_0^{-1}(0) - R(g_0)$ is $(T, s-1)$-simple. □

By an inductive procedure, Lemma (1.8) leads immediately to

(1.9) THEOREM. *Let $U \subset \boldsymbol{R}^n$ be allowable and such that T acts freely on $\overline{U}$, and let $f \in \mathscr{E}\mathscr{A}(\overline{U})$. Then there exists a generic map $g \in \mathscr{E}\mathscr{A}(\overline{U})$ such that $f \sim g$ in $\mathscr{E}\mathscr{A}(\overline{U})$*

Normal maps and homotopies

Let U be a fixed allowable (i.e., T-invariant open bounded) subset of $\boldsymbol{R}^n = \boldsymbol{R}^p \oplus \boldsymbol{R}^q$, $p + q = n$. For any $f : \overline{U} \to \boldsymbol{R}^n$, we write $f = (f_1, f_2)$, where $f_1 : \overline{U} \to \boldsymbol{R}^p$, $f_2 : \overline{U} \to \boldsymbol{R}^q$.

Given $\varepsilon > 0$, we let

$$U(\varepsilon) = \{x = (u, v) \in U \mid \|v\| < \varepsilon\}$$

and call a map $f \in \mathscr{E}\mathscr{A}(\overline{U})$ *ε-normal* if

$$x = (u, v) \in U(\varepsilon) \;\Rightarrow\; f_2(u, v) = v.$$

A map $f \in \mathscr{E}\mathscr{A}(\overline{U})$ is called *normal* if it is ε-normal for some $\varepsilon > 0$. More generally, a homotopy $h \in \mathscr{E}\mathscr{A}\mathscr{H}(\overline{U})$ is *normal* if there exists an $\varepsilon > 0$ such that each $h_t : \overline{U} \to \boldsymbol{R}^n$ is ε-normal. We let

$$\mathscr{N}\mathscr{E}\mathscr{A}(\overline{U}) = \{f \in \mathscr{E}\mathscr{A}(\overline{U}) \mid f \text{ is normal}\}.$$

(1.10) DEFINITION. Two maps $f, g \in \mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$ are called *homotopic in* $\mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$ (written $f \approx g$ in $\mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$) if there exists a normal homotopy $h \in \mathscr{E}\mathscr{A}\mathscr{H}(\overline{U})$ joining f and g.

Clearly, homotopy is an equivalence relation in $\mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$; we let $[\![f]\!]$ be the normal homotopy class of $f \in \mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$ and denote by $\mathscr{N}\mathscr{E}\mathscr{A}[\overline{U}]$ the set of all such classes.

The construction of the degree for maps in $\mathscr{E}\mathscr{A}(\overline{U})$ relies on the following

(1.11) THEOREM. *If U is an allowable open subset of $\boldsymbol{R}^n$, then the map*

$$\tau : \mathscr{N}\mathscr{E}\mathscr{A}[\overline{U}] \to \mathscr{E}\mathscr{A}[\overline{U}], \quad [\![f]\!] \mapsto [f],$$

is bijective.

PROOF. (i) τ *is surjective*: Let $f = (f_1, f_2) \in \mathscr{E}\mathscr{A}(\overline{U})$ be given. We have to find a $g \in \mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$ such that $f \sim g$ in $\mathscr{E}\mathscr{A}(\overline{U})$. We first note that $f_2(u, 0) = 0$ for all $(u, 0) \in \overline{U}$, because f is equivariant. Letting $s = \inf\{\|f(u, v)\| \mid (u, v) \in \partial U\} > 0$, choose an $\varepsilon > 0$ so that for all $(u, v) \in \overline{U}$,

$$\|v\| < 2\varepsilon \;\Rightarrow\; \|f_2(u, v)\| < s.$$

To establish our assertion, let $\lambda : \boldsymbol{R} \to \boldsymbol{I}$ be a smooth function such that

$$\lambda(t) = \begin{cases} 1 & \text{for } t < \varepsilon, \\ 0 & \text{for } t > 2\varepsilon, \end{cases}$$

and define a homotopy $h : \overline{U} \times \boldsymbol{I} \to \boldsymbol{R}^n$ by setting

$$h(u, v, t) = (f_1(u, v), t\lambda(\|v\|)v + (1 - t\lambda(\|v\|))f_2(u, v))$$

for $(u, v, t) \in \overline{U} \times I$. It can now be easily verified that h is a homotopy in $\mathscr{E}\mathscr{A}(\overline{U})$ joining $f(u,v) = h(u,v,0)$ and $g(u,v) = h(u,v,1)$. Clearly, g is normal, which proves the surjectivity of τ.

(ii) τ *is injective*: Given f, g in $\mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$ with $f \sim g$ in $\mathscr{E}\mathscr{A}(U)$, we need to show that there exists a normal homotopy joining f and g. The proof is analogous to that in (i) and is left to the reader.

Equivariant degree

We first define the degree for maps in $\mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$. As before, let $U \subset \boldsymbol{R}^n$ be allowable and $f = (f_1, f_2) \in \mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$. Set $U_0 = U \cap \boldsymbol{R}^p$ and, assuming $U_0 \neq \emptyset$, define $g_0 : \overline{U}_0 \to \boldsymbol{R}^p$ by $g_0(u) = f_1(u, 0)$; clearly, $g_0 \in \mathscr{A}(\overline{U}_0)$.

We now let

$$d_0 = \begin{cases} d(g_0, U_0) & \text{if } U_0 \neq \emptyset, \\ 0 & \text{if } U_0 = \emptyset. \end{cases}$$

Because f is normal, there exists an $\varepsilon > 0$ such that the formula

$$g_1(x) = f(x) \qquad \text{for } x \in \overline{U} - U(\varepsilon)$$

defines a map g_1 in $\mathscr{E}\mathscr{A}(\overline{U}_1)$, where $U_1 = U - \overline{U(\varepsilon)}$. Since T acts freely on $\overline{U}_1$, there exists an integer d_1 such that $d(g_1, U_1) = 2d_1$ ([1]).

We now define the T-*equivariant degree* $\deg_T(f, U)$ of f by setting

$$\deg_T(f, U) = (d_0, d_1) \in \boldsymbol{Z} \oplus \boldsymbol{Z}.$$

(1.12) THEOREM. *Let $\mathscr{N}$ denote the class of all maps $f \in \mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$, where U is an allowable subset of $\boldsymbol{R}^n$. Then there exists a function*

$$\deg_T : \mathscr{N} \to \boldsymbol{Z} \oplus \boldsymbol{Z}$$

assigning a degree $\deg_T(f, U)$ *to each* $f \in \mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$ *and satisfying:*

(1) (Existence) *If* $\deg_T(f, U) \neq 0$, *then* $Z(f) \neq \emptyset$.

(2) (Excision) *If $U_0 \subset U$ is an open T-invariant subset of U such that $Z(f) \subset U_0$, then* $\deg_T(f, U) = \deg_T(f|U_0, U_0)$.

(3) (Additivity) *If V_1, V_2 are two disjoint open T-invariant subsets of U such that $Z(f) \subset V_1 \cup V_2$, then*

$$\deg_T(f, U) = \deg_T(f|V_1, V_1) + \deg_T(f|V_2, V_2).$$

(4) (Homotopy) *If h is a homotopy in $\mathscr{N}\mathscr{E}\mathscr{A}(\overline{U})$, then* $\deg_T(h_t, U)$ *does not depend on $t \in I$.*

([1]) This follows from the fact that if g_1 is generic and $g_1(x_0) = 0$, then also $g_1(Tx_0) = 0$, and $\operatorname{sgn}\det Dg_1(Tx_0) = \operatorname{sgn}\det g_1(x_0)$ (because $g_1 = T \circ g_1 \circ T$ and $\det T = \pm 1$ imply $\det Dg_1(Tx_0) = \det Dg_1(x_0)$).

We remark that for a map $f \in \mathscr{NEA}(\overline{U})$ the "ordinary" degree $d(f, U)$ is determined by the pair (d_0, d_1) of integers via the formula

$$d(f, U) = d_0 + 2d_1 \quad \text{and} \quad d_0 = d(f, U(\varepsilon)).$$

The extension of the degree to arbitrary maps in $\mathscr{EA}(\overline{U})$ relies on Theorem (1.11). Starting off with f in $\mathscr{EA}(\overline{U})$, we select a map g in $\mathscr{NEA}(\overline{U})$ with $[g] = [f]$ and define

$$\mathrm{Deg}_T(f, U) = \deg_T(g, U).$$

By the above theorem, $\mathrm{Deg}_T(f, U)$ is independent of the choice of g. □

As a consequence of Theorem (1.12), we obtain the main result:

(1.13) THEOREM. *Let $\mathscr{E}$ denote the class of all maps $f \in \mathscr{EA}(\overline{U})$, where U is an allowable subset of $\boldsymbol{R}^n$, $n \geq 2$. Then $f \mapsto \mathrm{Deg}_T(f, U)$ defines a function $\mathrm{Deg}_T : \mathscr{E} \to \boldsymbol{Z} \oplus \boldsymbol{Z}$ satisfying:*
(1) Existence
(2) Excision
(3) Additivity
(4) (Homotopy) *If h is a homotopy in $\mathscr{EA}(\overline{U})$, then $\mathrm{Deg}_T(h_t, U)$ does not depend on $t \in \boldsymbol{I}$.* □

2. The Infinite-Dimensional E^+-Cohomology

Let E be an infinite-dimensional normed linear space and $H_E^{\infty-*}$ the corresponding finite-codimensional Čech cohomology. Let $V \subset E$ be a closed linear subspace of finite codimension m, and S_V the unit sphere in V The dimension axiom for $H_E^{\infty-*}$, together with the other Eilenberg–Steenrod axioms, implies that

$$H_E^{\infty-q}(S_V) \cong H^{q-m-1}(\text{point}) \qquad \text{for each } q \in \boldsymbol{Z}.$$

Therefore, the theory $H_E^{\infty-*}$ distinguishes spheres according to their codimension, exactly as the usual Čech cohomology distinguishes them according to their dimension.

But in an infinite-dimensional space, a sphere may have both infinite dimension and infinite codimension. The aim of this section is to outline the construction of a cohomology theory that can distinguish between such objects. For technical reasons we will describe this construction in the context of real Hilbert spaces.

Relative dimension

Let E be a fixed infinite-dimensional Hilbert space over $\boldsymbol{R}$, and $G(E)$ the *Grassmannian* of E, i.e., the collection of all closed linear subspaces of E.

We set

$$G_*(E) = \{V \in G(E) \mid \dim(V) < \infty\},$$
$$G^*(E) = \{V \in G(E) \mid \operatorname{codim}(V) < \infty\}.$$

If $V \in G(E)$, we let $V^\perp$ denote its orthogonal complement, and P_V the orthogonal projection of E onto V.

In order to compare two different elements of $G(E)$ we need the following

(2.1) DEFINITION. Two linear subspaces $V, W \in G(E)$ are said to be *commensurable* (written $V \sim W$) if the linear operator $P_V - P_W$ is completely continuous. Commensurability is an equivalence relation in $G(E)$, and if $V \sim W$, the *relative dimension* $\dim(V, W)$ of V with respect to W is the (possibly negative) integer defined by

$$\dim(V, W) = \dim(V \cap W^\perp) - \dim(V^\perp \cap W).$$

Note that the subspaces $V \cap W^\perp$ and $V^\perp \cap W$ are finite-dimensional, being the sets of fixed points of the completely continuous operators

$$P_{W^\perp} \circ P_V = (P_V - P_W) \circ P_V, \qquad P_{V^\perp} \circ P_W = (P_W - P_V) \circ P_W,$$

respectively.

We list a few simple examples and properties of commensurability:

(i) Any $V, W \in G_*(E)$ are commensurable, and

$$\dim(V, W) = \dim V - \dim W.$$

(ii) Any $V, W \in G^*(E)$ are commensurable, and

$$\dim(V, W) = \operatorname{codim} W - \operatorname{codim} V.$$

(iii) If W_r has dimension r, then $V \oplus W_r \sim V$ for any $V \in G(E)$ such that $V \cap W_r = \{0\}$, and $\dim(V \oplus W_r, V) = r$.

(iv) If W_s is an s-codimensional subspace of V, then $V \sim W_s$ and $\dim(W_s, V) = -s$.

(v) If $V \in G(E)$, $K \in \mathscr{L}(E, E)$ is completely continuous, and $T = I + K$, then $T(V) \in G(E)$ and $T(V) \sim V$.

The E^+-cohomology

We fix an orthogonal splitting $E = E^+ \oplus E^-$, where E^+ and E^- are infinite-dimensional closed linear subspaces of E. We will construct a cohomology theory, the E^+-cohomology $H^*_{E^+}$, such that the spheres in closed linear subspaces commensurable with E^- have nontrivial cohomology, depending on their relative dimension with respect to E^-.

Let τ_{E^+} be the product topology of the weak topology of E^+ and the strong topology of E^-; clearly, τ_{E^+} induces the weak topology on every closed linear subspace commensurable with E^+, and the strong topology

on every closed linear subspace commensurable with E^- In particular, any sphere in a subspace commensurable with E^- is τ_{E^+}-closed.

We now describe the basic category $\mathfrak{L}_{E^+}$ on which the E^+-cohomology will be defined. The objects of $\mathfrak{L}_{E^+}$ are the τ_{E^+}-closed bounded subsets of E, and its morphisms are the τ_{E^+}-continuous maps $f : X \to Y$ of the form

$$f(x) = x - F(x), \quad x \in X,$$

where the τ_{E^+}-closure of $F(X)$ is τ_{E^+}-compact. Such morphisms will be referred to as *E^+-compact fields*. Note that every linear compact field is also an E^+-compact field.

By an *E^+-compact homotopy* in $\mathfrak{L}_{E^+}$ is meant a τ_{E^+}-continuous map $h : X \times I \to Y$ of the form

$$h(x,t) = x - H(x,t), \quad x \in X,\ t \in I,$$

where the τ_{E^+}-closure of $H(X \times I)$ is τ_{E^+}-compact. Two E^+-compact fields $f, g : X \to Y$ are said to be *E^+-compactly homotopic* (written $f \simeq g$) if they can be joined by such a homotopy. This is an equivalence relation, and $\mathfrak{L}_{E^+}$ is an h-category. The symbol $\mathfrak{L}^2_{E^+}$ will denote the corresponding h-category of pairs.

Let $H^* = \{H^q, \delta^q\}$ be the Čech cohomology with coefficients in some abelian group G. We use here the integer grading, with the convention that $H^q(X) = 0$ for $q < 0$.

(2.2) DEFINITION. By an *E^+-cohomology* $H^*_{E^+} = \{H^q_{E^+}, \delta^q_{E^+}\}_{q \in Z}$ on $\mathfrak{L}^2_{E^-}$ (with coefficients in G) is meant a sequence of contravariant functors $(X, A) \mapsto H^q_{E^+}(X, A)$ (one for each $q \in Z$) from $\mathfrak{L}^2_{E^+}$ to **Ab** together with a sequence of natural transformations

$$\delta^q(X, A) : H^q_{E^+}(A) \to H^{q+1}_{E^+}(X, A)$$

satisfying the homotopy, exactness and strong excision axioms and the following property: if $V \subset E$ is a closed linear subspace commensurable with E^- and S_V is the unit sphere in V, then

$$H^q_{E^+}(S_V) \cong H^{q-m+1}(\text{point}) \quad \text{for } q \in Z,$$

where $m = \dim(V, E^-)$; we will refer to the above property as the *dimension axiom* for the E^+-cohomology.

We are now in a position to state the main result of this section:

(2.3) THEOREM. *For any abelian group G, there exists a cohomology theory $H^*_{E^+}$ on $\mathfrak{L}^2_{E^+}$ with coefficients in G.*

REMARK. The topology τ_{E^+}, the category $\mathfrak{L}_{E^+}$, and the cohomology theory $H^*_{E^+}$ all depend on the orthogonal splitting $E = E^+ \oplus E^-$. One could also

have used the subscript E^- in the notation. However, if one works in a Banach space, where a closed linear subspace need not have a topological complement, E^+ turns out to be the relevant subspace. For example, the topology τ_{E^+} could be defined as the weakest topology on E such that all the bounded linear functionals on E and the quotient projection $E \to E/E^+$ are continuous.

Construction of $H^*_{E^+}$

In the remainder of this section we will give a general idea of how the E^+-cohomology theory is constructed. The construction turns out to be similar to that of $H^{\infty-*}$; for obvious reasons, we will only sketch the arguments, pointing out the main differences with the previous case.

Let $G_*(E^-) = \{L_\alpha, L_\beta, \ldots\}$ be the set of all finite-dimensional linear subspaces of E^-. For notational convenience we establish a one-to-one correspondence between the symbols $\alpha, \beta, \ldots$ and the elements of $G_*(E^-)$. We write $\alpha \preceq \beta$ if $L_\alpha \subset L_\beta$ and denote by $\mathcal{L}$ the set $G_*(E^-)$ partially ordered by the relation $\preceq$; clearly, $\mathcal{L}$ is a directed set.

Choose arbitrarily an orientation of each $\alpha \in \mathcal{L}$ and observe that given $\alpha \preceq \beta$ with $d(\beta) = d(\alpha) + 1$ (where $d(\alpha) = \dim L_\alpha$), the orientations of L_α and L_β determine in a unique way two closed half-spaces $L^+_{\beta_\alpha}$, $L^-_{\beta_\alpha}$ of L_β such that

$$L^+_{\beta_\alpha} \cup L^-_{\beta_\alpha} = L_\beta \quad \text{and} \quad L^+_{\beta_\alpha} \cap L^-_{\beta_\alpha} = L_\alpha.$$

If $X \subset E$ and α, β are as above, define τ_{E^+}-compact sets

$$X_\alpha = X \cap (E^+ \oplus L_\alpha),$$
$$X^+_{\beta_\alpha} = X \cap (E^+ \oplus L^-_{\beta_\alpha}),$$
$$X^-_{\beta_\alpha} = X \cap (E^+ \oplus L^-_{\beta_\alpha}).$$

Now let (X, A) be an object of $\mathfrak{L}^2_{E^+}$. Since

$$(X_\beta, A_\beta) = (X^+_{\beta_\alpha} \cup X^-_{\beta_\alpha}, A^+_{\beta_\alpha} \cup A^-_{\beta_\alpha}),$$
$$(X_\alpha, A_\alpha) = (X^+_{\beta_\alpha} \cap X^-_{\beta_\alpha}, A^+_{\beta_\alpha} \cap A^-_{\beta_\alpha}),$$
$$A^+_{\beta_\alpha} = A_\beta \cap X^+_{\beta_\alpha}, \qquad A^-_{\beta_\alpha} = A_\beta \cap X^-_{\beta_\alpha},$$

we have the relative Mayer–Vietoris homomorphism (see (18.1.6))

$$\Delta_{\alpha\beta}(X, A) : H^{q+d(\alpha)}(X_\alpha, A_\alpha) \to H^{q+d(\beta)}(X_\beta, A_\beta).$$

Now, given any $\alpha \preceq \beta$ in $\mathcal{L}$, we define the homomorphism

$$\Delta_{\alpha\beta} : H^{q+d(\alpha)}(X_\alpha, A_\alpha) \to H^{q+d(\beta)}(X_\beta, A_\beta)$$

as follows: if $d(\beta) = d(\alpha) + 1$, we let $\Delta_{\alpha\beta}$ be the relative Mayer–Vietoris homomorphism $\Delta_{\alpha\beta}(X, A)$ as above; otherwise, we take a chain of consecutive

elements of $\mathcal{L}$ and define $\Delta_{\alpha\beta}$ to be the composition of the corresponding Mayer Vietoris homomorphisms.

Although the argument of Lemma (18.3.3), based on the Alexander-Pontrjagin duality, does not apply here, the functoriality of the Mayer–Vietoris homomorphism (18.1.8) can be used to prove directly that the definition of $\Delta_{\alpha\beta}$ does not depend on the choice of the chain in $\mathcal{L}$ joining α and β. Therefore, $\{H^{q+d(\alpha)}(X_\alpha, A_\alpha), \Delta_{\alpha\beta}\}_{\alpha\in\mathcal{L}}$ is a direct system of groups.

(2.4) Definition. For an object (X, A) in $\mathfrak{L}^2_{E^+}$, we define the abelian group

$$H^q_{E^+}(X, A) = \varinjlim_{\alpha\in\mathcal{L}} \{H^{q+d(\alpha)}(X_\alpha, A_\alpha), \Delta_{\alpha\beta}\}$$

to be the direct limit over $\mathcal{L}$ of the direct system constructed above.

As a first step to proving the dimension axiom, we shall compute the E^+-cohomology of the unit sphere in a closed subspace of a particular type. Let V^+ be a linear subspace of E^+ of finite dimension r, and let V^- be a closed linear subspace of E^- of finite codimension s. Then $V = V^+ \oplus V^-$ is closed, commensurable with E^-, and $\dim(V, E^-) = r - s$. Let S_V be the unit sphere in V, and let L_σ be an s-dimensional complement of V^- in E^- The direct limit used to define $H^q_{E^+}(S_V)$ can be restricted to $\mathcal{L}_0 = \{\alpha \in \mathcal{L} \mid \sigma \preceq \alpha\}$, which is a cofinal subset of $\mathcal{L}$. If $\alpha \in \mathcal{L}_0$, we see that $\dim V \cap (E^+ \oplus L_\alpha) = r + d(\alpha) - s$, and $S_\alpha = (S_V)_\alpha$ is the unit sphere in $V \cap (E^+ \oplus L_\alpha)$, so

$$H^{q+d(\alpha)}(S_\alpha) = \begin{cases} G & \text{if } q = r - s - 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that if $\alpha \preceq \beta$ in $\mathcal{L}_0$ and $d(\beta) = d(\alpha) + 1$, then the Mayer–Vietoris homomorphism $\Delta_{\alpha\beta} : H^{q+d(\alpha)}(S_\alpha) \to H^{q+d(\beta)}(S_\beta)$ is an isomorphism. From this we infer that

$$H^q_{E^+}(S_V) = \begin{cases} G & \text{if } q = r - s - 1, \\ 0 & \text{otherwise.} \end{cases}$$

The complete version of the dimension axiom will follow once we prove the functoriality with respect to E^+-compact fields: indeed, every closed subspace commensurable with E^- is the image of a subspace of the above kind under a linear invertible isometry of the form $I + T$, where T is completely continuous, and such a map is an invertible morphism in $\mathfrak{L}_{E^+}$.

The functoriality is proved in two steps: first for a restricted class of morphisms, and then extended to all morphisms in $\mathfrak{L}_{E^+}$ by continuity and homotopy arguments. To make this more precise, we need some terminology. We say that an E^+-compact field $f : X \to Y$, $f(x) = x - F(x)$, is an $(E^+ \oplus L_\alpha)$-*field* if the image of F is contained in $E^+ \oplus L_\alpha$; any such field (for some $\alpha \in \mathcal{L}$) will also be called an E^+-*finite field*. Two E^+-finite fields

$f, g : X \to Y$ that are $(E^+ \oplus L_\alpha)$-homotopic for some $\alpha \in \mathcal{L}$ are said to be E^+*-finitely homotopic.*

If $f : X \to Y$ is an $(E^+ \oplus L_{\alpha_0})$-field for some $\alpha_0 \in \mathcal{L}$, then for any $\alpha \succeq \alpha_0$ in $\mathcal{L}$, f maps X_α into Y_α, and hence $f_\alpha : X_\alpha \to Y_\alpha$ determines the homomorphisms

$$H^{q+d(\alpha)}(f_\alpha) : H^{q+d(\alpha)}(Y_\alpha) \to H^{q+d(\alpha)}(X_\alpha).$$

The functoriality of the Mayer–Vietoris homomorphism now implies that the family $\{H^{q+d(\alpha)}(f_\alpha)\}_{\alpha \succeq \alpha_0}$ is a direct system of homomorphisms from $\{H^{q+d(\alpha)}(Y_\alpha), \Delta_{\alpha\beta}(Y)\}$ to $\{H^{q+d(\alpha)}(X_\alpha), \Delta_{\alpha\beta}(X)\}$. The homomorphism

$$f^* : H^q_{E^+}(Y) \to H^q_{E^+}(X)$$

is defined as the direct limit of the above system over the cofinal subset $\mathcal{L}_0 = \{\alpha \in \mathcal{L} \mid \alpha_0 \preceq \alpha\}$ of $\mathcal{L}$. It is easy to verify that contravariant functoriality holds for E^+-finite fields and also that if two E^+-finite fields $f, g : X \to Y$ are E^+-finitely homotopic, then $f^* = g^*$.

The key tool for extending this definition to arbitrary E^+-compact fields is again the continuity property of $H^*_{E^+}$. If X is an object in $\mathfrak{L}_{E^+}$, then by an *approximating sequence* for X is meant a descending sequence $X_1 \supset X_2 \supset \cdots$ of objects in $\mathfrak{L}_{E^+}$ such that $X = \bigcap_{k=1}^{\infty} X_k$. The inclusion maps $i_k : X \to X_k$ and $i_{kl} : X_l \to X_k$, $l \geq k$, are $(E^+ \oplus \{0\})$-fields, so the diagram

$$\begin{tikzcd} H^q_{E^+}(X_k) \arrow[r, "i^*_{kl}"] \arrow[d, "i^*_k"'] & H^q_{E^+}(X_l) \arrow[dl, "i^*_l"] \\ H^q_{E^+}(X) & \end{tikzcd}$$

commutes. Therefore, we can consider the direct limit

$$\varinjlim_k i^*_k : \varinjlim\{H^q_{E^+}(X_k), i^*_{kl}\} \to H^q_{E^+}(X),$$

and the continuity property asserts that this is an isomorphism. This follows from the continuity of Čech cohomology, because the sets $X_k \cap (E^+ \oplus L_\alpha)$ are τ_{E^+}-compact, being τ_{E^+}-closed in a linear subspace that inherits the weak topology. Compactness is crucial here, and this explains why we are working with the topology τ_{E^+}.

Given an E^+-compact field $f : X \to Y$, by an *approximating system* for f is meant a sequence $\{Y_k, f_k\}$, where $\{Y_k\}$ is an approximating system for Y, and $f_k : X \to Y_k$ are E^+-finite fields such that f_k is E^+-compactly homotopic to $j_k \circ f$, while f_k is E^+-finitely homotopic to $i_{kl} \circ f_l$, where $j_k : Y \hookrightarrow Y_k$ and $i_{kl} : Y_l \hookrightarrow Y_k$ are the inclusions. It is not difficult to show that every E^+-compact field has an approximating system. Then the

commutative diagram

$$\begin{tikzcd} H^q_{E+}(Y_k) \arrow[r, "r^*_{kl}"] \arrow[d, "f_k^*"'] & H^q_{E+}(Y_l) \arrow[dl, "f_l^*"] \\ H^q_{E+}(X) & \end{tikzcd}$$

together with the continuity of H^*_{E+} allows us to make the following

(2.5) DEFINITION. The E^+-compact field $f : X \to Y$ induces the homomorphism

$$f^* = (\varinjlim_k f_k^*) \circ (\varinjlim_k j_k^*)^{-1} : H^q_{E+}(Y) \to H^q_{E+}(X)$$

between the E^+-cohomology groups.

One verifies that this definition does not depend on the choice of the approximating system, as in Proposition (18.2.6). Contravariant functoriality is proved as in (18.2.9), after we observe that $E^+ \oplus L_\alpha$ with the weak topology induced by τ_{E+} is a locally convex space, so the Tietze extension theorem holds for $(E^+ \oplus L_\alpha)$-valued maps. The construction of f^* can also be easily generalized to E^+-compact fields between pairs.

Let (X, A) be an object in $\mathfrak{L}^2_{E+}$. For each $\alpha \in \mathcal{L}$, we have the coboundary homomorphism

$$\delta^{q+d(\alpha)}(X_\alpha, A_\alpha) : H^{q+d(\alpha)}(A_\alpha) \to H^{q+d(\alpha)+1}(X_\alpha, A_\alpha).$$

Let $\alpha \preceq \beta$ be a relation in $\mathcal{L}$ such that $d(\beta) = d(\alpha) + 1$. An argument involving cone suspension shows that the diagram

$$\begin{tikzcd} H^{q+d(\alpha)}(A_\alpha) \arrow[r, "\delta^{q+d(\alpha)}(X_\alpha, A_\alpha)"] \arrow[d, "\Delta_{\alpha\beta}(A)"'] & H^{q+d(\alpha)+1}(X_\alpha, A_\alpha) \arrow[d, "\Delta_{\alpha\beta}(X,A)"] \\ H^{q+d(\beta)}(A_\beta) \arrow[r, "\delta^{q+d(\beta)}(X_\beta, A_\beta)"] & H^{q+d(\beta)+1}(X_\beta, A_\beta) \end{tikzcd}$$

anticommutes (by (18.1.10) and (18.5.7)). Hence $\{(-1)^{d(\alpha)}\delta^{q+d(\alpha)}(X_\alpha, A_\alpha)\}$ is a direct system of homomorphisms from $\{H^{q+d(\alpha)}(A_\alpha), \Delta_{\alpha\beta}(A)\}$ to $\{H^{q+d(\alpha)+1}(X_\alpha, A_\alpha), \Delta_{\alpha\beta}(X, A)\}$, so we can take direct limits.

(2.6) DEFINITION. Given an object (X, A) in $\mathfrak{L}^2_{E+}$, the homomorphism

$$\delta^q_{E+} = \varinjlim_{\alpha \in \mathcal{L}} \{(-1)^{d(\alpha)}\delta^{q+d(\alpha)}(X_\alpha, A_\alpha)\} : H^q_{E+}(A) \to H^{q+1}_{E+}(X, A)$$

is the *coboundary homomorphism for* H^*_{E+}.

The exactness of the long cohomology sequence and the naturality of the coboundary with respect to E^+-compact fields are proved as in (18.5.8) and (18.5.11).

3. Lefschetz Theorem for $\mathcal{NB}$-Maps of Compacta

We now turn to another generalization of the Lefschetz–Hopf theorem, by showing that with Čech homology it remains valid for the class of $\mathcal{NB}$-maps of arbitrary compacta having Čech homology of finite type; this class contains all self-maps of compact ANRs. In this section we consider only compact metric spaces and always use Čech homology over the rationals Q; when no confusion can arise, the Q will be omitted in the notation, so that the graded homology of X over Q will be denoted simply by $H_*(X) = \{H_n(X)\}$.

We begin by gathering the main features of Čech theory that will be needed, and first recall the continuity property of Čech homology on the category of compact pairs:

(3.1) PROPOSITION.
(a) *Let $(X_1, A_1) \supset (X_2, A_2) \supset \cdots$ be a descending sequence of compact pairs with $(X, A) = \bigcap_{i=1}^{\infty}(X_i, A_i)$. Then*

$$H_*(X, A) \cong \varprojlim\{H_*(X_k, A_k), (i_{kl})_*\},$$

where $(i_{kl})_*$ are the bonding homomorphisms.
(b) *Fix n and for each k, let $j_k : H_n(X, A) \to H_n(X_k, A_k)$ be induced by $(X, A) \subset (X_k, A_k)$. If $z \in H_n(X, A)$ is not zero, then there is an index $k(z)$ such that $j_k(z) \neq 0$ for all $k \geq k(z)$.* □

Proposition (3.1)(b) has the following, sometimes more useful formulation:

(3.2) PROPOSITION. *With the notation in* (3.1), *let $\{z_1, \ldots, z_s\}$ be a linearly independent set in $H_n(X, A)$. Then there exists an N such that:*
(a) *$\{j_k(z_1), \ldots, j_k(z_s)\}$ is linearly independent in $H_n(X_k, A_k)$ for all $k \geq N$,*
(b) *in particular, j_k is monic on the subspace $R_s = \langle z_1, \ldots, z_s\rangle$ generated by $\{z_1, \ldots, z_s\}$ for all $k \geq N$.*

PROOF. For each $k \in N$, let N_k be the kernel of $j_k|R_s : R_s \to H_n(X_k, A_k)$; then $\{N_k\}_{k=1}^{\infty}$ is a descending sequence of subspaces of $H_n(X, A)$. To prove our assertion, we need only show that one of them is zero. Indeed, otherwise there is an m such that $\{0\} \neq N_m = N_{m+1} = \cdots$. Now, by applying (3.1)(b) to a nontrivial element $z \in N_m$, we obtain a contradiction. □

A compact space is said to have *homology of finite type* if all its Betti numbers are finite, and almost all are zero. (Recall that the nth *Betti number* of a space X is the dimension of $H_n(X)$.)

We will need the following property of Čech homology:

(3.3) PROPOSITION. *Let Z be a compactum having homology of finite type. Then there exists an $\varepsilon = \varepsilon(Z)$ with the following property: For every compactum X, any two ε-close maps $f, g : X \to Z$ induce the same homomorphism $f_* = g_* : H_*(X) \to H_*(Z)$.*

PROOF. Let $H_n(Z) = \varprojlim\{H_n(N(\mathcal{U})), p_{\mathcal{U}\mathcal{V}}\}$, where $N(\mathcal{U})$ denotes the nerve of the finite open cover $\mathcal{U}$ of Z, and the $p_{\mathcal{U}\mathcal{V}}$ are the bonding morphisms; and let $p_\mathcal{U} : H_n(Z) \to H_n(N(\mathcal{U}))$ be the projection of this inverse limit to $H_n(N(\mathcal{U}))$. Because $H_n(Z)$ is finite-dimensional, a proof entirely similar to that of (3.2) shows that there is a finite open cover $\mathcal{U} = \{U_1, \ldots, U_s\}$ of Z such that $p_\mathcal{W} : H_n(Z) \to H_n(N(\mathcal{W}))$ is monic for each $\mathcal{W}$ refining $\mathcal{U}$. Use normality to find an open cover $\mathcal{V} = \{V_1, \ldots, V_s\}$ of Z such that $\overline{V_i} \subset U_i$ for each $i \in [s]$, and set $\varepsilon = \min \operatorname{dist}(\overline{V_i}, X - U_i)$.

Now let X be any compactum, and let $f, g : X \to Z$ be ε-close; then $g^{-1}(V_i) \subset f^{-1}(U_i)$ for each $i \in [s]$, so with the simplicial map ϑ determined by the vertex map $g^{-1}(V_i) \mapsto f^{-1}(U_i)$ and the obvious homomorphisms f_*, g_* (induced by simplicial maps $f_\#$, $g_\#$) we have the commutative diagram

$$\begin{tikzcd}
 & H_n[N(g^{-1}(\mathcal{V}))] \arrow[r, "g_*"] \arrow[dd, "\vartheta_*"] & H_n(N(\mathcal{V})) \arrow[dd, "p_{\mathcal{V}\mathcal{U}}"] & \\
H_n(X) \arrow[ur, "q_1"] \arrow[dr, "q_2"'] & & & H_n(Z) \arrow[ul, "p_\mathcal{V}"'] \arrow[dl, "p_\mathcal{U}"] \\
 & H_n[N(f^{-1}(\mathcal{U}))] \arrow[r, "f_*"] & H_n(N(\mathcal{U})) &
\end{tikzcd}$$

where the q_j are the projections in the spectrum for $H_n(X)$. Let $a \in H_n(X)$ and set $f_*(a) = b$, $g_*(a) = \widehat{b}$; then $g_*q_1(a) = p_\mathcal{V}(\widehat{b})$ and $f_*q_2(a) = p_\mathcal{U}(b)$, so by commutativity

$$p_\mathcal{U}(\widehat{b}) = p_{\mathcal{V}\mathcal{U}}p_\mathcal{V}(\widehat{b}) = p_{\mathcal{V}\mathcal{U}}g_*q_1(a) = f_*\vartheta q_1(a) = f_*q_2(a) = p_\mathcal{U}(b).$$

Since $p_\mathcal{U}$ is monic, this shows that $\widehat{b} = b$, and because a is arbitrary, we find that f and g induce the same homomorphism $H_n(X) \to H_n(Z)$. Since $H_n(Z) \neq 0$ for at most finitely many n, the result follows. □

(3.4) DEFINITION. Let X be a compactum.

(a) A *Borsuk presentation* $\mathcal{B} = \{Z_i \mid i \in N\}$ for X consists of a descending sequence $Z_1 \supset Z_2 \supset \cdots$ of compact ANRs such that $\bigcap_{i=1}^{\infty} Z_i = X$.

(b) A map $f : X \to X$ is called a *$\mathcal{B}$-map* if for each $\varepsilon > 0$ there is some Z_n and some $g_n : Z_n \to X$ such that $d(f(x), g_n(x)) < \varepsilon$ for all $x \in X$.

Since every compactum X can be embedded in the Hilbert cube and have arbitrarily small compact ANR neighborhoods there, each such X has Borsuk presentations.

For any compactum Y, the set of $\mathcal{B}$-maps $X \to Y$ is a nonempty closed subset in the function space Y^X (with the sup metric) containing the constant maps. Given a $\mathcal{B}$-map $f : X \to Y$ and any $g : Y \to Z$, the composition is also a $\mathcal{B}$-map; in particular, composition in X^X preserves the $\mathcal{B}$-property.

(3.5) THEOREM. *Let X be a compactum with homology of finite type and a Borsuk presentation $\mathcal{B} = \{Z_i\}$. If $f : X \to X$ is a $\mathcal{B}$-map with $\lambda(f) \neq 0$, then f has a fixed point.*

PROOF. Choose $\varepsilon(X)$ as in (3.3) and fix $\varepsilon > 0$ with $\varepsilon < \varepsilon(X)$. Find Z_n and a map $g_n = g : Z_n \to X$ with $d(gj(x), f(x)) < \varepsilon$ for all $x \in X$, where $j : X \hookrightarrow Z_n$. Since $gj, f : X \to X$ are ε-close, they induce the same homomorphism in homology, so $\lambda(jg) = \lambda(gj) = \lambda(f) \neq 0$. Because Z_n is an ANR, this implies that $jg : Z_n \to Z_n$ has a fixed point x_0, necessarily in X, so $d(f(x_0), x_0) = d(f(x_0), g(x_0)) < \varepsilon$. Thus f has an ε-fixed point for all sufficiently small $\varepsilon > 0$, which completes the proof. □

The continuous image of an ANR compactum is locally connected; since the compactum X in (3.4) may not be locally connected, the requirement for a $\mathcal{B}$-map that the approximating g_k send Z_k into X appears to be rather restrictive. We shall now consider a class of self-maps of compacta more general than $\mathcal{B}$-maps by requiring that the approximating g_k have values in small Z_n rather than in X.

(3.6) DEFINITION. Let X be a compactum and $\mathcal{B} = \{Z_k\}$ a Borsuk presentation of X. A map $f : X \to X$ is called an *$\mathcal{NB}$-map* if there exists an extension $\widehat{f} : Z_1 \to Z_1$ of f with the following property: for each $\varepsilon > 0$ there exists a Z_n such that for every $k \geq n$ there is a map $g_{nk} : Z_n \to Z_k$ with $d(\widehat{f}(x), g_{nk}(x)) < \varepsilon$ for all $x \in Z_n$.

Our aim now is to extend the Lefschetz–Hopf theorem to $\mathcal{NB}$-maps of arbitrary compacta having Čech homology of finite type.

The use of Borsuk presentations in the proof of our main result will be based on the following two general lemmas.

(3.7) LEMMA (Borsuk homology embedding). *Let $\{Z_i\}$ be a Borsuk presentation for a compactum X having homology of finite type. Consider*

the diagram

$$\begin{tikzcd} & H_*(Z_k) \arrow[d, "p_*"] \\ H_*(X) \arrow[r, "j_*"] & H_*(Z_i) \end{tikzcd}$$

where $k > i$ and j_*, p_* are induced by inclusions. Then there exists an N such that j_* is monic for all $i \geq N$; and for each such i, there exists a $k(i) \geq i$ such that $\operatorname{Im} p_* \subset \operatorname{Im} j_*$ *for all $k \geq k(i)$.*

PROOF. Choose a basis $\{z_1, \ldots, z_n\}$ for $H_*(X)$. It follows from (3.2) that there is a Z_N on which these generators are linearly independent, so j_* is monic for all $i \geq N$. Fix any such Z_i; because Z_i has homology of finite type, we find that for the inclusion $w : (Z_i, \emptyset) \to (Z_i, X)$ the image $\operatorname{Im} w_*$ is finitely generated, so if we let $\mu : (Z_i, X) \to (Z_i, Z_k)$ be the inclusion (with $k > i$), another application of (3.2) shows that there is a $k(i)$ such that $\mu_* : H_*(Z_i, X) \to H_*(Z_i, Z_k)$ is monic on $\operatorname{Im} w_*$ for all $k \geq k(i)$. Now consider the diagram

$$\begin{tikzcd} & H_*(Z_k) \arrow[d, "p_*"] & \\ H_*(X) \arrow[r, "j_*"] & H_*(Z_i) \arrow[r, "w_*"] \arrow[d, "\lambda_*"] & H_*(Z_i, X) \arrow[dl, "\mu_*"] \\ & H_*(Z_i, Z_k) & \end{tikzcd}$$

All maps being induced by inclusions, the triangle is commutative; and both the horizontal and vertical lines are exact. For any $y \in H_*(Z_k)$, we have $\lambda_* p_*(y) = 0$ by exactness, so by commutativity,

$$0 = \lambda_* p_*(y) = \mu_* w_* p_*(y);$$

because μ_* is monic on $\operatorname{Im} w_*$, this shows that $w_* p_*(y) = 0$, and therefore $p_*(y) \in \operatorname{Ker} w_* = \operatorname{Im} j_*$; since y was arbitrary, the proof is complete. □

(3.8) LEMMA (Borsuk's trace lemma). *Let X, Y, Z be compacta, each having homology of finite type, and let $j : X \to Z$ and $p : Y \to Z$ be maps such that in the diagram of induced maps*

$$\begin{tikzcd} & H_n(Y) \arrow[d, "p_n"] \\ H_n(X) \arrow[r, dashed, "j_n"] & H_n(Z) \end{tikzcd}$$

j_n is monic and $\operatorname{Im} p_n \subset \operatorname{Im} j_n$. Then for any $f : X \to X$ and $g : Z \to Y$, if jf and pgj induce the same homomorphism $H_n(X) \to H_n(Z)$, then

$$\operatorname{tr}(f_n) = \operatorname{tr}(p_n g_n) = \operatorname{tr}(g_n p_n).$$

PROOF. By considering the commutative diagram

$$\begin{tikzcd}
\operatorname{Im} j_n \arrow[dr, hook, "i"] & H_n(Y) \arrow[l, "\widehat{p}_n"'] \arrow[d, "p_n"] \\
H_n(X) \arrow[u, "\widehat{\jmath}_n"] \arrow[r, "j_n"] & H_n(Z)
\end{tikzcd}$$

we note that

(*) $$j_n = i\widehat{\jmath}_n, \qquad p_n = i\widehat{p}_n,$$

and therefore, by assumption,

(**) $$i\widehat{\jmath}_n f_n = j_n f_n = p_n g_n j_n = i\widehat{p}_n g_n j_n.$$

From (**), since i is monic, we obtain

(***) $$\widehat{\jmath}_n f_n = \widehat{p}_n g_n j_n.$$

Now, using (*)–(***) and the commutativity of the trace, we obtain

$$\begin{aligned}
\operatorname{tr}(f_n) &= \operatorname{tr}(\widehat{\jmath}_n^{-1}\widehat{\jmath}_n f_n) = \operatorname{tr}(\widehat{\jmath}_n f_n \widehat{\jmath}_n^{-1}) = \operatorname{tr}(\widehat{p}_n g_n j_n \widehat{\jmath}_n^{-1}) \\
&= \operatorname{tr}(\widehat{p}_n g_n i\widehat{\jmath}_n \widehat{\jmath}_n^{-1}) = \operatorname{tr}(\widehat{p}_n g_n i) = \operatorname{tr}(i\widehat{p}_n g_n) \\
&= \operatorname{tr}(p_n g_n) = \operatorname{tr}(g_n p_n),
\end{aligned}$$

and thus the proof is complete. □

We are now ready to prove the main result of this section.

(3.9) THEOREM (Borsuk–Dugundji). *Let X be a compactum with homology of finite type and a Borsuk presentation $\mathcal{B} = \{Z_i\}$. If $f : X \to X$ is an $\mathcal{NB}$-map with $\lambda(f) \neq 0$, then f has a fixed point.*

PROOF. Let N be a fixed integer. We start by finding a $Z_s \subset Z_N$ such that the inclusion $X \hookrightarrow Z_s$ induces a monic $w_* : H_*(X) \to H_*(Z_s)$. For this Z_s there is an $\varepsilon_s = \varepsilon(Z_s)$ satisfying (3.3). For $\varepsilon = \frac{1}{2}\min(\varepsilon_s, 1/N)$ we find a Z_n as in Definition (3.6); we can assume that $Z_n \subset Z_s$, so in particular, the inclusion $X \hookrightarrow Z_n$ also induces a monic $j_* : H_*(X) \to H_*(Z_n)$. Now choose $k > n$ so large (as we can by Lemma (3.7)) that in the diagram

$$\begin{tikzcd}
& H_*(Z_k) \arrow[d, "p_*"] & \\
H_*(X) \arrow[r, "j_*"] & H_*(Z_n) \arrow[r, "s_*"] & H_*(Z_s)
\end{tikzcd}$$

(with all homomorphisms induced by inclusions) we have $\operatorname{Im} p_* \subset \operatorname{Im} j_*$, and let $g_{nk} = g : Z_n \to Z_k$ be such that $d(\widehat{f}(z), g(z)) < \varepsilon$ for all $z \in Z_n$. Because the composites

$$X \overset{j}{\hookrightarrow} Z_n \xrightarrow{g} Z_k \overset{p}{\hookrightarrow} Z_n \overset{s}{\hookrightarrow} Z_s$$

and

$$X \xrightarrow{f} X \overset{j}{\hookrightarrow} Z_n \overset{s}{\hookrightarrow} Z_s$$

are ε_s-close, they induce the same homomorphism in homology, so that $s_* p_* g_* j_* = s_* j_* f_*$. Because $s_* j_* = w_*$ is monic, we find that s_* is monic on $\operatorname{Im} j_* \supset \operatorname{Im} p_*$, so $p_* g_* j_* = j_* f_* : H_*(X) \to H_*(Z_n)$. Now, by Borsuk's trace lemma (3.7) we conclude that $\lambda(pg) = \lambda(f) \neq 0$. Because Z_n is a Lefschetz space, it follows that $pg : Z_n \to Z_n$ has a fixed point z_N; we note that $z_N \in Z_k \subset Z_N$ and $d(\widehat{f}(z_N), z_N) < \varepsilon \leq 1/N$. Performing this construction for $N = 1, 2, \ldots$, we can, because $X = \bigcap_{N=1}^{\infty} Z_N$, choose a subsequence of $\{z_N\}$ converging to some $z_0 \in X$. Then $\widehat{f}(z_0) = z_0$, and since $\widehat{f}|X = f$, the proof is complete. □

(3.10) COROLLARY. *Let X be a connected compactum with homology of finite type and a Borsuk presentation $\mathcal{B}$.*

(a) *If X is acyclic, then any $\mathcal{NB}$-map $f : X \to X$ has a fixed point.*

(b) *If $\chi(X) \neq 0$, then any homeomorphism $h : X \to X$ that is an $\mathcal{NB}$-map has a periodic point.*

(c) *If the odd-dimensional Čech homology groups $H_{2i+1}(X; Q)$ are all zero, then any $\mathcal{NB}$-map $f : X \to X$ has a periodic point.*

4. Miscellaneous Results and Examples

A. Special classes of maps

In subsections A and B, in the context of the Lefschetz theory, we replace singular homology by Čech cohomology. Thus "Lefschetz maps", "Lefschetz spaces" etc. are understood to be taken with respect to Čech cohomology. Using the continuity of the Čech theory, we derive some general fixed point theorems, different from those previously established.

(A.1) (*Asymptotically regular maps*) Let X be a space and $f : X \to X$ a map. We say that f is *asymptotically regular* if there exists a family $\{K_\alpha \mid \alpha \in \mathcal{A}\}$ of nonempty compact subsets of X satisfying:

(i) $f(K_\alpha) \subset K_\alpha$ for each $\alpha \in \mathcal{A}$,

(ii) for any $\alpha, \beta \in \mathcal{A}$ there exists a $\gamma \in \mathcal{A}$ such that $K_\gamma \subset K_\alpha \cap K_\beta$,

(iii) for any $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{A}$ with $K_\beta \subset K_\alpha$ there exists a positive integer n such that $f^n(K_\alpha) \subset K_\beta$.

Observe that conditions (i)–(iii) imply that $K_0 = \bigcap_{\alpha \in \mathcal{A}} K_\alpha \neq \emptyset$ and $f(K_0) \subset K_0$.

(a) Let $f : X \to X$ be asymptotically regular. For $\alpha \in \mathcal{A}$, let $f_\alpha^* : H(K_\alpha) \to H(K_\alpha)$, $f_0^* : H(K_0) \to H(K_0)$ and $i_\alpha^* : H(K_\alpha) \to H(K_0)$ denote the homomorphisms of

cohomology groups ([1]) induced by the contractions $f_\alpha : K_\alpha \to K_\alpha$, $f_0 : K_0 \to K_0$ of f and by $i_\alpha : K_0 \hookrightarrow K_\alpha$. Show:

(i) i_α^* maps $N(f_\alpha^*)$ into $N(f_0^*)$, and hence determines a homomorphism $\tilde{i}_\alpha^*$ $H(K_\alpha)/N(f_\alpha^*) \to H(K_0)/N(f_0^*)$.

(ii) If $a \in H(K_\alpha)$ and $i_\alpha^*(a) \in N(f_0^*)$, then $a \in N(f_\alpha^*)$.

(iii) $\tilde{i}_\alpha^*$ is injective.

(b) Suppose $f_\alpha : K_\alpha \to K_\alpha$ is a Lefschetz map. Show:

(i) Given $a \in H(K_0)$, there exists $b \in H(K_\alpha)$ such that $i_\alpha^*(b) - a \in N(f_0^*)$.

(ii) $\tilde{i}_\alpha^*$ is surjective.

[For (b)(i): using the fact that f is asymptotically regular, consider the commutative diagrams

$$\begin{tikzcd}
K_\alpha \arrow[r, "f^n_{\alpha\beta}"] & K_\beta \\
K_0 \arrow[u, "i_\alpha"] \arrow[r, "f^n_0"] & K_0 \arrow[u, "i_\beta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H(K_\alpha) \arrow[d, "i_\alpha^*"'] & H(K_\beta) \arrow[l, "(f^n_{\alpha\beta})^*"'] \arrow[d, "i_\beta^*"] \\
H(K_0) & H(K_0) \arrow[l, "(f^n_0)^*"']
\end{tikzcd}$$

where $K_\beta \subset K_\alpha$ and $f^n(K_\alpha) \subset K_\beta$ for some $n \in N$.]

(c) Show:

(i) If f_0 is a Lefschetz map, then so also is f_α for each $\alpha \in \mathcal{A}$.

(ii) If f_α is Lefschetz for some $\alpha \in \mathcal{A}$, then so also is f_0.

(iii) If f_β is Lefschetz for some $\beta \in \mathcal{A} \cup \{0\}$, then the Lefschetz numbers $\Lambda(f_0)$ and $\Lambda(f_\alpha)$, $\alpha \in \mathcal{A}$, are all defined and equal.

(A.2) (*Maps in the class $\mathcal{K}_{as}^*$*) If X is a space and $f : X \to X$, we let $C_f = \bigcap_{i=1}^\infty f^n(X)$ denote the core of f, and by $\mathcal{V}(\overline{C}_f)$ the set of all open nbd's of $\overline{C}_f$ in X. We define the class $\mathcal{K}_{as}^*$ by the condition $f \in \mathcal{K}_{as}^*$ if and only if $f : X \to X$ is locally compact and $\overline{C}_f$ is nonempty and compact ([2]).

(a) Let X be a regular space and $f : X \to X$ be in $\mathcal{K}_{as}^*$. Show: For each $U \in \mathcal{V}(\overline{C}_f)$ there exists a $V_U \in \mathcal{V}(\overline{C}_f)$ such that:

(i) $V_U \subset U$,

(ii) $f(V_U) \subset V_U$,

(iii) $K_U = \overline{f(V_U)} \cup \overline{C}_f$ is compact.

[Given $U \in \mathcal{V}(\overline{C}_f)$, using local compactness of f, choose a $W \in \mathcal{V}(\overline{C}_f)$ such that $W \subset U$ and $K = \overline{f(W)}$ is compact. Consider the compact set $K \cap (X - W)$ and note that for each $x \in K \cap (X - W)$ there exists an $n_x \in N$ such that $x \in X - f^{n_x}(K)$, because $\bigcap_{n\in N} f^n(K) \subset X - (K \cap (X - W))$. From this, by compactness, $K \cap (X - W) \subset \bigcup_{i=1}^k (X - f^{n_{x_i}}(K))$ for some $n_{x_1}, \dots, n_{x_k} \in N$. Let $n = \max_{1 \le i \le k} n_{x_i}$ and set $V_U = V = W \cap f^{-1}(W) \cap \dots \cap f^{-(n+1)}(W)$.

Fix $x \in f(V)$ and observe that:

(i) $\bigcap_{i=1}^{n+1} f^i(W) \subset \bigcap_{i=0}^n f^i(K) \subset W \cap K$.

(ii) $f^i(x) \in W$ for $i = 0, 1, \dots, n$, and hence $x \in W \cap f^{-1}(W) \cap \dots \cap f^{-n}(W)$.

([1]) To simplify notation, we write $H(X)$ for $H^*(X; Q)$; we use the notation analogous to that of Section 1 of §15.

([2]) Observe that if $f \in \mathcal{K}_{as}^*$ and the orbits $\{O_x\}$ of f are relatively compact, then $f \in \mathcal{K}_{as}$ (cf. (15.6.1)).

(iii) Since $f^{n+1}(x) \in K$ and $f^{n+1}(x) = f^i(f^{n+1-i}(x)) \in f^i(W)$ for $i = 1, \dots, n+1$, we have

$$f^{n+1}(x) \in \Big(\bigcap_{i=1}^{n+1} f^i(W)\Big) \cap K \subset W \cap K \subset W,$$

and thus $x \in f^{-(n+1)}(W)$.

To conclude, note that (i) and (iii) imply $f(V) \subset V$, and since $f(V) \subset f(W) \subset K$, $\overline{f(V)}$ is compact.]

(b) Prove: If X is a regular space and $f : X \to X$ is in $\mathscr{K}^*_{\mathrm{as}}$, then f is asymptotically regular.

[Consider the family $\{K_U \mid U \in \mathscr{V}(\overline{C}_f)\}$ of compact sets and show:

(i) $f(K_U) \subset K_U$ for any $U \in \mathscr{V}(\overline{C}_f)$.

(ii) $\overline{C}_f = \bigcap\{K_U \mid U \in \mathscr{V}(\overline{C}_f)\}$.

(iii) For any $U, W \in \mathscr{V}(\overline{C}_f)$, there exists an $O \in \mathscr{V}(\overline{C}_f)$ such that $K_O \subset K_U \cap K_W$.

(iv) For any $U, W \in \mathscr{V}(\overline{C}_f)$, there exists a positive integer n with $f^n(K_U) \subset K_W$.

To establish (iv), given $W \in \mathscr{V}(\overline{C}_f)$, take an $x \in K_U$ and note that since the orbit $O_x \subset K_U$ is relatively compact, there exists an n_x such that $f^{n_x}(x) \in V_W \supset \overline{C}_f$, by (15.5.2). Deduce that the open sets $f^{-n_x}(V_W)$, $x \in K_U$, cover K_U, and therefore $K_U \subset \bigcup_{i=1}^m f^{-n_{x_i}}(V_W)$ for some $x_1, \dots, x_m \in K_U$. Take now $n = \max_{1\le i\le m}(n_{x_i} + 1)$ and verify that $f^n(K_U) \subset K_W$.]

(c) Let $f : X \to X$ be in $\mathscr{K}^*_{\mathrm{as}}$, and let $f_U : K_U \to K_U$, $\widehat{f}_U$ $V_U \to V_U$ denote the contractions of f. Prove: The map $\widehat{f}_U : V_U \to V_U$ is eventually compact.

[(i) Take $W \in \mathscr{V}(\overline{C}_f)$ such that $\overline{C}_f \subset W \subset \overline{W} \subset V_U$ and using (A.2) choose an n with $f^n(K_U) \subset K_W$; this implies $f^{n+1}(V_U) \subset f^n(K_U) \subset K_W$; from $\overline{f(V_W)} \subset \overline{W} \subset V_U$, deduce $f^{n+1}(V_U) \subset K_W \subset V_U$, showing that $\widehat{f}_U^{n+1}$ is compact.

(ii) For $x \in V_U$, $f(x) \in V_U$, take an open $W \subset \overline{W} \subset V_U$ such that $f(x) \in W$ By assumption, there is an open nbd O of x in X such that $\overline{f(O)}$ is compact. Since $O \cap f^{-1}(W)$ is an open nbd of x in V_U and $\overline{f(O \cap f^{-1}(W))} \subset \overline{f(O)} \cap \overline{W}$ is a compact subset of V_U, the desired conclusion follows.]

B. Čech cohomology and Lefschetz-type results

In this subsection by an *admissible Lefschetz space* is meant a regular space X such that each nonempty open subset U of X is a Lefschetz space.

(B.1) Let X be an admissible Lefschetz space, and let $f : X \to X$ be in $\mathscr{K}^*_{\mathrm{as}}$. Prove: For each $U \in \mathscr{V}(\overline{C}_f)$, the map $f_U : K_U \to K_U$ is Lefschetz, and $\mathrm{Fix}(f_U) \neq \emptyset$ whenever $\Lambda(f_U) \neq 0$.

[Consider the commutative diagram

$$\begin{tikzcd}
K_U & f(K_U)\cup f(V_U) \arrow[l] & \cdots \arrow[l] & f^n(K_U)\cup f(V_U) \arrow[l] & f(V_U) \arrow[l] \arrow[r] & V_U \\
K_U \arrow[u, "f_U"] \arrow[ur] & f(K_U)\cup f(V_U) \arrow[l] \arrow[u] & \cdots \arrow[l] & f^n(K_U)\cup f(V_U) \arrow[l] \arrow[u] \arrow[ur] & f(V_U) \arrow[l] \arrow[r] \arrow[u] & V_U \arrow[u, "\widehat{f}_U"'] \arrow[ul]
\end{tikzcd}$$

where the horizontal arrows are inclusions and the others are evident contractions of f. Use (A.2), (15.6.3), and the general properties of Lefschetz maps to conclude that f_U is a strongly Lefschetz map.]

(B.2) (*Lefschetz theorem for maps in $\mathscr{K}^*_{\rm as}$*) Let X be an admissible Lefschetz space and $f : X \to X$ a map in $\mathscr{K}^*_{\rm as}$. Prove:

(i) The contraction $\widehat{f} : \overline{C}_f \to \overline{C}_f$ of f is a Lefschetz map.

(ii) If $\Lambda(\widehat{f}) \neq 0$, then $\mathrm{Fix}(\widehat{f}) \neq \emptyset$ and $\mathrm{Fix}(f) \neq \emptyset$.

(B.3) Let X be an admissible Lefschetz space, and $f : X \to X$ in $\mathscr{K}^*_{\rm as}$. Prove: If $\overline{C}_f$ is contractible or Čech-acyclic, then f has a fixed point.

(B.4) Let X be a regular NES(compact), or more generally, ANES(compact) space. Prove: If $f : X \to X$ is in $\mathscr{K}^*_{\rm as}$ and $\overline{C}_f$ is contractible or acyclic, then f has a fixed point.

(The results of subsections A and B are taken from an unpublished manuscript by G. Fournier and A. Granas.)

5. Notes and Comments

Degree for equivariant maps in $\boldsymbol{R}^n$

If $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous and $U \subset \boldsymbol{R}^n$ open and bounded such that $f|\partial U : \partial U \to \boldsymbol{R}^n - \{0\}$, there is defined an integer $d(f, U)$, the topological degree of f with respect to U. By considering smaller classes of maps and smaller classes of sets, one may attempt to define finer topological invariants. A simplest example of such an invariant was presented in Section 1, whose results are taken from an unpublished manuscript of K. Gęba.

More generally, one can replace $\boldsymbol{R}^n$ by a finite-dimensional real representation V of a compact Lie group G and consider only G-equivariant maps and G-invariant open subsets. For example, in such a framework Ize–Massabò–Vignoli [1989] introduce a general G-equivariant degree; in their construction, $f : V \to W$ is a G-equivariant map between (possibly different) representations of G and the degree is defined as an element of a certain equivariant homotopy group of spheres. In the case $V = W$ this element determines an element of $B(G)$, the Burnside ring ([1]) of the group G. A more direct (but less general) construction of the equivariant degree for G-equivariant maps $f : V \to V$ was given by Gęba–Krawcewicz–Wu [1994].

For details about the equivariant degree and also the equivariant fixed point index, the reader is referred to Rubinsztein [1976], Dold [1983], Komiya [1988], Ulrich [1988], Kushkuley–Balanov [1996], the book of Krawcewicz–Wu [1997], and "Additional References" II(c).

The E^+-cohomology and its applications to Morse theory

The E^+-cohomology outlined in Section 2 and due to Abbondandolo [1997] provides a noteworthy approach to infinite-dimensional Morse theory. Morse

([1]) See Chapter IV of T. tom Dieck, *Transformation Groups*, de Gruyter, 1987.

theory deals with the study of critical points of a smooth function on a manifold M. When M is finite-dimensional, the main ingredient of the theory is that the presence of a critical point of Morse index q changes the homotopy type of the sublevel sets of f by attaching a q-dimensional ball along its boundary (see Milnor [1963]). This fact remains true for an infinite-dimensional Hilbert manifold, so Morse theory holds in this framework and it detects critical points with finite Morse index (see Palais [1963]). However, since every infinite-dimensional sphere in a Hilbert space is an AR, the Morse theory fails to detect critical points with infinite index.

Let E be a Hilbert space and $f : E \to \boldsymbol{R}$ a Morse functional on E (i.e., a C^2 function satisfying the Palais- Smale condition ([1]) and such that at every critical point x, the Hessian $D^2 f(x)$ is nondegenerate). We recall that the *Morse index* $m(x)$ of f at a critical point x is the dimension of the maximal subspace of E on which the form $D^2 f(x)$ is negative-definite. We say that f is *strongly indefinite* if all of its critical points have infinite Morse index.

One general class of strongly indefinite functionals consists of the functionals f on E of the form

$$f(x) = \frac{1}{2}\langle Lx, x\rangle + b(x), \tag{$*$}$$

where L is an invertible self-adjoint bounded operator on E, with infinite-dimensional positive and negative eigenspaces, E^+ and E^-, and b is a C^2 function satisfying: (i) b is weakly continuous, (ii) the gradient ∇b of b is continuous from the weak to the strong topology of E. Functionals of this type arise in a natural way in the study of periodic solutions of Hamiltonian systems, of wave equations, and in many other concrete variational problems (see Rabinowitz's tract [1986]).

Assumption (ii) implies that the Hessian of b at any point x, $D^2 b(x)$, is a completely continuous operator. Then it can be shown that the negative eigenspace V^- of $D^2 f(x) = L + D^2 b(x)$ is commensurable with E^-. If x is a critical point of f, then we define the *relative Morse index* of x by $E^+\text{-}m(x) = \dim(V^-, E^-)$.

The dimension axiom for the E^+-cohomology theory suggests that a Morse theory based on $H^*_{E^+}$, rather than on a standard homology or cohomology, should be able to detect the critical points of f. Many deformation arguments in Morse theory require the use of the negative gradient flow, i.e., the local flow obtained by integrating the vector field $-\nabla f$; in the present case, this local flow has the form

$$(x, t) \mapsto e^{-tL}x - K(x, t), \tag{$**$}$$

([1]) A function $f : E \to \boldsymbol{R}$ satisfies the *Palais Smale condition* if any sequence $\{x_i\}$ in E such that $\{f(x_i)\}$ is bounded and $\|df(x_i)\| \to 0$, contains a convergent subsequence.

where K is continuous from the weak to the strong topology. So the class of morphisms and homotopies in the h-category $\mathfrak{L}_{E^+}$ has to be enlarged to include maps of this form. Using Z_2 as the coefficient group ([1]), it is possible to prove the homotopy invariance of $H^*_{E^+}$ with respect to homotopies of the form

$$h(x,t) = A_t[x - K(x,t)],$$

where $t \mapsto A_t$ is a continuous path of invertible linear operators preserving the splitting $E^+ \oplus E^-$, and $(x,t) \mapsto x - K(x,t)$ is an E^+-compact homotopy. The flow $(**)$ belongs to this class of homotopies.

In this way, it is possible to prove generalized Morse relations: following an approach introduced by Conley (cf. Conley's monograph [1978], Conley–Zehnder [1984], Salamon [1985]), an object (X, A) of $\mathfrak{L}^2_{E^+}$ is said to be an *index pair* if A is positively invariant with respect to X for the negative gradient flow of f, and it is an exit set from X for this flow. If (X, A) is an index pair, and f has only nondegenerate critical points in $X - A$, then the following identity holds:

$$\sum_{\substack{x \in X - A \\ \nabla f(x) = 0}} \lambda^{E^+\text{-}m(x)} - \sum_{n \in Z} \lambda^n \dim H^n_{E^+}(X, A; Z_2) = (1 + \lambda) Q(\lambda),$$

where Q is a Laurent polynomial with positive coefficients.

Note that by assumption (i), the sublevel sets of f, defined by $f^a = \{x \in E \mid f(x) \leq a\}$, are τ_{E^+}-closed. However, they are not bounded, so in order to consider their E^+-cohomology, the class of objects in $\mathfrak{L}_{E^+}$ has to be enlarged. The use of cohomology theory with compact supports, instead of Čech cohomology, in the definition of $H^*_{E^+}$ allows one to consider also unbounded sets. If f satisfies the Palais–Smale condition and $a < b$ are regular values, one can take $X = f^b$ and $A = f^a$ in the above identity.

The idea of using a generalized cohomology to construct a suitable Morse theory for the study of functionals with critical points of infinite Morse index is due to Szulkin [1992]. The construction of the E^+-cohomology theory and its applications to Morse theory outlined in Section 2 are due to Abbondandolo [1997]. Applications of E^+-cohomology to Hamiltonian systems and elliptic systems can be found in Abbondandolo [2000]. For closely related results the reader may consult Kryszewski– Szulkin [1997], Gęba–Izydorek–Pruszko [1999], and Izydorek [2001].

([1]) The impossibility of using arbitrary coefficients is due to the fact that the group of invertible linear operators on an infinite-dimensional Hilbert space is connected. A similar feature appears in degree theory, when one attempts to extend the Leray–Schauder degree from compact fields to Fredholm maps.

Fixed points for $\mathcal{NB}$-maps

By concentrating on the nature of the map rather than on that of the underlying space, Borsuk [1975] showed that there is a general and fairly extensive class of self-maps of compacta for which the Lefschetz–Hopf theorem remains valid. In Section 3, following Dugundji [1976], a general and considerably clarified formulation of the Borsuk principles is presented; some further simplifications were suggested by C. Bowszyc. We remark that in Theorem (3.5), the choice of $\mathcal{B}$ is irrelevant; it is the existence of some $\mathcal{B}$ making f into a $\mathcal{B}$-map that is of importance. Using the Leray trace, Dugundji [1976] observed that Theorem (3.5) can be extended to suitable compact maps of arbitrary metric spaces.

THEOREM. *Let X be any metric space and $f : X \to X$ a compact map such that $\overline{f(X)}$ has homology of finite type. If $\overline{f(X)} = \bigcap_{i=0}^{\infty} Z_i$, where all $Z_i \subset X$ are compact ANRs, then f is a Lefschetz map, and $\Lambda(f) \neq 0$ implies that f has a fixed point.*

The hypothesis in Theorem (3.5) that X has homology of finite type can be eliminated by a procedure suggested by Borsuk, which associates a "Lefschetz set" $\mathbf{\Lambda}(f)$ with each $\mathcal{B}$-map $f : X \to X$: for any integers n, k, let

$$A(f; n, k) = \{\lambda(g) \mid g : Z_n \to X,\ d[g(x), f(x)] < 1/k \text{ for all } x \in X\}$$

and define

$$\mathbf{\Lambda}(f) = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{\infty} A(f; n, k).$$

THEOREM. *If X is any compactum, $f : X \to X$ a $\mathcal{B}$-map, and $0 \notin \mathbf{\Lambda}(f)$, then f has a fixed point.*

We remark that Gauthier [1980] extended the Borsuk–Dugundji results to arbitrary compact spaces, and Fenske [1982] constructed a fixed point index for maps with Borsuk-presentable fixed point set.

***

In this book we concentrated mainly on themes closely related to the Leray–Schauder theory. Accordingly, a number of special and important topics were, for obvious reasons, left out. Among these we mention:

(a) Nielsen theory.
(b) The Poincaré–Birkhoff theorem.
(c) Symplectic fixed points.
(d) The Lefschetz fixed point theorem for elliptic complexes.
(e) Extensions of degree theory to noncontinuous maps.
(f) Vertical fixed point theory.

We remark that an appropriate coverage of any of the above topics would require a separate volume.

(*a*) *Nielsen theory*

Let $f : X \to X$ be a map, and let $M(f)$ denote the minimum number of fixed points of all maps homotopic to f; observe that $M(f) = 0$ whenever there exists a map g homotopic to f with $\mathrm{Fix}(g) = \emptyset$. The Nielsen theory is a large and important chapter in topological fixed point theory, which is closely related to the Lefschetz theory; under suitable conditions the Nielsen theory can be used to obtain information about the number $M(f)$ for a given map f.

Let X be a compact ANR, and $f : X \to X$ a map. Then the set $\mathrm{Fix}(f)$ admits an equivalence relation $\sim_f$ that partitions $\mathrm{Fix}(f)$ into a finite number of closed-open subsets called *fixed point classes* of f. If Φ is such a class, we can use the local index theory (Theorem (1.1)) to define the index of Φ as follows:

Let $U \subset X$ be open such that $\Phi \subset U$ and $U \cap \mathrm{Fix}(f) = \emptyset$. Then the *index* $i(\Phi)$ of Φ is defined by $i(\Phi) = I(f, U)$. If $i(\Phi) \neq 0$, the class Φ is called *essential*; otherwise, it is *inessential*. The number $N(f)$ of essential fixed point classes is called the *Nielsen number* of f. It turns out that:

(i) If $f, g : X \to X$ are homotopic, then $N(f) = N(g)$.

(ii) If $\Phi_1, \ldots, \Phi_k$ are all the essential fixed point classes of f, then

$$\lambda(f) = \sum_{j=1}^{k} i(\Phi_j).$$

Clearly, in view of (ii), if $N(f) = 0$, then $\lambda(f) = 0$. There are, however, examples of manifolds that admit homeomorphisms f with $\lambda(f) = 0$ and $N(f) \geq 2$.

We now examine the situations where $[\lambda(f) = 0] \Rightarrow [N(f) = 0]$. Let (X, x_0) be a based compact connected ANR, and $f : X \to X$ a based map. Consider the based function space (X^X, f) and the evaluation map $e : X^X \to X$ given by $e(g) = g(x_0)$ for $g \in X^X$. Let

$$e_\pi : \pi_1(X^X, f) \to \pi_1(X, x_0)$$

be the induced homomorphism of the corresponding fundamental groups. The image $J(f)$ of the homomorphism e_π is called the *Jiang subgroup* of $\pi_1(X, x_0)$; it is independent of the base point and depends only on the homotopy class of f; the Jiang subgroup of the identity id_X is denoted simply by $J(X)$.

THEOREM (Jiang [1964]). *Let X be a compact ANR, and $f : X \to X$ a map such that $J(f) = \pi_1(X)$. Then:*

(i) *all fixed point classes of f have the same index,*
(ii) $\lambda(f) = 0$ *implies* $N(f) = 0$.

By observing that $J(X) \subset J(f)$ for any $f : X \to X$, it follows that the condition $J(X) = \pi_1(X)$ implies the validity of the Jiang theorem for all self-maps f of X.

Say that a compact ANR X is *of type* (W) if for any $f : X \to X$ its homotopy class contains at least one g having precisely $N(f)$ fixed points. With this terminology the following theorem holds: *Let X be a compact ANR of type (W), and $f : X \to X$ a map satisfying the Jiang condition $J(f) = \pi_1(X)$. Then $\lambda(f) \neq 0$ if and only if every map homotopic to f has a fixed point.*

For details and more general or related results see the books of R.F. Brown [1971], Jiang [1983] and Kiang [1989]; some references on Nielsen theory are also given in "Additional References" III(a).

(b) *The Poincaré–Birkhoff theorem*

In some cases, a map $f : X \to X$ of a polyhedron may have fixed points even if $\lambda(f) = 0$. An example of special importance is given by:

POINCARÉ–BIRKHOFF TWIST THEOREM. *Let $X = \{(r, \delta) \in R^2 \mid a \leq r \leq b\}$ be an annulus and $f : X \to X$ an area-preserving homeomorphism such that there exist continuous maps $\delta \mapsto \alpha(\delta)$ and $\delta \mapsto \beta(\delta)$ satisfying $\alpha(\delta) < \delta$, $\beta(\delta) > \delta$, $f(a, \delta) = (a, \alpha(\delta))$, $f(b, \delta) = (b, \beta(\delta))$. Then f has at least two fixed points.*

This result was conjectured by Poincaré [1912] in his attempt to use fixed points in the search for periodic solutions in celestial mechanics, and was proved by Birkhoff [1913].

A similar result can be proved for the sphere S^2: *If $f : S^2 \to S^2$ is an orientation and area-preserving homeomorphism, then f has at least two fixed points.* For a proof, one can use the fact that the index $J(f, p)$ of a fixed point p for such a map in two dimensions is always ≥ 1, so that the conclusion follows from the index formula

$$\chi(S^2) = \sum \{J(f, p) \mid p \in \mathrm{Fix}(f)\} = 2.$$

J. Franks established the following generalization of the Poincaré–Birkhoff theorem:

THEOREM. *Let $X = \{(r, \delta) \in R^2 \mid a \leq r \leq b\}$ be an annulus and $f : X \to X$ a diffeomorphism such that:*
(i) *f is area-preserving,*
(ii) *f is isotopic to identity,*
(iii) $\mathrm{Fix}(f) \neq \emptyset$.
Then f has infinitely many periodic points in the interior of X.

For a proof of this result and its applications to the study of geodesics on S^2, the reader is referred to Franks [1992]. For other generalizations and related results, see "Additional References" III(b) and Hofer–Zehnder [1994].

(c) *Symplectic fixed points*

We now comment on symplectic fixed point results, which arise in the context of classical mechanics, and can be regarded as extensions of the Poincaré–Birkhoff theorem. We begin by describing some background and introduce the requisite terminology ([1]).

The equations of motion in classical mechanics arise as minima of various action integrals. The action integral in classical Hamiltonian mechanics is defined on the phase space $\boldsymbol{R}^{2n}$, where a point $z = (p, q) \in \boldsymbol{R}^{2n}$ describes the momentum $p \in \boldsymbol{R}^n$ and position $q \in \boldsymbol{R}^n$ of a particle. The Hamiltonian $H(p, q, t)$, which may be a function of time, is the total energy (kinetic plus potential) and the equations of motion, which arise as minima of the action integral

$$\Phi_H(z) = \int_{t_0}^{t_1} (\langle p, \dot{q}\rangle - H(p, q, t))\, dt,$$

can be written in the form

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}$$

or

$$J_0 \dot{z} = \nabla H_t(z),$$

where J_0 is the $2n \times 2n$ matrix $\left(\begin{smallmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{smallmatrix}\right)$.

All the above notions extend in a natural way to the category of symplectic manifolds. Let (M, ω) be a symplectic manifold, where M is a compact smooth manifold of even dimension $2n$ and ω is a nondegenerate 2-form on M ([2]). A smooth time-dependent Hamiltonian $H : M \times \boldsymbol{R} \to \boldsymbol{R}$ determines a Hamiltonian vector field X_H on M by the formula $\omega(X_H, \cdot) = dH_t(\cdot)$. The vector field X_H generates a Hamiltonian flow $\varphi_H^t \in \mathrm{Diff}(M)$ satisfying

$$\frac{d}{dt}\varphi_H^t(z) = X_H(\varphi_H^t(z), t), \qquad \varphi_H^0 = \mathrm{id}_M .$$

By a *Hamiltonian map* is meant a diffeomorphism ψ that is, by definition, the time 1-map of a Hamiltonian flow $\{\varphi_H^t\}$ on M, i.e., $\psi = \varphi_H^1$.

([1]) For the terminology used in these comments and for further reading, see Arnold [1980], McDuff–Salamon [1995], Hofer–Zehnder [1994], and M. Schwarz [1993].

([2]) This means that the nth exterior power ω^n does not vanish at any point of M; then the form $\mathrm{vol} = (1/n!)\omega^n$ is the canonical volume form on M.

Let M be a compact smooth manifold. We let $\operatorname{Arn}(M)$ (resp. $\operatorname{Crit}(M)$) denote the minimal number of critical points for any smooth function (resp. Morse function) $\varphi : M \to \boldsymbol{R}$; if $f : M \to M$ we let $\#\operatorname{Fix}(f)$ denote the number of fixed points of f.

ARNOLD CONJECTURE. Let (M, ω) be a compact symplectic manifold, and let $f : M \to M$ be any Hamiltonian diffeomorphism. Then:

(a) $\#\operatorname{Fix}(f) \geq \operatorname{Arn}(M)$,

(b) if f has only nondegenerate fixed points, then

$$\#\operatorname{Fix}(f) \geq \operatorname{Crit}(M) \geq \sum_{k=0}^{2n} b_k(M).$$

Except for certain special cases, the conjecture turned out to be too difficult to prove in the original form (a).

We remark that at the Moscow Congress in 1966, V. Arnold stated his conjecture for the ordinary torus $T^2 = S^1 \times S^1$; even in this case, the problem proved to be very difficult and was resolved only in 1983 by C. Conley and E. Zehnder.

Assume that $H \quad M \times \boldsymbol{R} \to \boldsymbol{R}$ is periodic in time ($H(x, t) = H(x, t+1)$) and let ψ be the 1-map of the Hamiltonian flow $\{\varphi_H^t\}$ on M. Then p is a fixed point of ψ if and only if $p = x(0)$ is the initial condition of a solution of the Hamiltonian equation which is periodic of period 1; any such solution is called a *forced oscillation*. This implies that the Arnold conjecture can be rephrased in terms of dynamical systems as a conjecture about the existence of a lower bound on the number of forced oscillations, depending only on the topological invariants of the manifold M.

In the special case of $M = T^{2n}$ Conley–Zehnder [1983] proved the Arnold conjecture by establishing the following:

THEOREM. *For each time-dependent Hamiltonian vector field on the standard torus T^{2n} which is 1-periodic in time, the associated flow has at least $2n+1$ periodic orbits.*

We remark that the detailed proof of this result (see Hofer–Zehnder [1994]) is quite sophisticated; the techniques used include the Lusternik–Schnirelmann theory, the Conley index, and the mod 2 degree for Fredholm maps.

Completely new ideas to the understanding of Arnold's conjecture were introduced by A. Floer. Using a new approach to infinite-dimensional Morse theory, Floer established in 1989 the Arnold conjecture in the following weakened form:

THEOREM (Floer [1989]). *Let (M, ω) be a symplectic manifold with the trivial homotopy group $\pi_2(M)$, and let $f : M \to M$ be a Hamiltonian diffeomorphism with nondegenerate fixed points. Then $\#\operatorname{Fix}(f) \geq \operatorname{cup}(M) + 1$, where $\operatorname{cup}(M)$ is the mod 2 cuplength of M* (^{1}).

(^{1}) Recall that the *mod 2 cuplength* $\operatorname{cup}(M)$ of a manifold M is the minimal number N such that for any cohomology classes $\alpha_1, \dots, \alpha_N \in H^*(M; Z_2)$ with $\deg \alpha_i \geq 1$, their cup product $\alpha_1 \cup \cdots \cup \alpha_N$ is zero, it can be proved that $\operatorname{Cat}(M) \geq \operatorname{cup}(M)$.

Since 1989, the work of Floer has been extended by several authors. The following two recent results are of special interest and importance:

(i) In 1998, G. Liu and G. Tian, and independently K. Fukaya and K. Ono, proved Arnold's conjecture for general symplectic manifolds, assuming that all fixed points of a Hamiltonian diffeomorphisms $f : M \to M$ are nondegenerate (see Liu–Tian [1998] and Fukaya–Ono [1999]).

(ii) In 1999, Yu. Rudyak proved the original Arnold conjecture for arbitrary symplectic manifolds satisfying the Floer condition $\pi_2(M) = 0$ (see Rudyak [1999] and also his papers in "Additional References" III(b)).

(d) The Lefschetz fixed point theorem for elliptic complexes

This theorem, due to Atiyah–Bott [1967], [1968] provides a remarkable refinement of the Lefschetz–Hopf formula in the setting of smooth manifolds.

Let M be a compact C^∞ manifold with cotangent bundle $\pi : T^*M \to M$. If E, F are smooth complex vector bundles over M, by a *differential operator* from E to F is meant a linear map $d : \Gamma(E) \to \Gamma(F)$ between the spaces of smooth cross-sections which in local coordinates is given by a matrix of partial differential operators with smooth coefficients. If d is of order k, the terms of order k define a bundle map, the *symbol* $\sigma_k(d) : \pi^*E \to \pi^*F$, where π^*E is the vector bundle over T^*M induced by $\pi : T^*M \to M$. By an *elliptic complex* E on M is meant a sequence $E_0, E_1, \dots, E_n$ of smooth vector bundles over M and a sequence of differential operators $d_i : \Gamma(E_i) \to \Gamma(E_{i+1})$ such that

(i) $d_{i+1} \circ d_i = 0$ for $i \in [n-1]$,

(ii) the sequence

$$\cdots \to \pi^*(E_k) \xrightarrow{\sigma(d_k)} \pi^*(E_{k+1}) \to \cdots$$

is exact over the complement of the 0-section of T^*M.

In view of (i) the cohomology groups $H^i(\Gamma E) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$ are well defined and it is a consequence of (ii) that they are finite-dimensional. By an *endomorphism* T of the elliptic complex E is meant a sequence of linear maps $T_i : \Gamma(E_i) \to \Gamma(E_i)$ such that $d_i T_i = T_{i+1} d_i$. Such an endomorphism induces endomorphisms $H^i(T) : H^i(\Gamma E) \to H^i(\Gamma E)$ for all i. In analogy with the Lefschetz number we let

$$L(T) = \sum (-1)^i \operatorname{tr} H^i(T).$$

Let $f : M \to M$ be a smooth map with graph transversal to the diagonal in $M \times M$ (this implies that each fixed point $p \in \operatorname{Fix}(f)$ is nondegenerate, i.e., $\det(1 - df_p) \neq 0$, where df_p is the induced map on T_pM, and that $\operatorname{Fix}(f)$ is finite). A *lifting* φ of f over the elliptic complex E is a family of bundle

morphisms $\varphi_k : f^*E_k \to E_k$ such that for T_k defined to be the composite

$$\Gamma(E_k) \xrightarrow{f^*} \Gamma(f^*E_k) \xrightarrow{\Gamma\varphi_k} \Gamma(E_k)$$

we have $T_{k+1}d_k = d_kT_k$. With this terminology M. Atiyah and R. Bott established the following:

THEOREM. *Let E be an elliptic complex over M, $f : M \to M$ a smooth map with graph transversal to the diagonal, φ a lifting of f, and $T : \Gamma E \to \Gamma E$ the endomorphism induced by φ. Then the Lefschetz number $L(T)$ is given by the formula*

$$L(T) = \sum_{p \in \mathrm{Fix}(f)} \nu(p),$$

where

$$\nu(p) = \sum_k (-1)^k \frac{\mathrm{tr}(\varphi_{k,p})}{|\det(1 - df_p)|}.$$

This theorem and its proof are quite sophisticated. The techniques developed for the proof have had in fact a great influence on the development of several areas of topology and differential geometry. There are a number of significant applications. For the de Rham complex of exterior differential forms on a smooth manifold, it is the classical Lefschetz–Hopf index theorem. Further specific cases yield precise descriptions for isometries on Riemannian manifolds. As an example one obtains the following

THEOREM. *Let M be a compact, connected, oriented smooth manifold, and $f : M \to M$ a smooth map satisfying $f^{p^k} = 1_M$ for some k and odd prime p. Then the number of fixed points of f is ≥ 2.*

This theorem was also proved by very different techniques by Conner–Floyd [1964]. Good references for related results and further reading are Berline-Getzler–Vergne [1992], Kotake [1969], and Bott [1987], [1988].

(e) *Extensions of degree theory to noncontinuous maps*

We shall now describe three extensions of degree theory to certain classes of noncontinuous maps.

1° *Degree for VMO maps.* For simplicity, we shall only consider an extension of the Brouwer degree for maps from S^n to S^n. First we gather some definitions and results about the class BMO of functions of *bounded mean oscillation.*

Suppose $f : \boldsymbol{R}^n \to \boldsymbol{R}$ is integrable over compact sets in $\boldsymbol{R}^n$ and B is any ball in $\boldsymbol{R}^n$ with volume $|B|$. By definition, f belongs to BMO if

$$\|f\|_{\mathrm{BMO}} = \sup_{B\subset \boldsymbol{R}^n} \frac{1}{|B|}\int_B |f(x) - f_B|\,dx < \infty.$$

where $f_B = |B|^{-1}\int_B f(x)\,dx$ is the *mean* of f over B: $\|f\|_{\mathrm{BMO}}$ is the *BMO norm* of f, and the class $\mathrm{BMO}(\boldsymbol{R}^n)$ modulo adding constants is a Banach space under $\|f\|_{\mathrm{BMO}}$ or under the equivalent norm

$$\|f\|_* = \sup_{B\subset \boldsymbol{R}^n} \frac{1}{|B|^2}\int_B\int_B |f(x) - f(y)|\,dx\,dy.$$

L^∞ functions lie in $\mathrm{BMO}(\boldsymbol{R}^n)$, but they are not dense; for example, $g(x) = \log|x|$ belongs to $\mathrm{BMO}(\boldsymbol{R}^n)$ but cannot be locally approximated by bounded functions near the origin.

The definition of BMO makes sense as soon as there are proper notions of integral and ball in a space. Thus for example, we have the Banach space $\mathrm{BMO}(\boldsymbol{S}^n)$ of the BMO functions $f : \boldsymbol{S}^n \to \boldsymbol{R}$. Functions in the Sobolev class $H^{1,p}(\boldsymbol{S}^n)$ belong to $\mathrm{BMO}(\boldsymbol{S}^n)$, and the space $\mathrm{BMO}(\boldsymbol{S}^n)$ is contained in $L^p(\boldsymbol{S}^n)$ for $1 \le p < \infty$; the functions in $C^0(\boldsymbol{S}^n)$ are also not dense in $\mathrm{BMO}(\boldsymbol{S}^n)$.

Let $\mathrm{VMO}(\boldsymbol{S}^n)$ be the closure of $C^0(\boldsymbol{S}^n)$ in $\mathrm{BMO}(\boldsymbol{S}^n)$; the elements $f \in \mathrm{VMO}(\boldsymbol{S}^n)$ are the functions of *vanishing mean oscillation* on $\boldsymbol{S}^n$. These functions are characterized by the property that

$$\lim_{|B|\to 0} \frac{1}{|B|}\int_B |f(x) - f_B|\,dx = 0.$$

We now define the function space $\mathrm{VMO}(\boldsymbol{S}^n, \boldsymbol{S}^n)$ on which an extension of the Brouwer degree will be defined.

Let $f : \boldsymbol{S}^n \to \boldsymbol{R}^k$ be a map. Then we say that:

(a) $f = (f_1, \ldots, f_k)$ belongs to $\mathrm{VMO}(\boldsymbol{S}^n, \boldsymbol{R}^k)$ whenever each f_i is in $\mathrm{VMO}(\boldsymbol{S}^n)$;

(b) $f \in \mathrm{VMO}(\boldsymbol{S}^n, \boldsymbol{S}^n)$ if $f \in \mathrm{VMO}(\boldsymbol{S}^n, \boldsymbol{R}^{n+1})$ and $f(x) \in \boldsymbol{S}^n$ a.e.

We remark that if $f \in \mathrm{VMO}(\boldsymbol{S}^n, \boldsymbol{S}^n)$, then there exists the smallest closed set $\operatorname{Ess} R(f) \subset \boldsymbol{S}^n$ such that $f(x) \in \operatorname{Ess} R(f)$ a.e.

By a *homotopy in* $\mathrm{VMO}(\boldsymbol{S}^n, \boldsymbol{S}^n)$ is meant a continuous family $\{h_t\}$ of maps in $\mathrm{VMO}(\boldsymbol{S}^n, \boldsymbol{S}^n)$.

The following results of H. Brézis and L. Nirenberg extend the Brouwer degree and theorems of Borsuk and Hopf to the class of VMO maps:

Generalized Brouwer Theorem. *There exists an integer-valued function $f \mapsto \deg(f)$ defined for $f \in \mathrm{VMO}(\boldsymbol{S}^n, \boldsymbol{S}^n)$ such that:*

(I) *If* $\deg(f) \neq 0$, *then* $\operatorname{Ess} R(f) = S^n$

(II) (Homotopy) *If two maps f and g are homotopic in* $\mathrm{VMO}(S^n, S^n)$, *then* $\deg(f) = \deg(g)$.

GENERALIZED BORSUK THEOREM. *If $f \in$ VMO(S^n, S^n) is an odd map, then* $\deg(f)$ *is odd.*

GENERALIZED HOPF THEOREM. *If f, g are in* VMO(S^n, S^n) *and* $\deg(f) = \deg(g)$, *then the maps f and g are homotopic in* VMO(S^n, S^n).

The definition of $\deg(f)$ for a map $f \in \mathrm{VMO}(S^n, S^n)$ is based on a density argument: If $\varepsilon > 0$ is small, then $f_\varepsilon : S^n \to R^{n+1}$ defined by

$$f_\varepsilon(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} f(y)\, dy$$

(where $B_\varepsilon(x)$ is the ε-ball around x) is continuous and $\|f - f_\varepsilon\|_{\mathrm{BMO}} \to 0$ as $\varepsilon \to 0$. For sufficiently small $\varepsilon > 0$, f_ε maps S^n into $R^{n+1} - \{0\}$ and one defines $\deg(f)$ as the Brouwer degree of $f_\varepsilon : S^n \to R^{n+1} - \{0\}$. For details, the reader is referred to Brézis–Nirenberg [1995].

The above theory of degree can be extended to VMO maps between arbitrary smooth manifolds without boundary. If M and N are such manifolds, Brézis and Nirenberg proved that there exists a bijection between the 0-homotopy sets $\pi_0(C(M, N))$ and $\pi_0(\mathrm{VMO}(M, N))$. In fact, the more general result of A. Abbondandolo (see "Additional References" III(c)) asserts that the inclusion $i : C(M, N) \hookrightarrow \mathrm{VMO}(M, N)$ is a homotopy equivalence.

We remark that using approximation properties of the VMO maps one can also extend the degree for continuous maps $f \in C_0(\overline{U})$ to the VMO maps defined on $\overline{U}$ for a bounded domain $U \subset R^n$ This more general theory can be found in Brézis–Nirenberg [1996]. For a survey of some of the above results the reader is referred to Brézis [1997]; for related and more general results see "Additional References" III(c).

2° *Degree for Sobolev maps* ([1]). Let M and N be smooth compact oriented Riemannian manifolds of dimension $n \geq 2$, and let dx and dy denote the volume forms on M and N, respectively, that are induced by the orientation and metric tensors of the corresponding manifolds; for convenience, we also assume that $\int_N dy = 1$.

Let $f : M \to N$ be a smooth map. Then, according to the well known formula (going back to Kronecker), the Brouwer degree of f is given by

$$(*) \qquad d(f; M, N) = \int_M f^{\#}\eta,$$

([1]) For the terminology used in these comments and further reading see Iwaniec–Martin [2001].

where $f^{\#}\eta$ is the pull-back of a smooth n-form η on N. One can express the degree in equivalent form by the formula

$$(**) \qquad d(f; M, N) = \int_M J(x, f)\, dx,$$

where the Jacobian determinant $J(x, f)$ is the pull-back of the volume form, $J(x, f)dx = f^{\#}(dy)$.

If $f : M \to N$ is not smooth, but in the Sobolev class $W^{1,n}(M, N)$, then the Jacobian $J(x, f)$ exists almost everywhere and is L^1-integrable; one can therefore define the degree $d(f; M, N)$ of f again by the formula $(**)$. This case, however, is covered by the Brézis–Nirenberg theory because $W^{1,n}(M, N) \subset \mathrm{VMO}(M, N)$. Since, as shown by F. Bethuel, for $1 \le p < n$, $W^{1,p}(M, N)$ need not necessarily be smaller than $\mathrm{VMO}(M, N)$, the natural question emerged whether one can make sense of the degree formula $(**)$ for maps with nonintegrable Jacobian.

T. Iwaniec and his collaborators have developed, in fact, a viable degree theory for maps in some classes slightly below the Sobolev class $W^{1,n}(M, N)$. One typical example is the Orlicz–Sobolev space $W^{1,P}(M, N)$, where $P = P(t) = t^n/\log(e + t)$. It consists of the maps $f : M \to N$ whose differential $Df : TM \to TN$ satisfies ([1])

$$\int_M \frac{|Df(x)|^n}{\log(e + |Df(x)|)}\, dx < \infty.$$

Another example is the Marcinkiewicz Sobolev space $W^{1,n}_{\mathrm{weak}}(M, N)$, which consists of the maps such that

$$\mathrm{meas}\{x \in M \mid |Df(x)| > t\} = o(t^n).$$

For a map in any of the above classes, the integral $\int_M J(x, f)\, dx$, appearing in $(**)$, exists in a somewhat weaker sense: it will actually converge provided f is orientation-preserving, i.e., $J(x, f) \ge 0$ almost everywhere in M. But this can only be done in the case when the cohomology $H^k(N)$ of the target manifold is nonzero for some $1 \le k < n$. If this is the case, the formula

$$d(f; M, N) = \lim_{\varepsilon \to 0+} \int_M \frac{J(x, f)}{|Df(x)|^{\varepsilon}}\, dx$$

defines the degree, which is invariant under homotopy within each of the above classes of maps. For details of the above theory, the reader is referred to Greco–Iwaniec–Sbordone–Stroffolini [1995] and Hajłasz–Iwaniec–Malý–Onninen [2003].

([1]) It can be proved that each Orlicz–Sobolev class $W^{1,P}(S^n, S^n)$ with $P = P(t) = o(t^n)$ contains an orientation-preserving map with nonintegrable Jacobian. This implies that there is no integral formula for the degree of a map in any such class.

For results concerned with homotopy properties of Sobolev maps, the reader is referred to articles of H. Brézis and his collaborators in "Additional References" III(c). For some applications of the degree to the problem of local invertibility of Sobolev maps, see the book of Fonseca-Gangbo [1995].

3° *Degree in Wiener spaces.* Let $(H, \|\cdot\|)$ be a (separable) Hilbert space and ν the standard cylindrical measure on H. It is known that:

(i) H equipped with a norm weaker than the original one can be completed to a Banach space W,

(ii) there exists on W a Radon measure μ that extends the measure ν over W

The triple (W, H, μ) is called an *abstract Wiener space* ([1]) with *Cameron-Martin space* H, and μ is the *Wiener measure*; the injection $j : H \hookrightarrow W$ is continuous and completely continuous.

In 1986 E. Getzler, using the framework of the Malliavin calculus (cf. P. Malliavin, *Stochastic Analysis*, Springer, 1977) introduced a notion of degree for suitable maps, not necessarily continuous, of a Banach space into itself. More precisely, let (W, H, μ) be an abstract Wiener space and $D_p^s(W, H)$ be the Malliavin Sobolev space, of order s and exponent p, of H-valued functions $v : W \to H$. By a *Wiener map* $T : W \to W$ is meant a transformation $v \mapsto v - F(v)$, where $F \in D^1(W, H) = \bigcap_{p>1} D_p^1(W, H)$. Getzler [1986] proved that under suitable conditions (too technical to be stated here), one can assign to a Wiener map an integer, called the *degree* of T, which, in the case when $F : W \to H$ is smooth, may be interpreted as the sum of the signs of the derivative of T over all points of a regular fiber. For details and some applications (for example to the existence of solutions of stochastic partial differential equations), see "Additional References" III((d) and also an exposition of the theory in Üstünel–Zakai [2000].

(*f*) *Vertical fixed point theory*

Let B be a space ([2]). By a *fiber space* over B is meant a space E together with a surjective map $p : E \to B$; for each $b \in B$, the *fiber* over b is the subset $E_b = f^{-1}(b)$. Clearly, with the obvious definition of morphisms, the fiber spaces over B form a category.

([1]) Let $W = C_0([0, 1])$ be the Banach space of continuous functions $u : [0, 1] \to \boldsymbol{R}$ with $u(0) = 0$, equipped with the sup norm. For $u \in C_0([0, 1])$ and $t \in [0, 1]$, let $W_t(u) = u(t)$; then it can be shown that there exists a unique probability measure μ on W such that $(t, u) \mapsto W_t(u)$ is a Wiener process. Let H denote the Hilbert subspace of W consisting of the absolutely continuous functions with square integrable derivative and the norm $|u|^2 = \int_0^1 |u'(s)|^2\, ds$; H is obviously dense in W, and the triple (W, H, μ) is the simplest example of an abstract Wiener space.

([2]) In this section we assume for convenience that all spaces under consideration are metrizable.

For U an open subset of E, a map $f : U \to E$ is called *vertical* if $pf = p$ on U. For such a map, the fixed point set $\mathrm{Fix}(f)$ is then also a fiber space over B; for each $b \in B$, we have $f_b : U_b \to U_b$, where $U_b = p^{-1}(b) \cap U$, f_b is the restriction of f and $\mathrm{Fix}(f) = \bigcup_{b\in B} \mathrm{Fix}(f_b)$.

The vertical fixed point theory is concerned with the existence of fixed points of $f : U \to E$ that cannot be eliminated using fiber-preserving homotopies of f. We now briefly comment on some basic results of this theory established by Dold [1974] in the framework of the category ENR_B.

A fiber space $p : E \to B$ is called an ENR_B (ENR over B) if for some open set V in $B \times \boldsymbol{R}^n$ there are maps $E \xrightarrow{i} V \xrightarrow{r} E$ (over B) such that $ri = \mathrm{id}_E$; then, in particular, the fibers of p are ENRs. For example, it can be proved that: *If E and B are ENRs, then a proper map $p : E \to B$ is an ENR_B if and only if p is a Hurewicz fibration.*

We now make the following assumptions:

(i) $p : E \to B$ is an ENR over a locally compact space B,

(ii) $f : U \to E$ is a vertical map such that the restriction $p|\mathrm{Fix}(f) : \mathrm{Fix}(f) \to B$ is proper,

(iii) h is a given general cohomology theory.

In this setting, Dold assigns to f an index $I(f)$ with values in $h^0(B)$. If $f : E \to E$ and B is a point, then $I(f)$ coincides with the Lefschetz number. The elements of $h^0(B)$ that occur as indices are precisely the spherical stable ones, i.e., belong to the image of $\varepsilon : \pi^0(B\oplus\mathrm{pt}) \to h^0(B)$, where ε is a natural transformation from stable cohomotopy theory to h^0. The Dold index has all the familiar properties (additivity, commutativity, homotopy invariance etc.) as in the classical case of $B =$ point. A detailed exposition of Dold's theory together with an extension to fiberwise ANRs was given in Crabb–James [1998]; this extension contains as a special case the index for compact maps of ANRs presented in Chapters IV and V.

For related and more general results the reader is referred to "Additional References" III(e).

Appendix: Preliminaries

This appendix contains a condensed review of those parts of topology, functional analysis, and algebra that are used in the book. It is included to provide an easily available reference for both the necessary background material and the terminology. For the proofs of the results presented in this appendix the reader is referred to "General Reference Texts" in the bibliography.

A. Generalities

We use the standard set-theoretic and logical symbols: $\emptyset$ for the empty set, $\bigcup$ for the union of sets, $\bigcap$ for intersection, $-$ for difference, and $\subset$ for inclusion. The Cartesian product of two sets A and B is denoted by $A \times B$, and the product of an indexed family by $\prod_{i \in I} A_i$. The symbol $x \in A$ indicates that x is an element of the set A; the negation of a membership assertion is indicated by $\notin$. The subset of all elements x of a set X that have some property P (i.e., for which $P(x)$ holds) is denoted by $\{x \in X \mid P(x)\}$. The connectives "implies" and "if and only if" are denoted by $\Rightarrow$ and $\Leftrightarrow$, respectively. The set of all subsets of a set X is denoted by 2^X.

Notation

The following fixed notations are used:

$\boldsymbol{N}$ = set of natural numbers,
$\boldsymbol{Z}$ = ring of integers,
$\boldsymbol{R}$ = field of real numbers,
$\boldsymbol{C}$ = field of complex numbers,
$\boldsymbol{Q}$ = field of rational numbers,
$\boldsymbol{R}^n$ = Euclidean n-space with $\|r\| = \sqrt{\sum r_i^2}$,

K^n = Euclidean n-ball = $\{x \in \boldsymbol{R}^n \mid \|x\| \le 1\}$,
S^{n-1} = Euclidean $(n-1)$-sphere = $\{x \in \boldsymbol{R}^n \mid \|x\| = 1\}$,
I = closed unit interval $[0,1]$,
I^n = n-cube = $\{x \in \boldsymbol{R}^n \mid 0 \le x_i \le 1 \text{ for } 1 \le i \le n\}$,
I^∞ = Hilbert cube, i.e., the countable infinite product of closed unit intervals.

Maps

Given two sets X and Y, a map $f : X \to Y$ from X to Y assigns to every $x \in X$ a well determined $f(x) \in Y$; the set of all maps from X to Y is denoted by Y^X. If $f : X \to Y$ and $g : Y \to Z$, the *composite map* $g \circ f : X \to Z$ is defined by $x \mapsto f(g(x))$. If $A \subset X$ is a subset, the *inclusion* $i : A \to X$ is defined by $i(a) = a$ for all $a \in A$; if $A = X$, the inclusion i becomes the *identity map* id_X on X. The composite of $i : A \to X$ with $f : X \to Y$ is denoted by $f|A : A \to Y$ and is called the *restriction* of f to A; we say that f *extends* $f|A$ over X. For a map $f : X \to Y$, if $A \subset X$ and $B \subset Y$, the *image* $f(A)$ of A and the *counterimage* $f^{-1}(B)$ of B are given by

$$f(A) = \{f(x) \mid x \in A\}, \qquad f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

A map $f : X \to Y$ is *injective* if $f(x) = f(x')$ implies $x = x'$; it is *surjective* if $f(X) = Y$; a map that is both injective and surjective is said to be *bijective*. If $f : X \to Y$ and $g : Y \to X$ with $gf = \mathrm{id}_X$, then f is injective and g is surjective. For a bijective $f : X \to Y$, the inverse of f is denoted by $f^{-1} : Y \to X$; it is characterized by the identities $f^{-1} \circ f = \mathrm{id}_X$, $f \circ f^{-1} = \mathrm{id}_Y$.

If $x \in X = \prod_{i \in I} X_i$, we write $x = \{x_i\}_{i \in I}$ and call $x_i \in X_i$ the ith coordinate of x; the map $p_i : X \to X_i$ given by $x \mapsto x_i$ is the *projection* of X onto the axis X_i.

A family $\{f_i\}_{i \in I}$ of maps $f_i : X \to Y_i$ is *separating* provided for any $x_1, x_2 \in X$ there is a map f_j with $f_j(x_1) \ne f_j(x_2)$. If $\{f_i\}_{i \in I}$ is separating, then $x \mapsto \{f_i(x)\}_{i \in I}$ is an injective map of X into the product $\prod_{i \in I} Y_i$.

Order

A binary relation $\preceq$ in a set X is called a *preorder* if (a) $x \preceq x$ for every $x \in X$, (b) $x \preceq y$ and $y \preceq z$ implies $x \preceq z$; it is called a *partial order* if also (c) $x \preceq y$ and $y \preceq x$ implies $x = y$. A set X together with a preorder (respectively partial order) $\preceq$ is called a *preordered* (respectively *partially ordered*) set. A subset A of a preordered set $(X, \preceq)$ is *cofinal* in X if for each $x \in X$ there is an $a \in A$ with $x \preceq a$. An element $x_0 \in X$ is called *maximal* in X if there is no $x \in X$ with $x \ne x_0 \preceq x$; a minimal element is defined

similarly. An element $x_0 \in X$ is an *upper bound* for a set $A \subset X$ if $a \preceq x_0$ for all $a \in A$. The partially ordered set is *totally ordered* (or a *chain*) if for each $x, y \in X$, either $x \preceq y$ or $y \preceq x$; it is *well ordered* if each nonempty subset has a minimal element.

The axiom of choice, which is assumed throughout the book, is used in one of the following forms:

(A.1) KURATOWSKI-ZORN LEMMA. *A partially ordered set in which every totally ordered subset has an upper bound contains at least one maximal element.*

(A.2) ZERMELO THEOREM. *Every set can be well ordered.*

B. Topological Spaces

A *topological space* is a pair (X, τ) consisting of a set X and a collection $\tau = \{U\}$ of subsets of X called *open sets* such that:

(i) any union and any finite intersection of open sets are open,

(ii) the whole space X and the empty set $\emptyset$ are open.

The collection τ is a *topology* for X. The sets complementary to the open sets are called *closed sets*. Any intersection and any finite union of closed sets are closed; X and $\emptyset$ are closed. The *closure* $\bar{A}$ of $A \subset X$ is the intersection of all closed sets that contain A. The *interior* $\operatorname{Int}(B)$ of B is the union of all open sets that are subsets of B. A is closed if and only if $A = \bar{A}$, and B is open if and only if $B = \operatorname{Int}(B)$. The set $\partial A = \bar{A} \cap \overline{X - A}$ is the *boundary* of A, and its elements are *boundary points* of A. If $A \subset B \subset X$, then A is *dense* in B if $\bar{A} = B$. A space X is *separable* if it contains a countable set dense in X.

A space X is *connected* if it is not expressible as the union of two nonempty disjoint open sets. In any space X, the union of all connected sets containing an $x \in X$ is called the *component of* x in X; the set of components of the points of X forms a partition of X, and the members of this partition are called the *components of* X. Each component is a maximal connected subset of X and is closed in X.

A *base* for the topology of a topological space is a collection $\beta \subset \tau$ of sets such that the open sets are all the unions of members of β. A collection $\sigma \subset \tau$ is a *subbase* for τ if the family of all finite intersections of members of σ forms a base for τ

A space with more than one point may be topologized in different manners. A topology τ_1 is *weaker* than a topology τ_2 if $\tau_1 \subset \tau_2$. The weakest topology is the one in which the whole space and the empty set are the only open sets. The strongest topology is the *discrete topology* in which every subset is open.

Continuity

Let X and Y be topological spaces. A map $f : X \to Y$ is *continuous* if whenever U is open in Y, $f^{-1}(U)$ is open in X. Evidently if $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is the composite map $gf : X \to Z$. If for a bijective map $f : X \to Y$ both f and its inverse f^{-1} are continuous, then f is called *bicontinuous* or a *homeomorphism*. If such a map exists for two spaces X and Y, then these spaces are called *homeomorphic*. A map $f : X \to Y$ is an *embedding* if f maps X homeomorphically onto $f(X)$. An embedding $f : X \to Y$ is *closed* (respectively *open*) if $f(X)$ is closed (respectively open) in Y. A continuous map $f : X \to Y$ is *open* (respectively *closed*) provided f maps open (respectively closed) sets in X onto open (respectively closed) sets in Y. If $A \subset X$, a continuous $r : X \to A$ with $r|A = \mathrm{id}_A$ is called a *retraction* of X onto A; if such an r exists, then A is called a *retract* of X.

A function $f : X \to \boldsymbol{R}$ on a topological space X is *lower* (respectively *upper*) *semicontinuous* if $\{x \in X \mid f(x) \leq r\}$ (respectively $\{x \in X \mid f(x) \geq r\}$) is closed for each real r. The sum of two and the sup of any family of lower semicontinuous functions is lower semicontinuous; the inf of any family of upper semicontinuous functions is upper semicontinuous.

Relations between topologies may be conveniently expressed in terms of continuous maps. If τ_1 and τ_2 are topologies on X, then $\tau_1 \subset \tau_2$ if and only if for any space Y every τ_1-continuous map $f : X \to Y$ is also τ_2-continuous.

Neighborhoods and separation axioms

Given a point x (respectively a set A) in a topological space X, the set U is a *neighborhood* of x (respectively of A) if there is an open set V such that $x \in V \subset U$ (respectively $A \subset V \subset U$). We often abbreviate "neighborhood" to "nbd".

Let $\mathcal{N}_x$ be the set of neighborhoods of a point $x \in X$. Then:

(i) $x \in U$ for all $U \in \mathcal{N}_x$,
(ii) if $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_x$, then $U \cap V \in \mathcal{N}_x$,
(iii) if $U \in \mathcal{N}_x$ and $U \subset V$, then $V \in \mathcal{N}_x$,
(iv) if $U \in \mathcal{N}_x$, there is a $V \in \mathcal{N}_x$ with $U \in \mathcal{N}_y$ for all $y \in V$

Assume now that X is an arbitrary set such that for each $x \in X$ there is defined a nonempty family $\mathcal{N}_x$ of subsets of X satisfying conditions (i)–(iv). Then there is exactly one topology τ in X that converts $\mathcal{N}_x$ into the set of neighborhoods of x for every x ($U \in \tau$ if for each $x \in U$ there is some $V \in \mathcal{N}_x$ with $V \subset U$). A subset $\mathcal{N}_x' \subset \mathcal{N}_x$ is a *base of neighborhoods* of x provided given $U \in \mathcal{N}_x$ there is a $V \in \mathcal{N}_x'$ with $V \subset U$.

A topological space X is *Hausdorff* if any two distinct points of X have disjoint neighborhoods. A space X is *normal* if any two disjoint nonempty closed subsets of X admit disjoint neighborhoods. A space X is *regular* if

for each $x \in X$ and each nbd V of x, there exists a nbd U of x such that $\overline{U} \subset V$; and X is *completely regular* if for each $x \in X$ and closed set A not containing x, there is a continuous $f : X \to \boldsymbol{I}$ with $f(x) = 1$ and $f|A = 0$.

In this book all topological spaces are assumed to be Hausdorff.

(B.1) URYSOHN LEMMA. *Let X be normal, and let A and B be two disjoint nonempty closed subsets of X. Then there is a continuous $\lambda : X \to \boldsymbol{I}$ such that $\lambda(a) = 1$ for $a \in A$ and $\lambda(b) = 0$ for $b \in B$.*

(B.2) THEOREM (Tietze–Urysohn). *Let X be normal, $A \subset X$ closed, and $f : A \to \boldsymbol{R}$ continuous. Then there is an $f^* : X \to \boldsymbol{R}$ continuous that extends f over X.*

Generating new spaces from old

From a given space, others are constructed by means of certain standard procedures. The simplest are relativization, products, and identification.

Let X be a topological space and $A \subset X$; A can be made into a topological space by taking the intersections with A of the open sets in X to be the open sets in A; in this topology, the closed sets are the intersections with A of closed sets in X.

Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The *Tychonoff product* of the X_i is the set $\prod_{i\in I} X_i$, with a basis for the topology taken as the collection of all products $\prod U_i$, where U_i is open in X_i and $U_i = X_i$ for all but a finite number of indices.

Let X be a topological space and R an equivalence relation in X. Let X/R be the set of equivalence classes and $f : X \to X/R$ the function taking $x \in X$ to its equivalence class. Then X/R is made a topological space by making U open if and only if $f^{-1}(U)$ is open in X. The space X/R is the *quotient space* of X by the relation R.

These are special cases of two general methods of forming new spaces. Let X be a set together with a family of topological spaces $\{X_i\}_{i\in I}$. Given a family $\{f_i : X \to X_i\}_{i\in I}$ of maps, the *projective* (or *weak*) *topology* on X determined by the family $\{X_i, f_i\}_{i\in I}$ is the weakest topology for which each f_i is continuous; this topology is characterized by the property: if Y is any topological space, then a map $g : Y \to X$ is continuous if and only if each $f_j \circ g : Y \to X_j$ is continuous ($j \in I$). Dually, given $\{g_i : X_i \to X\}_{i\in I}$, the *inductive topology* on X determined by $\{X_i, g_i\}_{i\in I}$ is the strongest topology for which each g_i is continuous; this topology is characterized by the property: if Y is any topological space, then a map $g : X \to Y$ is continuous if and only if every $g \circ g_j : X_j \to Y$ is continuous ($j \in I$). For example, given a family $\{K\}$ of subspaces of a topological space X, the inductive topology determined by the inclusion maps $K \to X$ is called the *topology generated by the family* $\{K\}$.

Compactness

A *covering* of a topological space X is a collection $\{U\}$ of sets whose union is X. If all the sets of a covering $\{U\}$ are open, then $\{U\}$ is called an *open* covering. A *finite* covering is one consisting of finitely many sets. A family of sets has the *finite intersection property* if any finite number of sets in the family have a nonempty intersection.

A topological space is *compact* if it is Hausdorff and if every open covering contains a finite subcovering. Equivalently, a space is compact if every family of closed sets with the finite intersection property has a nonempty intersection. A subset of a space is *compact* if it is compact when regarded as a subspace; it is *relatively compact* if its closure is compact. A topological space X is *locally compact* if every point of X has a compact neighborhood.

Some elementary but frequently used properties of compact spaces are: (a) every compact space is normal; (b) a compact subset of a Hausdorff space is closed; (c) the continuous image (in a Hausdorff space) of a compact space is compact; (d) an injective map with a compact domain is an embedding; (e) a lower (respectively upper) semicontinuous function on a compact space attains its infimum (respectively supremum).

(B.3) THEOREM (Tychonoff). *The topological product of any family of compact spaces is compact.*

Generalized convergence

A preordered set $(\mathscr{D}, \preceq)$ is *directed* if for each pair $x, y \in \mathscr{D}$ there is some z such that $x \preceq z$ and $y \preceq z$.

Let $\mathscr{D}$ be a directed set. A *net* $\{x_\alpha\}_{\alpha\in\mathscr{D}}$ in a set X is any function $x : \mathscr{D} \to X$. A net $\{y_\beta\}_{\beta\in\mathscr{D}_0}$ with a domain $\mathscr{D}_0$ is a *subnet* of $\{x_\alpha\}_{\alpha\in\mathscr{D}}$ if there is a function $N : \mathscr{D}_0 \to \mathscr{D}$ such that (i) $y_\beta = x_{N(\beta)}$ for each $\beta \in \mathscr{D}_0$; (ii) for each $\alpha \in \mathscr{D}$ there is a $\beta \in \mathscr{D}_0$ such that $N(\gamma) \succeq \alpha$ whenever $\gamma \succeq \beta$.

Let $\{x_\alpha\}_{\alpha\in\mathscr{D}}$ be a net in a topological space X. The net $\{x_\alpha\}_{\alpha\in\mathscr{D}}$ *converges* to a point $x \in X$ (written $\lim_{\alpha\in\mathscr{D}} x_\alpha = x$) provided for each nbd U of x there is an $\alpha_0 \in \mathscr{D}$ such that for all $\alpha \succeq \alpha_0$ we have $x_\alpha \in U$.

(B.4) THEOREM.

(i) *If A is a subset of a topological space X, then $x \in \overline{A}$ if and only if there is a net in A converging to x.*

(ii) *X is compact if and only if each net in X has a subnet converging to some point of X.*

(iii) *X is Hausdorff if and only if each net in X converges to at most one point.*

(iv) *$f : X \to Y$ is continuous if and only if $\lim_{\alpha\in\mathscr{D}} x_\alpha = x$ implies $\lim_{\alpha\in\mathscr{D}} f(x_\alpha) = f(x)$ for each net $\{x_\alpha\}_{\alpha\in\mathscr{D}}$ in X.*

Metrizable spaces

A *metric* on a set X is a function $d : X \times X \to \boldsymbol{R}$ such that:

(i) $d(x, y) \geq 0$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) = d(y, x)$,

(iv) $d(x, y) \leq d(x, z) + d(z, y)$.

The topology τ that has as a base the collection of all open balls

$$B(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\} \quad \text{for all } x \in X,\ \varepsilon > 0$$

is said to be *generated* by d. Two metrics d_1, d_2 on a set X are *equivalent* if they generate the same topology τ on X. A topological space (X, τ) is *metrizable* if there is a metric on X that generates τ. Every metrizable space (X, τ) admits a metric d that generates τ and satisfies $d(x, y) \leq 1$ for all $x, y \in X$.

Several useful properties of the metrizable spaces can be formulated in the sequential language: (i) In a metrizable (X, τ), $x_n \to x$ if and only if $d(x_n, x) \to 0$ for any metric d that generates the topology τ; (ii) a map $f : X \to Y$ between metrizable spaces is continuous if and only if $x_n \to x$ implies $f(x_n) \to f(x)$ for any sequence $\{x_n\}$ in X; (iii) a metrizable (X, τ) is compact if and only if it is *sequentially compact*: any sequence $\{x_n\}$ in X contains a subsequence $\{x_{n_k}\}$ that converges to some $x \in X$.

Let d be a metric on a set X. A sequence $\{x_n\}$ of points in X is a *d-Cauchy sequence* if for every $\varepsilon > 0$ there is an integer $K(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for $n, m > K(\varepsilon)$. The metric d is *complete* if every d-Cauchy sequence is convergent. A space (X, τ) is *topologically complete* if there is a complete metric d that generates τ.

A pair (X, d) consisting of a set X and a metric d on X is a *metric space* and $d(x, y)$ is the *distance* between x and y. Every subset of a metric space X is itself a metric space with metric the same as in X. Given a metric space (X, d) the following formulas define the *distance between a point and a set*, the *distance between two sets*, and the *diameter of a set*, respectively:

$$d(x, A) = \inf_{y \in A} d(x, y), \qquad \operatorname{dist}(A, B) = \inf_{x \in A,\, y \in B} d(x, y),$$
$$\delta(A) = \sup_{x, y \in A} d(x, y).$$

A *bounded set* in a metric space is one having a finite diameter. A map $f : X \to Y$ between metric spaces is an *isometric embedding* if $d_X(x, y) = d_Y(f(x), f(y))$ for any $x, y \in X$. An *isometry* is an isometric embedding that is onto. Two metric spaces (X, d_X) and (Y, d_Y) are *isometric* if there is an isometry $f : X \to Y$.

A metric space (X, d) is *complete* if d is a complete metric for X. A subset of a complete metric space is complete if and only if it is closed.

(B.5) THEOREM (Hausdorff). *Every metric space (X, d) admits an isometric embedding $f : X \to Y$ into a complete metric space Y such that $f(X)$ is dense in Y. The space Y (called the completion of X) is unique up to isometry.*

(B.6) THEOREM (Cantor). *Let (X, d) be complete and $\{A_n\}$ a decreasing sequence of nonempty closed sets in X such that $\delta(A_n) \to 0$. Then the intersection $\bigcap_{n=1}^{\infty} A_n$ contains exactly one point.*

(B.7) THEOREM (Baire). *Let X be either a complete metric space or a locally compact space. If $X = \bigcup_{n=1}^{\infty} A_n$, where each A_n is a closed set, then some A_n must contain a nonempty open set.*

A finite subset $N = \{x_1, \ldots, x_n\}$ of a metric space (X, d) is an *ε-net* (where $\varepsilon > 0$) if $X \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$; (X, d) is *totally bounded* if it contains an ε-net for every $\varepsilon > 0$. Every totally bounded space is separable.

(B.8) THEOREM (Hausdorff). *A metric space is compact if and only if it is complete and totally bounded.*

Paracompactness of metric spaces

If $\{U\}$ and $\{V\}$ are two coverings of a space X, then $\{U\}$ is a *refinement* of $\{V\}$ if for each U there is some V with $U \subset V$.

An open covering $\{U_\lambda \mid \lambda \in \Lambda\}$ of a space X is called *neighborhood finite* if each $x \in X$ has a neighborhood V such that $\operatorname{card}\{\lambda \mid U_\lambda \cap V \neq \emptyset\}$ is finite. A space X is *paracompact* if every open covering has an open nbd-finite refinement. Clearly, every compact space is paracompact.

(B.9) THEOREM (Stone). *Every metrizable space is paracompact.*

PROOF. Let d be a metric for X; for any $A \subset X$ write $(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\}$, the ε-nbd of A. We first show that any countable open cover $\{U_i\}$ of X has an open nbd-finite refinement $\{V_i\}$ with $V_i \subset U_i$ for each i. Let $\varphi_i : X \to \boldsymbol{I}$ be the function $x \mapsto \min(1, d(x, X - U_i))$; define $V_1 = U_1$ and for each $n > 1$ let

$$V_n = U_n \cap \bigcap_{i=1}^{n-1} \{x \mid \varphi_i(x) < 1/n\},$$

which is an open set contained in U_n. The family $\{V_i\}$ is an open cover: given $x \in X$, let $U_{n(x)}$ be the first set U_i to which it belongs; then $\varphi_i(x) = 0$ for each $i < n(x)$, so $x \in V_{n(x)}$. Moreover, x has a nbd meeting at most

finitely many V_i: for $\varphi_{n(x)}(x) = s > 0$, so $\{y \mid \varphi_{n(x)}(y) > s/2\}$ is a nbd of x meeting no V_k with $k > \max\{n(x), 2/s\}$. We now prove the theorem.

Let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover; we can assume that the indexing set Λ is well ordered. For each fixed integer $n = 1, 2, \ldots$, define a transfinite sequence of closed sets as follows:

$$H_{n,1} = \{x \mid d(x, X - U_1) \geq 1/n\}, \quad \ldots,$$
$$H_{n,\lambda} = \{x \mid d(x, X - U_\lambda) \geq 1/n\} \cap \Big\{x \,\Big|\, d\Big(x, \bigcup_{\mu<\lambda} H_{n,\mu}\Big) \geq 1/n\Big\}.$$

These sets have the following properties:

(a) $(H_{n,\lambda}, 1/n) \subset U_\lambda$ for each (n, λ),

(b) if $\mu < \lambda$, then $d(H_{n,\mu}, H_{n,\lambda}) \geq 1/n$,

(c) $\bigcup_n \bigcup_\lambda H_{n,\lambda} = X$.

(a) and (b) are immediate from the observation that if $y \in H_{n,\lambda}$, then $(y, 1/n) \subset U_\lambda$ and $(y, 1/n) \cap H_{n,\mu} = \emptyset$ for all $\mu < \lambda$. For (c), given $x \in X$, find the first U_λ containing x and choose n so large that $(x, 1/n) \subset U_\lambda$; then $x \in H_{n,\lambda}$: otherwise, there would be a $\mu < \lambda$ and a $y \in H_{n,\mu}$ with $d(x, y) < 1/n$, showing $x \in (y, 1/n) \subset U_\mu$ and contradicting the choice of λ.

Now let $G_{n,\lambda} = (H_{n,\mu}, 1/(3n))$; the open sets $\{G_{n,\lambda}\}$ form a covering for X, since $H_{n,\lambda} \subset G_{n,\lambda}$, and therefore the open sets $Q_n = \bigcup_\lambda G_{n,\lambda}$ give a countable cover for X. According to what we have already proved, $\{Q_n\}$ has a nbd-finite open refinement $\{Q'_n\}$ with $Q'_n \subset Q_n = \bigcup_\lambda G_{n,\lambda}$ for each n. Therefore, the family

$$V_{n,\lambda} = \{Q'_n \cap G_{n,\lambda} \mid n = 1, 2, \ldots; \ \lambda \in \Lambda\}$$

forms an open cover (since each x belongs to some Q'_n and $Q'_n \subset \bigcup_\lambda G_{n,\lambda}$). Moreover, $V_{n,\lambda} \subset U_\lambda$ for each (n, λ). Finally, $\{V_{n,\lambda}\}$ is nbd-finite: given $x \in X$, since Q'_n is nbd-finite, there is a nbd W and an integer N with $W \cap Q'_n = \emptyset$ for all $n \geq N$; moreover, $(x, 1/(6N))$ can meet at most one $G_{n,\lambda}$ for each $n \leq N$ because $d(G_{n,\mu}, G_{n,\lambda}) \geq 1/(3n)$ whenever $\mu < \lambda$. Thus, $W \cap (x, 1/(6N))$ meets at most N sets $V_{n,\lambda}$, which completes the proof. □

Observe that with the construction of the sets $V_{n,\lambda}$, we have established another property of open coverings in metric spaces that is of importance in applications.

(B.10) COROLLARY. *Let X be a metric space and $\{U_\lambda \mid \lambda \in \Lambda\}$ an open covering. Then $\{U_\lambda\}$ has a σ-discrete open refinement (i.e., a refinement that can be represented as the union of a countable number of families $\mathfrak{A}_n = \{V_{n,\lambda} \mid \lambda \in \Lambda\}$ of open sets, where for each n every $x \in X$ has a nbd meeting at most one set of $\mathfrak{A}_n$).*

PROOF. The sets $V_{n,\lambda} = Q'_n \cap (H_{n,\lambda}, 1/(3n))$ satisfy (i) $V_{n,\lambda} \subset U_\lambda$ for all (n,λ); (ii) $d(V_{n,\lambda}, V_{n,\mu}) \geq 1/(3n)$ whenever $\mu < \lambda$, so the $\{V_{n,\lambda}\}$ are a pairwise disjoint family for each n; (iii) $\bigcup_n \bigcup_\lambda V_{n,\lambda} = X$; and (iv) for each n, the family $\{V_{n,\lambda} \mid \lambda \in \Lambda\}$ is nbd-finite. □

For any space Y, the *support* $\operatorname{supp}(f)$ of a map $f : Y \to \boldsymbol{R}$ is the closed set $\overline{\{y \in Y \mid f(y) \neq 0\}}$. Let Y be a space and $\{U_\lambda \mid \lambda \in \Lambda\}$ an open covering of Y. By a *partition of unity subordinate to* $\{U_\lambda\}$ is meant a family of continuous maps $\kappa_\lambda : Y \to \boldsymbol{I}$ for $\lambda \in \Lambda$ such that:

(i) $\operatorname{supp}(\kappa_\lambda) \subset U_\lambda$ for each λ,

(ii) $\{\operatorname{supp}(\kappa_\lambda) \mid \lambda \in \Lambda\}$ is a nbd-finite closed covering of Y,

(iii) $\sum_\lambda \kappa_\lambda(y) = 1$ for all $y \in Y$.

Let X be paracompact, and let $\{U_\lambda \mid \lambda \in \Lambda\}$ be any open covering of X. Then:

(a) $\{U_\lambda\}$ has an open nbd-finite refinement $\{V_\lambda \mid \lambda \in \Lambda\}$ with $\overline{V}_\lambda \subset U_\lambda$ for each λ.

(b) $\{U_\lambda\}$ has an open refinement $\{V_\mu \mid \mu \in M\}$ with the property: for each $x \in X$, the union $\bigcup\{V_\mu \mid x \in V_\mu\}$ is contained in some U_λ. (This is called an open *barycentric refinement*.)

(c) $\{U_\lambda\}$ has a subordinate partition of unity.

Some general embedding theorems

(B.11) THEOREM.

(a) (Urysohn) *Any compact metric space admits an embedding into the Hilbert cube* $\boldsymbol{I}^\infty$.

(b) (Tychonoff) *Any compact space can be embedded in a Tychonoff cube, i.e., a (possibly uncountable) product of unit intervals.*

(B.12) THEOREM (Kuratowski). *Any metric space* (Y,d) *can be isometrically embedded in the Banach space* $C(Y)$ *of all bounded continuous real-valued functions on* Y *taken with the* sup *norm.*

(B.13) THEOREM (Arens–Eells). *Any metric space* (Y,d) *can be isometrically embedded as a closed subset in a normed linear space.*

PROOF ([1]). Let Σ be the set of all finite nonempty subsets of Y. Taking Σ with the discrete topology, let $C(\Sigma)$ be the Banach space of all bounded (continuous) real-valued functions on Σ equipped with the sup norm. We first embed Y isometrically into $C(\Sigma)$.

Choose a point $p \in Y$, and for each $y \in Y$ let $f_y : \Sigma \to \boldsymbol{R}$ be the function $f_y(\xi) = d(y,\xi) - d(p,\xi)$. Then $f_y \in C(\Sigma)$ because

$$|f_y(\xi)| = |d(y,\xi) - d(p,\xi)| \leq d(y,p)$$

([1]) This proof is due to Toruńczyk [1972].

shows that it is bounded. The map $\lambda : Y \to C(\Sigma)$ given by $y \mapsto f_y$ is an isometric embedding because

$$\|f_y - f_z\| = \sup_{\xi} |d(y, \xi) - d(z, \xi)| \leq d(y, z)$$

and the sup is attained with $\xi = \{z\} \in \Sigma$. Thus $\|\lambda(y) - \lambda(z)\| = d(y, z)$ and λ is isometric.

Now we observe that (a) $f_p(\xi) \equiv 0$; (b) for each $y \in Y$, we have $f_y(\xi) = 0$ whenever $\xi \supset \{y, p\}$.

In particular, $\lambda(Y)$ contains the origin of $C(\Sigma)$. Let L be the linear span of $\lambda(Y)$ in $C(\Sigma)$; clearly, L is a normed linear space which need not be closed in $C(\Sigma)$; but $\lambda(Y) \subset L$, and we now show that $\lambda(Y)$ is closed in L.

Let $g \in L - \lambda(Y)$; then $g = \sum_{i=1}^{n} \alpha_i f_{y_i}$ for suitable real α_i and $y_i \in Y$. Because $g \notin \lambda(Y)$, we have $g \neq f_{y_i}$ for $i = 1, \ldots, n$; to show that $\lambda(Y)$ is closed in L, it is sufficient to show that $B(g, \delta) \cap \lambda(Y) = \emptyset$ for some $\delta > 0$. Let $\delta > 0$ be smaller than

$$\min\left\{\tfrac{1}{2}\|g\|, \tfrac{1}{2}\|g - f_{y_1}\|, \ldots, \tfrac{1}{2}\|g - f_{y_n}\|\right\}$$

and assume $\|g - f_y\| < \delta$ for some $f_y \in \lambda(Y)$; then for this f_y we would have $\|f_y - f_{y_i}\| \geq \delta$ and $\|f_y\| = \|f_y - f_p\| \geq \delta$; therefore, because λ is isometric, $d(y, y_i) \geq \delta$ and $d(y, p) \geq \delta$. But $\|g - f_y\| \geq |g(\xi) - f_y(\xi)|$ for every $\xi \in \Sigma$; in particular, for $\xi = \{y_1, \ldots, y_n, p\}$ we have $g(\xi) = 0$ by (b), so

$$\|g - f_y\| \geq |f_y(\xi)| = d(y, \{y_1, \ldots, y_n, p\}) \geq \delta,$$

contradicting the assumption that $\|g - f_y\| < \delta$. Thus $B(g, \delta) \cap \lambda(Y) = \emptyset$, and since $g \in L - \lambda(Y)$ was arbitrary, the proof is complete. □

Set-valued maps

Let X, Y be topological spaces. A map $S : X \to 2^Y$ is called a *set-valued map*; the sets Sx are the *values* of S. The *image* of an $A \subset X$ is $S(A) = \bigcup\{Sx \mid x \in A\}$, and the *graph* of S is $G_S = \{(x, y) \in X \times Y \mid y \in Sx\}$. The *inverse* $S^{-1} : Y \to 2^X$ and the *dual* $S^* : Y \to 2^X$ of S are the maps $y \mapsto S^{-1}y = \{x \in X \mid y \in Sx\}$ and $y \mapsto S^*y = X - S^{-1}y$. The values of S^{-1} (respectively of S^*) are called the *fibers* (respectively the *cofibers*) of S.

Note that S is *surjective* (i.e., $S(X) = Y$) if and only if its fibers $S^{-1}y$ are all nonempty. By a *fixed point* of a set-valued map $S : X \to 2^Y$ is meant a point $x_0 \in X$ for which $x_0 \in Sx_0$. Clearly, if S has a fixed point, then so does S^{-1}

The *union* $S \cup T$ (respectively *intersection* $S \cap T$) of maps $S, T : X \to 2^Y$ is the map $x \mapsto Sx \cup Tx$ (respectively $x \mapsto Sx \cap Tx$). The *composition* of $S : X \to 2^Y$ and $G : Y \to 2^Z$ is the map $x \mapsto G(Sx)$. An $S : X \to 2^Y$ is *point-compact* if each Sx is compact.

A map $S : X \to 2^Y$ is *upper semicontinuous* (written u.s.c.) if $\{x \in X \mid Sx \subset U\}$ is open in X for each open $U \subset Y$; it is *lower semicontinuous* (written l.s.c.) if $\{x \in X \mid Sx \subset B\}$ is closed in X for each closed $B \subset Y$; it is *continuous* if it is both u.s.c. and l.s.c.

The following results are standard:

(a) A point-compact u.s.c. map has a closed graph.
(b) The point-compact u.s.c. image of a compact set is compact.
(c) The union of finitely many, intersection of any family, composition of any two point-compact u.s.c. maps is point-compact u.s.c.
(d) If $S : X \to 2^Y$ is point-compact u.s.c., and $T : X \to 2^Y$ is any map with a closed graph, then $S \cap T$ is point-compact u.s.c.
(e) Let $S : X \to 2^Y$ be point-compact u.s.c., let $\{x_\alpha\}$ be a net in X, and let $y_\alpha \in Sx_\alpha$ for each α. If $x_\alpha \to x_0$ and $y_\alpha \to y_0$, then $y_0 \in Sx_0$.

If $S : X \to 2^Y$ is any set-valued map, then a function $\varphi : X \to Y$ satisfying $\varphi(x) \in Sx$ for each $x \in X$ is called a *selection* for S.

(B.14) THEOREM (Michael). *Let X be paracompact, E a Banach space, and $S : X \to 2^E$ a map with each Sx a nonempty closed convex set. If S is lower semicontinuous, then S has a continuous selection $\varphi : X \to E$.*

C. Linear Topological Spaces

Vector spaces

We will use $\boldsymbol{K}$ to denote either the field $\boldsymbol{R}$ of real numbers or the field $\boldsymbol{C}$ of complex numbers. A *vector space* E over the field $\boldsymbol{K}$ is an abelian group E together with a map $\boldsymbol{K} \times E \to E$ (in which the image of (λ, x) is written λx) satisfying

$$\begin{aligned} \lambda(x+y) &= \lambda x + \lambda y, & \lambda(\mu x) &= (\lambda\mu)x, \\ (\lambda+\mu)x &= \lambda x + \mu x, & 1x &= x, \end{aligned}$$

for all $x, y \in E$ and $\lambda, \mu \in \boldsymbol{K}$. The field $\boldsymbol{K}$ is called the *scalar field* of E, and the map $\boldsymbol{K} \times E \to E$ the *scalar multiplication*; $\boldsymbol{K}$ is itself a vector space over $\boldsymbol{K}$ if the scalar multiplication is defined to be the multiplication in $\boldsymbol{K}$. We say E is a *real* (respectively *complex*) vector space if $\boldsymbol{K} = \boldsymbol{R}$ (respectively $\boldsymbol{K} = \boldsymbol{C}$).

A subgroup $M \subset E$ is called a *linear subspace* if $\lambda x \in M$ for all $\lambda \in \boldsymbol{K}$ and $x \in M$; the quotient group, with scalar multiplication defined by setting $\lambda(x+M) = \lambda x + M$ for each coset, is also a vector space over $\boldsymbol{K}$, called the *quotient space* E/M.

If a, b are elements of E, the set $\{x \mid x = (1-\lambda)a + \lambda b,\ 0 \le \lambda \le 1\}$ is called the *line segment* joining a to b; a subset $C \subset E$ is *convex* if for

each pair $a, b \in C$, the line segment joining them lies in C. For any subset $A \subset E$, the intersection of all convex sets containing A is called the *convex hull* $\operatorname{conv}(A)$ of A; it is the "smallest" convex set containing A and can be described directly in terms of A as

$$\operatorname{conv}(A) = \left\{ y \;\middle|\; y = \sum_{i=1}^{n} \lambda_i a_i;\ a_i \in A,\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1;\ n \text{ arbitrary} \right\}.$$

The intersection of all linear subspaces containing A is called the *linear span* of A and is denoted by $\operatorname{span}(A)$; it is the smallest linear subspace containing A and can be described as

$$\operatorname{span}(A) = \left\{ y \;\middle|\; y = \sum_{i=1}^{n} \lambda_i a_i;\ \lambda_i \in \boldsymbol{K},\ a_i \in A;\ n \text{ arbitrary} \right\}.$$

A *flat* (or *linear variety*) in E is a subset of the form $L + a$, where L is a linear subspace of E and $a \in E$. A *wedge* in E is a convex set P such that $\lambda P \subset P$ for each real $\lambda \geq 0$. A *cone* in E is a wedge C such that $x \in C$, $x \neq 0$ implies $-x \notin C$.

A finite set $\{a_1, \ldots, a_n\}$ in E is *linearly independent* if the equation

$$\lambda_1 a_1 + \cdots + \lambda_n a_n = 0$$

is satisfied only by $\lambda_1 = \cdots = \lambda_n = 0$; an arbitrary subset $S \subset E$ is called linearly independent if each of its finite subsets is linearly independent. A maximal linearly independent set $S \subset E$ is called a *basis*; each element of E can then be written uniquely as a finite sum $x = \sum_{i=1}^{s} \lambda_i s_i$, where $\lambda_i \in \boldsymbol{K}$ and $s_i \in S$. Every vector space has a basis, and all bases have the same cardinality; if this is a finite cardinality n, then we call E an *n-dimensional vector space* over $\boldsymbol{K}$ and write $\dim E = n$; otherwise, we write $\dim E = \infty$. If $A \subset E$ is a linear subspace, then any basis for A can be enlarged to a basis for E.

Two linear subspaces M, N of E are *complementary* if each element of E can be written uniquely as $z = x + y$, where $x \in M$ and $y \in N$; we then write $E = M \oplus N$ and call E the *direct sum* of M and N. Every linear subspace $M \subset E$ has at least one complementary subspace, and all these subspaces have the same dimension, called the *codimension* of M in E. If $\dim E < \infty$, then $\dim E = \dim M + \operatorname{codim} M$.

Let E, F be two vector spaces over the same field $\boldsymbol{K}$. A map $T : E \to F$ is called *linear* (or a *linear operator*) if $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for all $x, y \in E$ and $\lambda, \mu \in \boldsymbol{K}$. Regarding the scalar field as a vector space, a linear map $f : E \to \boldsymbol{K}$ is called a *linear functional*. If M is a linear subspace of E, then the map $E \to E/M$ given by $x \mapsto x + M$ is linear. For any linear operator $T : E \to F$, its *kernel* $\operatorname{Ker} T = \{x \mid Tx = 0\} \subset E$ and its *image* $\operatorname{Im} T = \{Tx \mid x \in E\} \subset F$ are linear subspaces, and T is injective

if and only if $\operatorname{Ker} T = 0$. We call a linear operator $T : E \to F$ *monic* (or a *monomorphism*) if it is injective, *epic* (or an *epimorphism*) if it is surjective, and an *isomorphism* if it is bijective; the inverse of a bijective linear operator is necessarily linear.

A sequence

$$\cdots \to E_i \xrightarrow{T_i} E_{i+1} \xrightarrow{T_{i+1}} E_{i+2} \to \cdots$$

of linear spaces and linear operators is *exact* if the image of each operator is precisely the kernel of the next operator. Because every short exact sequence $0 \to E_1 \xrightarrow{T_1} E_2 \xrightarrow{T_2} E_3 \to 0$ of vector spaces and linear operators splits, it follows that $E_2 \cong E_1 \oplus E_3$; as in the case of abelian groups, since any linear operator $T : E \to F$ gives rise to a short exact sequence $0 \to \operatorname{Ker} T \to E \to \operatorname{Im} T \to 0$, we have $E/\operatorname{Ker} T \cong \operatorname{Im} T$.

Linear topological spaces

A *linear topological space* is a vector space E over $\boldsymbol{K}$ equipped with a Hausdorff topology such that the mappings $(x, y) \mapsto x + y$ of $E \times E$ to E and $(\lambda, x) \mapsto \lambda x$ of $\boldsymbol{K} \times E$ to E (with the usual topology on $\boldsymbol{K}$) are continuous. In this book, by a linear topological space we mean a linear topological space over $\boldsymbol{R}$; in some places where complex spaces appear this is stated explicitly.

Some useful general properties of linear topological spaces are:

(i) A linear topological space is always completely regular.

(ii) (F. Riesz) A linear topological space is locally compact if and only if it is finite-dimensional.

(iii) (Tychonoff) If $\dim E = n < \infty$, then E is linearly homeomorphic to $\boldsymbol{R}^n$ (a linear homeomorphism is established by choosing a basis $\{x_1, \ldots, x_n\} \subset E$ and setting $\sum_{i=1}^n \lambda_i x_i \mapsto (\lambda_1, \ldots, \lambda_n) \in \boldsymbol{R}^n$).

Minkowski functionals. Seminorms

Let E be a linear topological space. A set $A \subset E$ is:

(i) *balanced* (or *circled*) if $\lambda A \equiv \{\lambda x \mid x \in A\} \subset A$ for all $|\lambda| \leq 1$,

(ii) *absorbing* if for each $x \in E$ there is an $r > 0$ with $x \in \lambda A$ for all $|\lambda| \geq r$,

(iii) *bounded* if for each neighborhood V of 0 there is a real $\lambda = \lambda(V)$ such that $A \subset \lambda V$

Any neighborhood V of 0 contains a balanced neighborhood of 0; and if V is also convex, then V contains a balanced convex neighborhood of 0.

Let K be any convex, balanced, absorbing set in E. The function $p_K : E \to \boldsymbol{R}$ defined by

$$p_K(x) = \inf\{\lambda > 0 \mid x \in \lambda K\}$$

is called the *Minkowski functional* of K, and has the properties:

(a) $\quad p_K(x+y) \leq p_K(x) + p_K(y) \quad$ for all $x, y \in E$,

(b) $\quad p_K(\lambda x) = |\lambda| p_K(x) \quad$ for all $x \in E$ and $\lambda \in \boldsymbol{R}$.

Moreover, p_K is continuous if and only if 0 belongs to the interior of K. Conversely, let $p : E \to \boldsymbol{R}$ be any function with properties (a) and (b) and let $W = \{x \mid p(x) < 1\}$; then W is convex, balanced, and absorbing, and p is the Minkowski functional of W.

A function $p : E \to \boldsymbol{R}$ satisfying (a) and (b) is called a *seminorm* on E; if (b) holds only for all $\lambda \geq 0$, then p is called a *sublinear functional*. If $p : E \to \boldsymbol{R}$ is a seminorm, then the set $\{x \mid p(x) = 0\}$ is a linear subspace and if it consists only of 0, the function p is called a *norm*. The following result is frequently useful: If $p : E \to \boldsymbol{R}$ is a seminorm and $\{x \mid p(x) < 1\}$ is bounded, then p is in fact a norm.

(C.1) (Hahn-Banach principle) *Let $p : E \to \boldsymbol{R}$ be a sublinear functional on the linear space E, let M be a subspace of E, and let $f : M \to \boldsymbol{R}$ be a linear functional such that $f(x) \leq p(x)$ on M. Then f extends to a linear functional $F : E \to \boldsymbol{R}$ such that $F(x) \leq p(x)$ on E.*

Normed linear spaces

If $p : E \to \boldsymbol{R}$ is a norm, then the map $d : E \times E \to \boldsymbol{R}$ defined by $d(x, y) = p(x - y)$ is a metric on E; the metric topology on E determined by d is called the topology determined by the norm p. A *normed linear space* is a linear topological space whose topology is determined by a norm. Two norms $p, q : E \to \boldsymbol{R}$ determine the same topology if and only if there are real numbers a, b such that $p(x) \leq aq(x) \leq bp(x)$ for all $x \in E$. A norm used to determine the topology of a normed linear space is denoted by $\|x\|$.

(i) A linear operator $T : E \to F$ between normed linear spaces is continuous either at every point of E or at no point of E. It is continuous on E if and only if there is a constant M such that $\|Tx\| \leq M\|x\|$ for every $x \in E$.

(ii) (Hahn Banach) If E is a normed linear space and $x_0 \neq 0$, then there exists a continuous linear functional $f : E \to \boldsymbol{R}$ with $\|f(x)\| \leq \|x\|$ for all $x \in E$ and $f(x_0) = \|x_0\|$.

Locally convex spaces

A linear topological space is *locally convex* if every neighborhood of 0 contains a convex neighborhood of 0. Every locally convex topology on a vector space is determined by some family $\{p_\alpha \mid \alpha \in \mathscr{A}\}$ of seminorms having the property that $p_\alpha(x) = 0$ for all $\alpha \in \mathscr{A}$ if and only if $x = 0$; in the topology determined by such a family of seminorms, a set V is open if and only if for

each $v \in V$ there exists some $\varepsilon > 0$ and finitely many $\alpha_1, \dots, \alpha_n \in \mathscr{A}$ such that $\bigcap_{i=1}^n \{x \mid p_{\alpha_i}(x - v) < \varepsilon\} \subset V$.

Let A be a subset of a locally convex space E. A point $x \in A$ is an *extreme point* of A if it is not contained in the interior of any line segment having its endpoints in A. The *convex closure* $\operatorname{Conv} A$ of A is the smallest closed convex subset containing A.

(C.2) THEOREM (Krein–Milman).
(a) *If A is a compact convex subset of a locally convex space E, then A is the convex closure of its set of extreme points.*
(b) *If B is any compact subset of E, and if* $\operatorname{Conv} B$ *is also compact, then all the extreme points of* $\operatorname{Conv} B$ *belong to B.*

The *weak topology* in a locally convex space E is the smallest topology for which all the linear functionals $f : E \to \boldsymbol{R}$ that are continuous in the original topology remain continuous. The weak topology, which is in general smaller than the original one, also makes E into a locally convex linear space. We speak of weakly open sets, weak compactness, etc. when referring to the weak topology.

(C.3) THEOREM (Mazur). *In a locally convex space E, a convex set $C \subset E$ is weakly closed if and only if it is closed.*

Banach spaces

A normed linear space is called a *Banach space* if the metric determined by its norm is complete. The Hausdorff completion of any normed linear space is a Banach space; in particular, every normed linear space can be mapped by a linear isometry onto a dense subset of a Banach space.

If E, F are linear topological spaces, then the set of all continuous linear operators from E to F, with the operations

$$(S+T)(x) = S(x) + T(x) \quad \text{and} \quad (\lambda T)(x) = \lambda T(x),$$

forms a linear space $\mathscr{L}(E, F)$. If both E and F are normed linear spaces, then $\mathscr{L}(E, F)$ is normed by setting $\|T\| = \sup\{\|T(x)\| \mid \|x\| = 1\}$, and if F is a Banach space, this norm makes $\mathscr{L}(E, F)$ into a Banach space.

(C.4) THEOREM (Mazur). *Let E be a Banach space and $A \subset E$ a relatively compact subset. Then its convex closure* $\operatorname{Conv} A$ *is compact.*

(C.5) THEOREM. *Let E, F be Banach spaces and $T \in \mathscr{L}(E, F)$.*
(a) (Banach) *If T is bijective, then T^{-1} is continuous.*
(b) (Schauder) *If T is surjective, then T is an open mapping.*

(C.6) (Closed graph theorem) *Let E, F be Banach spaces. A linear operator $T : E \to F$ is continuous if and only if its graph is closed.*

(C.7) THEOREM (Banach Steinhaus). *Let E be a Banach space, let F be a normed linear space, and let $\{T_n : E \to F\}$ be a sequence of continuous linear operators.*

(a) *If* $\sup_n \|T_n(x)\| < \infty$ *for each $x \in E$, then* $\sup_n \|T_n\| < \infty$.

(b) *If* $\lim_n T_n(x)$ *exists for each $x \in E$, then $T : E \to F$ defined by* $T(x) = \lim_n T_n(x)$ *is a continuous linear operator.*

Let E, F be Banach spaces, and $U \subset E$ open. A map $f : U \to F$ is *differentiable* at $x \in U$ if there is an operator $T_x \in \mathscr{L}(E, F)$ such that

$$f(y) - f(x) = T_x(y - x) + R(x, y),$$

where

$$\lim_{\|y-x\|\to 0} \frac{\|R(x, y)\|}{\|y - x\|} = 0.$$

The linear operator T_x is called the *derivative* of f at x, and is written $Df(x)$; if $Df(x)$ exists, then f is continuous at x. We say that f is of *class* C^1 on U if $Df(z)$ exists for each $z \in U$ and the map $z \mapsto Df(z)$ of U into $\mathscr{L}(E, F)$ is continuous. If U is convex, f is C^1 on U, and $\|Df(x)\| \leq K$ for all $x \in U$, then

$$\|f(y) - f(z)\| \leq K\|y - z\| \quad \text{for all } y, z \in U$$

(the mean value theorem).

Weak topology in Banach spaces. Dual spaces. Reflexive spaces

Let E be a Banach space. A sequence $\{x_n\}$ in E converges weakly to an element $x \in E$ (i.e., converges in the weak topology) if $f(x_n) \to f(x)$ for every norm continuous linear functional; in particular, norm convergence implies weak convergence.

(C.8) THEOREM. *Let E be a Banach space.*

(a) *If the sequence $\{x_n\}$ converges weakly to x, then* $\sup\{\|x_n\|\} < \infty$.

(b) (Eberlein) *A subset of E is weakly compact if and only if it is weakly sequentially compact.*

(c) (Krein–Šmulian) *If $A \subset E$ is weakly compact, then the weak closure of* conv A *is also weakly compact.*

Let E be a Banach space. The Banach space $\mathscr{L}(E, \boldsymbol{R})$ of all continuous linear functionals is called the *dual space* (or *conjugate space*) of E, and is denoted by E^*. For $f \in E^*$ and $x \in E$, we write $\langle f, x\rangle$ instead of $f(x)$; observe that the map $E^* \times E \to \boldsymbol{R}$ given by $(f, x) \mapsto \langle f, x\rangle$ is bilinear, and that $|\langle f, x\rangle| \leq \|f\| \cdot \|x\|$.

The dual of E^* is called the *second dual* of E. Corresponding to each $x \in E$ there is a unique $F_x \in E^{**} = \mathscr{L}(E^*, \boldsymbol{R})$ defined by $F_x(f) = \langle f, x\rangle$;

the map $J : E \to E^{**}$ given by $x \mapsto F_x$ is an isometric embedding of E into E^{**}, called the *canonical embedding*. A Banach space is *reflexive* if the canonical embedding is surjective.

(C.9) THEOREM.
(a) *If E is reflexive, then E^* is also reflexive.*
(b) (Mazur–Šmulian) *A Banach space E is reflexive if and only if every closed bounded convex subset of E is weakly compact.*
(c) (Eberlein) *A Banach space E is reflexive if and only if each bounded sequence in E contains a weakly convergent subsequence.*

For the dual E^* of a Banach space E, there is another weak topology, called the *weak-star topology*: this has as subbase of neighborhoods of 0 the family

$$\{W_x \mid x \in E\}, \quad \text{where} \quad W_x = \{f \in E^* \mid \langle f, x\rangle < 1\}.$$

For reflexive spaces, the weak and weak-star topologies coincide.

(C.10) THEOREM (Banach–Alaoglu). *Let E be a Banach space. Then the closed unit ball of E^* is compact in the weak-star topology.*

Riesz–Schauder theory

Let E, F be normed linear spaces. An operator $T \in \mathscr{L}(E,F)$ is called *Fredholm* if $\operatorname{Im} T$ is closed in F and both $\operatorname{Ker} T$ and $\operatorname{Coker} T = F/\operatorname{Im} T$ are of finite dimension. We call

$$\operatorname{Ind} T = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T$$

the (*Fredholm*) *index* of T. The subset of $\mathscr{L}(E,F)$ of Fredholm operators of index m is denoted by $\Phi_m(E,F)$.

An operator $T \in \mathscr{L}(E,F)$ is called *completely continuous* ([1]) if T sends bounded sets in E to relatively compact sets in F The set of all such operators is denoted by $\mathscr{K}(E,F)$; when $E = F$ we write $\mathscr{K}(E) = \mathscr{K}(E,E)$.

Some elementary but frequently used properties of completely continuous operators are:

(i) A completely continuous operator is continuous.
(ii) $\mathscr{K}(E,F)$ is a linear subspace of $\mathscr{L}(E,F)$.
(iii) If F is a Banach space, then $\mathscr{K}(E,F)$ is closed in $\mathscr{L}(E,F)$.
(iv) If E is a Banach space and $T \in \mathscr{K}(E)$, then $I - T \in \Phi_0(E,E)$.

Let E be a real Banach space and $T \in \mathscr{K}(E)$. We call

$$r(T) = \{\mu \in \boldsymbol{R} \mid \operatorname{Ker}(I - \mu T) \neq \{0\}\}$$

([1]) Note that such operators are now often called "compact", which is in conflict with our terminology of §6.

the set of *characteristic values* of T. It is clear that $0 \notin r(T)$ and $\mu \in r(T) \Leftrightarrow \mu^{-1}$ is an eigenvalue of T.

(C.11) (Riesz Schauder) *Let E be a real Banach space and $T \in \mathscr{K}(E)$.*

1° *Any closed interval $[a,b] \subset \boldsymbol{R}$ contains only a finite number of characteristic values of T.*

2° *If $\mu \notin r(T)$, then $I - \mu T$ is invertible.*

3° *If $\mu \in r(T)$, then:*

(a) *there exists the smallest $s \in \boldsymbol{N}$ (called the exponent of μ) such that*

$$N = \operatorname{Ker}(I - \mu T)^s = \operatorname{Ker}(I - \mu T)^{s+k} \quad \text{for } k \geq 1,$$
$$M = \operatorname{Im}(I - \mu T)^s = \operatorname{Im}(I - \mu T)^{s+k} \quad \text{for } k \geq 1,$$

(b) *N and M are closed subspaces of E invariant under T,*

(c) *the subspace $N = \bigcup_{l=1}^{\infty} \operatorname{Ker}(I - \mu T)^l$ is of finite dimension; the integer $m_\mu = \dim N$ is called the algebraic multiplicity of μ.*

Hilbert spaces

Denote the complex conjugate of $z \in \boldsymbol{C}$ by $\bar{z}$. Let E be a complex Banach space. A map $E \times E \to \boldsymbol{C}$ (the image of the pair (x,y) being denoted simply by (x,y)) is called an *inner product* on E if:

(a) $(x, \lambda y + \mu z) = \bar{\lambda}(x,y) + \bar{\mu}(x,z)$ for all $x, y \in E$ and $\lambda, \mu \in \boldsymbol{C}$,

(b) $(x,y) = \overline{(y,x)}$ for all $x, y \in E$,

(c) $(x,x) \geq 0$ and $(x,x) \neq 0$ if $x \neq 0$.

An inner product in a real Banach space is a real-valued map $E \times E \to \boldsymbol{R}$ having the properties (a)–(c) (so that the bars can be omitted).

It follows that $\|x\| = \sqrt{(x,x)}$ is a norm. A real (or complex) Banach space with norm obtained from an inner product is called a *Hilbert space*. We list their main properties:

(i) A norm in a (real or complex) Banach space E is derivable from an inner product if and only if it satisfies the parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(ii) The inner product on a Hilbert space H is continuous on $H \times H$, and $|(x,y)| \leq \|x\| \cdot \|y\|$ for all $x, y \in H$.

(iii) (F. Riesz) Given any continuous linear functional $f : H \to \boldsymbol{K}$ on the Hilbert space H, there exists a unique $y \in H$ such that $f(x) = (x,y)$ for all $x \in H$.

(iv) Every Hilbert space is reflexive.

Let H be a Hilbert space. A subset $A \subset H$ is called *orthogonal* if $(x,y) = 0$ for all $x, y \in A$ with $x \neq y$; if also $\|x\| = 1$ for each $x \in A$, then A is called *orthonormal*.

Some function spaces

Let Ω be an open bounded subset of $\boldsymbol{R}^n$. If $s = (s_1, \dots, s_n)$ is an n-tuple of nonnegative integers, we write $D^s = D_1^{s_1} \dots D_n^{s_n}$, where $D_j = \partial/\partial x_j$. Then D^s is a differential operator of order $|s| = \sum_{i=1}^n s_i$.

1. *The space* $C(\overline{\Omega})$. The space of continuous real-valued functions on $\overline{\Omega}$ denoted by $C(\overline{\Omega})$ is a Banach space under the norm $\|u\|_0 = \sup_{x\in\overline{\Omega}} |u(x)|$. A subset $K \subset C(\overline{\Omega})$ is *equicontinuous* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x_1 - x_2\| < \delta$ implies $|u(x_1) - u(x_2)| < \varepsilon$ for every $x_1, x_2 \in \overline{\Omega}$ and every $u \in K$. A subset of $C(\overline{\Omega})$ is relatively compact if and only if it is bounded and equicontinuous (Arzelà–Ascoli theorem).

2. *The spaces* $L^p(\Omega)$. Let $1 \le p < \infty$ and denote by $L^p(\Omega)$ the class of measurable real-valued functions u on Ω for which $\int_\Omega |u(x)|^p\, dx < \infty$. We identify in $L^p(\Omega)$ functions that are equal almost everywhere on Ω. Then $L^p(\Omega)$ is a Banach space under the norm

$$\|u\|_{L^p} = \left\{ \int_\Omega |u(x)|^p\, dx \right\}^{1/p}$$

A set $K \subset L^p(\Omega)$ is relatively compact if and only if it is bounded in L^p norm and $\int_\Omega |u(x+h) - u(x)|^p\, dx \to 0$ as $h \to 0$ uniformly for $u \in K$ (M. Riesz). The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int_\Omega u(x)v(x)\, dx.$$

3. *The spaces* $C^k(\overline{\Omega})$. Given an integer $k \ge 1$ let $C^k(\overline{\Omega})$ be the set of all real-valued functions u defined in $\overline{\Omega}$ whose partial derivatives of order $\le k$ all exist and are continuous ([1]). The class $C^k(\overline{\Omega})$ is a Banach space under one of the equivalent norms

$$\|u\|_k = \sum_{|s|\le k} \sup_{x\in\overline{\Omega}} |D^s u(x)| = \sum_{|s|\le k} \|D^s u\|_0,$$

$$|u|_k = \max\{\|D^s u\|_0 \mid |s| \le k\}.$$

The embedding $(C^{k+1}(\overline{\Omega}), \|\ \|_{k+1}) \to (C^k(\overline{\Omega}), \|\ \|_k)$ is completely continuous.

4. *The Hölder spaces* $C^{k+\alpha}(\overline{\Omega})$. A function $u : \overline{\Omega} \to \boldsymbol{R}$ is Hölder continuous with exponent α $(0 < \alpha < 1)$ provided we have

$$\|u\|_\alpha = \|u\|_0 + \sup_{\substack{x,y\in\overline{\Omega}\\ x\ne y}} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha} < \infty.$$

([1]) We assume that each $u \in C^k(\overline{\Omega})$ admits a C^k extension over some nbd of $\overline{\Omega}$ in $\boldsymbol{R}^n$.

For an integer $k \geq 1$ and $0 < \alpha < 1$, let $C^{k+\alpha}(\overline{\Omega})$ be the set of all functions u that together with their derivatives up to order k are Hölder continuous with exponent α. The class $C^{k+\alpha}(\overline{\Omega})$ is a Banach space under one of the equivalent norms

$$\|u\|_{k+\alpha} = \sum_{|s|\leq k} \|D^s u\|_\alpha, \qquad |u|_{k+\alpha} = \max\{\|D^s u\|_\alpha \mid |s| \leq k\}.$$

The embedding $(C^{k+\alpha}(\overline{\Omega}), \| \ \|_{k+\alpha}) \to (C^{n+\beta}(\overline{\Omega}), \| \ \|_{n+\beta})$ is completely continuous if $k + \alpha > n + \beta$ (Schauder).

5. *The Sobolev spaces* $W^{k,p}(\Omega)$. Denote by $C^k(\Omega)$ the set of all functions u defined in Ω whose partial derivatives of order $\leq k$ all exist and are continuous in Ω. For $u \in C^k(\Omega)$ we let

$$\|u\|_{k,p} = \Big\{ \sum_{|s|\leq k} \|D^s u\|^p_{L^p} \Big\}^{1/p} = \Big\{ \sum_{|s|\leq k} \int_\Omega |D^s u(x)|^p \, dx \Big\}^{1/p}$$

and set

$$\widehat{C}^{k,p}(\Omega) = \{u \in C^k(\Omega) \mid \|u\|_{k,p} < \infty\}.$$

We denote by $W^{k,p}(\Omega)$ the completion of $\widehat{C}^k(\Omega)$ with respect to the norm $\| \ \|_{k,p}$. Note that $\{u_m\}$ is a Cauchy sequence in $\widehat{C}^{k,p}(\Omega)$ if and only if

$$\|D^s u_m - D^s u_k\|_p \to 0 \quad \text{as } m, k \to \infty \ (0 \leq |s| \leq k).$$

Because $L^p(\Omega)$ is complete, there exist functions $u^s \in L^p(\Omega)$ such that $\|D^s u_m - u^s\|_{L^p} \to 0$. We can assign to $u^0 \in W^{k,p}(\Omega)$ the functions u^s $(0 < |s| \leq k)$ and call these functions the "derivatives" of u^0. One verifies that $W^{k,2}(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int_\Omega \sum_{0\leq|s|\leq k} D^s u(x) D^s v(x) \, dx, \qquad u, v \in W^{k,2}(\Omega).$$

D. Algebraic Preliminaries

A set G together with a map $(x, y) \mapsto x + y$ from $G \times G$ to G is called an *abelian group* whenever:

(i) $x + (y + z) = (x + y) + z$ for all $x, y, z \in G$,
(ii) $x + y = y + x$ for all $x, y \in G$,
(iii) there exists an element $0 \in G$ such that $x + 0 = x$ for all $x \in G$,
(iv) for each $x \in G$ there is an $x' \in G$ such that $x + x' = 0$.

The element 0 is unique, as also is the inverse x' of each x; x' is written $-x$, and $x+(-y)$ is written $x-y$. A set $A \subset G$ is called a *subgroup* of G if $x-y \in A$ for all $x, y \in A$; the set $G/A = \{z+A \mid z \in G\}$, with addition defined on the cosets $z+A = \{z+a \mid a \in A\}$ by setting $(x+A)+(y+A) = (x+y)+A$, is also

an abelian group, called the *quotient group* G/A. If $\{G_i\}_{i\in I}$ is a collection of abelian groups, then their *direct product* $\prod G_i$ is the abelian group structure defined on the Cartesian product of $\{G_i\}$ by setting $\{x_i\}+\{y_i\} = \{x_i+y_i\}$. The subgroup $\bigoplus G_i \subset \prod G_i$, consisting of the elements $\{x_i\}$ with only a finite number of nonzero coordinates, is the *direct sum* of $\{G_i\}$.

Let G, H be two abelian groups. A map $\varphi : G \to H$ is called a *homomorphism* if $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in G$. If A is a subgroup of G, then the map $G \to G/A$ given by $x \mapsto x + A$ is a homomorphism. We call a homomorphism *monic* (or a *monomorphism*) if it is an injective map, *epic* (or an *epimorphism*) if it is surjective, and an *isomorphism* if it is bijective. If $\varphi : G \to H$ is an isomorphism, then the groups G and H are called *isomorphic* and we write $G \cong H$.

For every homomorphism $\varphi : G \to H$, its *kernel* $\operatorname{Ker}\varphi = \{x \in G \mid \varphi(x) = 0\} \subset G$ and its *image* $\operatorname{Im}\varphi = \varphi(G) \subset H$ are subgroups; we note that φ is monic if and only if $\operatorname{Ker}\varphi = \{0\}$, and it is epic if its *cokernel* $\operatorname{Coker}\varphi = H/\operatorname{Im}\varphi$ is the trivial group $\{0\}$.

Exact sequences

A sequence

$$\cdots \to G' \to G \to G'' \to \cdots$$

of abelian groups and homomorphisms is *exact* if the image of each homomorphism is precisely the kernel of the next homomorphism. For example, if $\varphi : G \to H$ is a homomorphism, then with $i : \operatorname{Ker}\varphi \hookrightarrow G$ being the inclusion and $p : H \to \operatorname{Coker}\varphi$ the quotient map, the sequence

$$0 \to \operatorname{Ker}\varphi \xrightarrow{i} G \xrightarrow{\varphi} H \xrightarrow{p} \operatorname{Coker}\varphi \to 0$$

is exact.

An exact sequence of the form

$$(*) \qquad 0 \to G' \xrightarrow{\varphi} G \xrightarrow{\psi} G'' \to 0$$

is called a *short exact sequence*. Since every homomorphism $\varphi : G \to H$ gives rise to an obvious short exact sequence $0 \to \operatorname{Ker}\varphi \to G \to \operatorname{Im}\varphi \to 0$, we have $G/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi$.

Observe that any 5 consecutive terms in a long exact sequence

$$G_1 \xrightarrow{\varphi_1} G_2 \to G_3 \to G_4 \xrightarrow{\varphi_4} G_5$$

give rise to a short exact sequence

$$0 \to \operatorname{Coker}\varphi_1 \to G_3 \to \operatorname{Ker}\varphi_4 \to 0.$$

A short exact sequence $(*)$ is said to be *split* if one of the following equivalent conditions is satisfied.

(i) φ has a *left inverse* φ', that is, $\varphi'\varphi = 1_{G'}$,
(ii) ψ has a *right inverse* ψ', that is, $\psi\psi' = 1_{G''}$,
(iii) $G \cong G' \oplus G''$.

The following behavior of exact sequences is frequently used:

(D.1) 5-LEMMA. *In the commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "v_1"] & A_2 \arrow[r] \arrow[d, "v_2"] & A_3 \arrow[r] \arrow[d, "v_3"] & A_4 \arrow[r] \arrow[d, "v_4"] & A_5 \arrow[d, "v_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

of abelian groups and homomorphisms, with exact rows, if v_1, v_2, v_4, v_5 *are all isomorphisms, then* v_3 *is also an isomorphism.*

(D.2) WHITEHEAD-BARRATT LEMMA. *If the commutative diagram of abelian groups and homomorphisms*

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "\alpha_i"] & B_i \arrow[r, "g_i"] \arrow[d, "\beta_i"] & C_i \arrow[r, "h_i"] \arrow[d, "\gamma_i", "\cong"'] & A_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\alpha_{i+1}"] & B_{i+1} \arrow[r] \arrow[d, "\beta_{i+1}"] & \cdots \\
\cdots \arrow[r] & \hat{A}_i \arrow[r, "\hat{f}_i"] & \hat{B}_i \arrow[r, "\hat{g}_i"] & \hat{C}_i \arrow[r, "\hat{h}_i"] & \hat{A}_{i+1} \arrow[r, "\hat{f}_{i+1}"] & \hat{B}_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

has exact rows and the γ_i *are isomorphisms, then the sequence*

$$\cdots \to A_i \xrightarrow{(\alpha_i, -f_i)} \hat{A}_i \oplus B_i \xrightarrow{\hat{f}_i + \beta_i} \hat{B}_i \xrightarrow{h_i \gamma_i^{-1} \hat{g}_i} A_{i+1} \to \cdots$$

is exact.

Free abelian groups

An abelian group G is called *free* with *basis* $\{g_\alpha\}_{\alpha \in \mathcal{A}}$, where $\mathcal{A}$ is some index set, if each $g \in G$ can be uniquely written as a finite sum with integral coefficients

$$\sum_{\alpha \in \mathcal{A}} \lambda_\alpha g_\alpha \qquad (\lambda_\alpha \in Z,\ \text{almost all } \lambda_\alpha = 0).$$

Some frequently used properties of free abelian groups are:

(a) If H is an abelian group and G is free, then any map $g_\alpha \mapsto h_\alpha$ of the basis $\{g_\alpha\}$ of G into H extends to a unique homomorphism $\varphi : G \to H$ by setting $\varphi(\sum \lambda_\alpha g_\alpha) = \sum \lambda_\alpha h_\alpha$.
(b) Every abelian group is isomorphic to a quotient of a free abelian group.
(c) If F is free, then any short exact sequence $0 \to G' \to G \to F \to 0$ splits (hence $G \cong G' \oplus F$).
(d) Every subgroup of a free abelian group is free.

Let X be any set of objects. The set $F(X)$ of all formal finite linear combinations $\sum c_i x_i$ of elements of X with $c_i \in Z$ forms the *free abelian group generated by* X. Because the elements of X form a basis in $F(X)$, every map from X to a set Y induces a unique homomorphism $\varphi : F(X) \to F(Y)$.

Tensor and torsion products

Let A and C be abelian groups, and denote by $A \circ C$ the free abelian group generated by all symbols $a \circ c$, where $a \in A$, $c \in C$. Let $R \subset A \circ C$ be the subgroup generated by all elements $(a + a') \circ c - a \circ c - a' \circ c$ and $a \circ (c + c') - a \circ c - a \circ c'$. The quotient group $A \circ C/R$ is called the *tensor product* of A and C, and is denoted by $A \otimes C$; the symbol $a \otimes c$ represents the coset $a \circ c + R$. Thus, $A \otimes C$ is a group in which an element of A can be "multiplied" by an element of C, the "multiplication" being distributive.

Let G be an abelian group, and $f : A \times C \to G$ a mapping of the Cartesian product of A and C into G such that $f(a + a', c) = f(a, c) + f(a', c)$ and $f(a, c + c') = f(a, c) + f(a, c')$ for all a, a', c, c'. Then there is a unique homomorphism $\omega : A \otimes C \to G$ such that $\omega(a \otimes c) = f(a, c)$: for, since $A \circ C$ is free abelian, there is a unique $\omega' : A \circ C \to G$ with $\omega'(a \circ c) = f(a, c)$, and by the linearity of f in each variable, $\omega'(R) = 0$, so we can pass to the quotient. Thus, a homomorphism of $A \otimes C$ into any group G is defined uniquely by giving the images of the symbols $a \otimes c$ provided that these images are additive in a and in c.

We now list some properties of the tensor product:

(a) If $a = 0$ or $c = 0$, then $a \otimes c = 0$; the converse need not be true.

(b) $n(a \otimes c) = (na) \otimes c = a \otimes nc$ for all $n \in Z$.

(c) $\otimes$ is commutative, associative, and

$$\Big(\bigoplus_\alpha A_\alpha\Big) \otimes C = \bigoplus_\alpha (A_\alpha \otimes C).$$

(d) $A \otimes Z \cong A$ and $Z/nZ \otimes Z/mZ \cong Z/(m,n)Z$, where (m, n) is the greatest common divisor of m and n.

(e) Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence. Then for any abelian group G, the sequence $A \otimes G \xrightarrow{\alpha \otimes 1} B \otimes G \xrightarrow{\beta \otimes 1} C \otimes G \to 0$ is also exact. Note that in contrast to α, the homomorphism $\alpha \otimes 1$ need not be monic. However, if either C or G is torsion-free, or if A is a direct summand in B, then $\alpha \otimes 1$ is also monic.

Let A be an abelian group. By a *free presentation* of A is meant a short exact sequence $0 \to R \xrightarrow{\alpha} F \xrightarrow{\beta} A \to 0$, where F (and R) are free abelian groups. This gives the exact sequence

$$R \otimes G \xrightarrow{\alpha \otimes 1} F \otimes G \xrightarrow{\beta \otimes 1} A \otimes G \to 0$$

for any abelian group G. The group $\operatorname{Ker}(\alpha \otimes 1)$ is called the *torsion product* of A and G, and is denoted by $\operatorname{Tor}(A, G)$. We list some of its properties:

(a) The group $\operatorname{Tor}(A, G)$ is independent of the particular free presentation used for A, and is functorial in A and G.

(b) $\operatorname{Tor}(A, G) \cong \operatorname{Tor}(G, A)$.

(c) $\operatorname{Tor}(\bigoplus_\alpha A_\alpha, G) \cong \bigoplus_\alpha \operatorname{Tor}(A_\alpha, G)$.

(d) $\operatorname{Tor}(A, G) = 0$ if either A or G is free, and

$$\operatorname{Tor}(Z/nZ, Z/mZ) \cong Z/(n,m)Z.$$

(e) If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is any exact sequence of abelian groups, then for any abelian group G, there is an exact sequence

$$\begin{aligned} 0 \to \operatorname{Tor}(A, G) \to \operatorname{Tor}(B, G) \to \operatorname{Tor}(C, G) \\ \to A \otimes G \xrightarrow{\alpha\otimes 1} B \otimes G \xrightarrow{\beta\otimes 1} C \otimes G \to 0. \end{aligned}$$

Graded abelian groups

A *graded abelian group* C is a family $\{C_n \mid n \in Z\}$ of abelian groups with $C_n = 0$ for $n < 0$; a *subgroup* of the graded group C is a graded group $B = \{B_n \mid n \in Z\}$, where B_n is a subgroup of C_n for each n; the *quotient* of these graded groups is the graded group $C/B = \{C_n/B_n \mid n \in Z\}$.

The *direct sum* of the graded groups $C = \{C_n \mid n \in Z\}$ and $D = \{D_n \mid n \in Z\}$ is the graded group $C \oplus D = \{C_n \oplus D_n \mid n \in Z\}$. For reasons coming from algebraic topology, the tensor product $C \otimes D$ of graded groups has a special grading: it is the graded group $T = \{T_n \mid n \in Z\}$, where

$$T_n = \bigoplus_{i=0}^{n} C_i \otimes D_{n-i}.$$

A *homomorphism* $f : C \to D$ of graded groups is a family of homomorphisms $f_n : C_n \to D_{n+r}$, one for each $n \in Z$, where r is a fixed (positive or negative) integer called the *degree* of f

Chain complexes and homology

A *chain complex* $C = \{C_n, \partial_n \mid n \in Z\}$ is a graded abelian group together with an endomorphism $\partial : C \to C$ of degree -1 such that $\partial\partial = 0$. In other words, we are given a sequence $\{\partial_n : C_n \to C_{n-1} \mid n \in Z\}$ of homomorphisms such that $\partial_n \partial_{n+1} = 0$ for all n. The map ∂ (as well as its components) is called the *differential* (or the *boundary map*); we frequently omit the index n on ∂_n. Elements of C_n are called *n-chains*, elements of $Z_n(C) = \operatorname{Ker} \partial_n$ are *n-cycles*, and elements of $B_n(C) = \operatorname{Im} \partial_{n+1}$ are *n-boundaries*.

A map $f : C \to C'$ of chain complexes (or a *chain map*) is a map of degree zero of graded abelian groups such that $f\partial = \partial f$, where the last ∂ denotes

the differential in C'. Thus a chain map is a family $\{f_n : C_n \to C'_n \mid n \in Z\}$ of homomorphisms such that $\partial_{n+1} f_{n+1} = f_n \partial_{n+1}$ for all n. Two chain maps $f, g : C \to C'$ are *chain homotopic* if there is a map $h : C \to C'$ of graded abelian groups of degree $+1$ such that $\partial h + h\partial = f - g$; the h is then called a *chain homotopy* from f to g, and we write $f \sim g$.

Let $C = \{C_n, \partial_n\}$ be a chain complex. Since for each $n \in Z$ the requirement $\partial_n \partial_{n+1} = 0$ is equivalent to $B_n(C) \subset Z_n(C)$, we can associate with C the graded abelian group

$$H_*(C) = \{H_n(C)\}, \quad \text{where} \quad H_n(C) = Z_n(C)/B_n(C) \text{ for } n \in Z.$$

The group $H_n(C)$ (respectively $H_*(C)$) is called the nth (respectively the *graded*) *homology group* of C. Every chain map $f : C \to C'$ naturally induces a map $f_* : H_*(C) \to H_*(C')$ of degree zero; thus $H_*(-)$ becomes a functor, called the *homology functor*, from the category of chain complexes to the category of graded abelian groups. We remark that if $f \sim g$, then $f_* = g_*$.

Long exact sequence

A sequence $0 \to C' \to C \to C'' \to 0$ of chain complexes and chain maps is called *exact* if the sequence $0 \to C'_n \to C_n \to C''_n \to 0$ of abelian groups is exact for each n.

(D.3) THEOREM. *If $0 \to C' \to C \to C'' \to 0$ is an exact sequence of chain complexes, then there is a long exact sequence in homology:*

$$\cdots \to H_n(C') \to H_n(C) \to H_n(C'') \xrightarrow{\partial} H_{n-1}(C') \to \cdots.$$

[The *connecting homomorphism* ∂ is constructed as follows: Let $x \in C''_n$ be a cycle with homology class $[x]$. Lift x to a $y \in C_n$; then ∂y has zero image in C''_{n-1}, hence it comes from C'_{n-1}. By construction, $\partial[x] = [\partial y] \in H_{n-1}(C')$.]

A *cochain complex* $C = \{C^n, \delta^n\}$ is a graded abelian group C together with an endomorphism $\delta : C \to C$ of degree $+1$ with $\delta\delta = 0$. Again δ is called the *codifferential* (or *coboundary operator*). Maps of cochain complexes (or *cochain maps*) are defined analogously to chain maps. For a cochain complex $C = \{C^n, \delta^n\}$ we define its *graded cohomology group* $H^*(C) = \{H^n(C)\}$ by

$$H^n(C) = \operatorname{Ker} \delta^n / \operatorname{Im} \delta^{n-1}, \qquad n \in Z.$$

With the analogous definition of induced maps, $H^*(-)$ then becomes a covariant *cohomology functor* from the category of cochain complexes to the category of graded abelian groups.

Direct systems of abelian groups and their limits

In this subsection, by "group" we understand an additive abelian group. Let $\mathcal{D}$ be a directed set and $\{G_\alpha \mid \alpha \in \mathcal{D}\}$ a family of groups. Assume that for any $\alpha \preceq \beta$ in $\mathcal{D}$, there is given a homomorphism $\pi_{\alpha\beta} : G_\alpha \to G_\beta$ such that:

(i) $\pi_{\alpha\alpha} = \mathrm{id}$,
(ii) $\pi_{\alpha\gamma} = \pi_{\beta\gamma}\pi_{\alpha\beta}$ for $\alpha \preceq \beta \preceq \gamma$.

Then the family $\{G_\alpha\} = \{G_\alpha, \pi_{\alpha\beta} \mid \alpha \in \mathcal{D}\}$ is called a *direct system of groups* over $\mathcal{D}$.

For convenience, any image of a $g_\alpha \in G_\alpha$ under any of the homomorphisms $\pi_{\alpha\beta}$ is called a *successor* of g_α. Let S be the set-theoretic union of all groups G_α, and call $g_1, g_2 \in S$ *equivalent* ($g_1 \sim g_2$) whenever they have a common successor in the system. This is indeed an equivalence relation in S, and its equivalence classes are called *bundles*.

The *direct limit group* $G^\infty = \varinjlim\{G_\alpha, \pi_{\alpha\beta}\}$ is defined as follows: The elements of G^∞ are the bundles of S, and addition of bundles $[x_1]+[x_2] = [x]$ is given by choosing a G_α in which both $[x_1]$, $[x_2]$ have representatives x_1, x_2 and letting $[x]$ be the bundle containing $x_1 + x_2$. This addition makes G^∞ an abelian group, the zero of G^∞ being the bundle containing the zeros of all G_α. Assigning to every $x_\alpha \in G_\alpha$ the bundle $[x_\alpha]$ containing it, we get a canonical homomorphism $u_\alpha : G_\alpha \to G^\infty$. These homomorphisms have the following properties:

(i) $u_\alpha = u_\beta\pi_{\alpha\beta}$ whenever $\alpha \preceq \beta$,
(ii) for any $\alpha \preceq \beta$, we have $u_\alpha(x_\alpha) = u_\beta(x_\beta)$ if and only if there is a γ such that $\alpha, \beta \preceq \gamma$ and $\pi_{\alpha\gamma}(x_\alpha) = \pi_{\beta\gamma}(x_\beta)$.

The family $u = \{u_\alpha : G_\alpha \to G^\infty \mid \alpha \in \mathcal{D}\}$ is called the *universal transformation* of $\{G_\alpha\}$ into G^∞ We call $g_\alpha \in G_\alpha$ a *representative* of $g \in G^\infty$ whenever $u_\alpha(g_\alpha) = g$.

The main properties of the pair $(G^\infty, \{u_\alpha\})$ are collected in the following

(D.4) THEOREM. *Let L be an abelian group and $\{f_\alpha : G_\alpha \to L \mid \alpha \in \mathcal{D}\}$ a family of homomorphisms such that $f_\alpha = f_\beta\pi_{\alpha\beta}$ whenever $\alpha \preceq \beta$. Then:*

(I) *there is a unique homomorphism $f : G^\infty \to L$ such that:*
(a) $fu_\alpha = f_\alpha$ *for each* $\alpha \in \mathcal{D}$,
(b) $\mathrm{Im}\, f = \bigcup \mathrm{Im}\, f_\alpha$,
(c) $\mathrm{Ker}\, f = \bigcup u_\alpha(\mathrm{Ker}\, f_\alpha)$,

(II) *f is an isomorphism if and only if the following two conditions hold:*
(i) $L = \bigcup \mathrm{Im}\, f_\alpha$,
(ii) *if $g_\alpha \in \mathrm{Ker}\, f_\alpha$, then $g_\alpha \in \mathrm{Ker}\, \pi_{\alpha\beta}$ for some $\beta \succeq \alpha$.*

(D.5) THEOREM. *Let $\{G_\alpha\}$, $\{\check{G}_\alpha\}$ be two direct systems over $\mathcal{D}$, and let $\{h_\alpha : G_\alpha \to \check{G}_\alpha \mid \alpha \in \mathcal{D}\}$ be a family of homomorphisms such that $h_\beta\pi_{\alpha\beta} = \check{\pi}_{\alpha\beta}h_\alpha$ if $\alpha \preceq \beta$. Then there exists a unique homomorphism $h^\infty : G^\infty \to \check{G}^\infty$ with the property that $h^\infty u_\alpha = \check{u}_\alpha h_\alpha$ for all $\alpha \in \mathcal{D}$.*

(D.6) THEOREM. *Let $\{C_\alpha\}$ be a direct system of chain complexes over the directed set $\mathcal{D}$. Then for each $n = 0, 1, \ldots,$*

$$\varinjlim_{\alpha} H_n(C_\alpha) = H_n(\varinjlim_{\alpha} C_\alpha).$$

(D.7) THEOREM. *Let $\{G'_\alpha\}$, $\{G_\alpha\}$, and $\{G''_\alpha\}$ be direct systems over the same directed set $\mathcal{D}$, and assume that for each $\alpha \in \mathcal{D}$ there is an exact sequence $G'_\alpha \to G_\alpha \to G''_\alpha$, where the maps commute with those defining the direct systems. Then the induced sequence*

$$\varinjlim G'_\alpha \to \varinjlim G_\alpha \to \varinjlim G''_\alpha$$

is also exact.

(D.8) THEOREM. *Let $\mathcal{N}$ and $\mathcal{L}$ be directed sets. Define an order on $\mathcal{N} \times \mathcal{L}$ by $(\alpha, k) \preceq (\beta, l)$ provided $\alpha \preceq \beta$ and $k \preceq l$. Let $G_{\alpha,k}$ be a direct system over $\mathcal{N} \times \mathcal{L}$. Then:*

(a) *the maps $G_{\alpha,k} \to \varinjlim_k G_{\alpha,k} \to \varinjlim_\alpha(\varinjlim_k G_{\alpha,k})$ induce an isomorphism*

$$\varinjlim_{\alpha,k} G_{\alpha,k} \cong \varinjlim_{\alpha}(\varinjlim_{k} G_{\alpha,k}),$$

(b) *there is a natural isomorphism*

$$\varinjlim_{\alpha}(\varinjlim_{k} G_{\alpha,k}) \cong \varinjlim_{k}(\varinjlim_{\alpha} G_{\alpha,k}).$$

Recall that $\mathcal{D}' \subset \mathcal{D}$ is *cofinal* in $\mathcal{D}$ if for any $\alpha \in \mathcal{D}$ there is a $\beta \in \mathcal{D}'$ such that $\alpha \preceq \beta$; a cofinal $\mathcal{D}' \subset \mathcal{D}$ is also a directed subset of $\mathcal{D}$. The following result permits us to compare the direct limits when the direct systems are taken over different directed sets.

(D.9) THEOREM. *If $\mathcal{D}' \subset \mathcal{D}$ is cofinal in $\mathcal{D}$, then for any direct system $\{G_\alpha \mid \alpha \in \mathcal{D}\}$ we have*

$$\varinjlim\{G_\alpha \mid \alpha \in \mathcal{D}\} \cong \varinjlim\{G_\beta \mid \beta \in \mathcal{D}'\}.$$

Inverse systems of abelian groups and their limits

Let $\mathcal{D}$ be a directed set and $\{G_\alpha \mid \alpha \in \mathcal{D}\}$ a family of groups. Assume that for any $\alpha \preceq \beta$ in $\mathcal{D}$, there is given a homomorphism $\mu_{\beta\alpha} : G_\beta \to G_\alpha$ such that:

(i) $\mu_{\alpha\alpha} = \mathrm{id}$,

(ii) $\mu_{\gamma\alpha} = \mu_{\beta\alpha}\mu_{\gamma\beta}$ for $\alpha \preceq \beta \preceq \gamma$.

Then the family $\{G_\alpha, \mu_{\beta\alpha}\}$ is called an *inverse system of groups* over $\mathcal{D}$. The *inverse limit group* $G_\infty = \varprojlim\{G_\alpha, \mu_{\beta\alpha}\}$ is defined as follows: An element $g \in G_\infty$ is a system $\{g_\alpha\}$ of elements $g_\alpha \in G_\alpha$, one for each $\alpha \in \mathcal{D}$, which

match in the sense that if $\alpha \preceq \beta$, then $g_\alpha = \mu_{\beta\alpha}(g_\beta)$; we call such a system $\{g_\alpha\}$ a *thread* and g_α is the *representative* of the thread in G_α. Addition in G_∞ is defined in the obvious manner: if $g = \{g_\alpha\}$, $g' = \{g'_\alpha\} \in G_\infty$, then $\{g_\alpha + g'_\alpha\}$ is also a thread, and we set $g + g' = \{g_\alpha + g'_\alpha\}$. This is easily seen to make G_∞ an abelian group, the zero being the thread consisting of the zeros of all G_α.

For each α, mapping a thread to its representative in G_α yields a canonical homomorphism $p_\alpha : G_\infty \to G_\alpha$ with the following properties:

(i) $g = g'$ in G_∞ if and only if $p_\alpha(g) = p_\alpha(g')$ for all $\alpha \in \mathcal{D}$,

(ii) $p_\alpha = \mu_{\beta\alpha} p_\beta$ whenever $\alpha \preceq \beta$.

E. Categories and Functors

Several simple notions about categories and functors are needed in order to express, quickly and precisely, the various mathematical frameworks within which we work.

A *category* **K** is a class (not necessarily a set!) of *objects* (written $A, B, C, \ldots$) together with:

1° a family of sets $\mathbf{K}(A, B)$, one for each ordered pair of objects (the elements of $\mathbf{K}(A, B)$ are called *morphisms* from A to B in **K** and denoted by $f : A \to B$ or $A \xrightarrow{f} B$), and

2° a function $C : \mathbf{K}(A, B) \times \mathbf{K}(B, C) \to \mathbf{K}(A, C)$, one for each ordered triple of objects ($C(f, g)$ is written $g \circ f$ or gf and called the *composition* of f and g), satisfying the following two conditions:

(a) The composition law is associative: if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(b) For each object B there is a morphism $1_B \in \mathbf{K}(B, B)$ such that $f \circ 1_B = f$ for every $f : B \to C$, and $1_B \circ g = g$ for every $g : A \to B$.

We list a few examples:

(i) The class **K** of all sets, with $\mathbf{K}(A, B)$ being the set of all maps of A into B with the customary composition law, obviously forms a category **Ens**.

(ii) The class **K** of all topological spaces, with $\mathbf{K}(A, B)$ the set of all continuous maps of A into B and with the usual composition law, forms a category **Top**.

(iii) For any category **K**, the *opposite category* $\mathbf{K}^*$ is that having the same objects as **K**, morphisms $\mathbf{K}^*(A, B) = \mathbf{K}(B, A)$, and the same composition law as in **K**.

(iv) There are many categories in which all objects have some type of algebraic structure, such as the category **Ab** of abelian groups (in which all morphisms are homomorphisms) or that of vector spaces, or modules over a fixed ring, whose definitions are obvious.

If $u : A \to B$ and $v : B \to A$ are morphisms in a category **K** such that

$$v \circ u = 1_A,$$

then v is called a *left inverse* of u, and u is a *right inverse* of v. If u admits a left inverse v_l and a right inverse v_r, then u is called an *isomorphism*. In this case $v_l = v_l(uv_r) = (v_lu)v_r = v_r$, and the object $v_l = v_r$, denoted by u^{-1}, is called the *inverse* of u. Two objects $A, B \in \mathbf{K}$ are *equivalent* (written $A \cong B$) if there is an isomorphism $u : A \to B$ in **K**; the relation is indeed an equivalence relation in **K**. The above notion of equivalence yields the customary concept of isomorphism in **Ens**, **Top** and **Ab**.

When several morphisms between various objects in a category are considered simultaneously, it is convenient to display the morphisms as arrows in a diagram, such as

$$\mathcal{D} = \begin{tikzcd} A \arrow[r, "u"] & B \\ C \arrow[u, "f"] \arrow[r, "g"] & D \arrow[u, "v"'] \end{tikzcd}$$

The diagram $\mathcal{D}$ is said to be *commutative* (or to commute) if $uf = vg$. More generally, a diagram commutes if for every ordered pair (X, Y) of objects in that diagram, all chains (if any) of consecutive arrows running from X and ending at Y compose to the same morphism of X into Y

Given a diagram

$$\begin{tikzcd} B & C \\ A \arrow[u, "u"] \arrow[ur, "v"'] & \end{tikzcd}$$

we say that a morphism $g : B \to C$ *completes* the diagram, and write

$$\begin{tikzcd} B \arrow[r, dashed, "g"] & C \\ A \arrow[u, "u"] \arrow[ur, "v"'] & \end{tikzcd}$$

whenever the latter diagram commutes. The same terminology applies to more complicated diagrams; a morphism that completes a given diagram is usually denoted by a dashed arrow.

Let $\mathbf{C}$ be a category, and let $\alpha_1 : X_1 \to Z$, $\alpha_2 : X_2 \to Z$ be a pair of morphisms with a common codomain Z. We say that a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\beta_2"] \arrow[d, "\beta_1"'] & X_2 \arrow[d, "\alpha_2"] \\
X_1 \arrow[r, "\alpha_1"] & Z
\end{tikzcd}$$

is a *pull-back* for (α_1, α_2) if any commutative diagram of the form

$$\begin{tikzcd}
\widehat{P} \arrow[dr, dashed, "\gamma"] \arrow[drr, "\widehat{\beta}_2"] \arrow[ddr, "\widehat{\beta}_1"'] & & \\
& P \arrow[r, "\beta_2"'] \arrow[d, "\beta_1"] & X_2 \arrow[d, "\alpha_2"] \\
& X_1 \arrow[r, "\alpha_1"] & Z
\end{tikzcd}$$

can always be completed by a unique morphism γ. The uniqueness of the factorization through the pull-back implies that up to an isomorphism in the category $\mathbf{C}$, the object P is uniquely determined by the pair (α_1, α_2).

Natural transformations of functors

Let $\mathbf{K}$, $\mathbf{L}$ be two categories. A (*covariant*) *functor* $T : \mathbf{K} \to \mathbf{L}$ is a rule that assigns to each object $A \in \mathbf{K}$ an object $T(A) \in \mathbf{L}$ and to each morphism $u \in \mathbf{K}(A, B)$ a morphism $T(u) \in \mathbf{L}(T(A), T(B))$ so that

(a) $T(1_A) = 1_{T(A)}$ for each $A \in \mathbf{K}$,

(b) $T(uv) = T(u)T(v)$ whenever uv is defined.

We list simple but important examples:

(i) The *forgetful functor* $U : \mathbf{Top} \to \mathbf{Ens}$. The function that assigns to each topological space its underlying set and to each continuous map the underlying set-theoretic map, determines a functor U from $\mathbf{Top}$ to $\mathbf{Ens}$ called the *forgetful functor*. Similarly, we have a forgetful functor from $\mathbf{Ab}$ to $\mathbf{Ens}$.

(ii) *The functor* $\mathbf{F} : \mathbf{Ens} \to \mathbf{Ab}$. For any set A, let $\mathbf{F}(A)$ be the free abelian group generated by A. The defining property of free abelian groups implies that if $f : A \to B$ is any function, then there is a unique homomorphism $\mathbf{F}f : \mathbf{F}(A) \to \mathbf{F}(B)$ with the property that $(\mathbf{F}f)i = jf$, where $i : A \to \mathbf{F}(A)$ and $j : B \to \mathbf{F}(B)$ are the inclusions.

(iii) *The functor* $\mathbf{S}_M : \mathbf{Top} \to \mathbf{Ens}$. For a fixed space M, we define $\mathbf{S}_M : \mathbf{Top} \to \mathbf{Ens}$ by

$$\mathbf{S}_M(X) = \mathbf{Top}(M, X), \qquad X \in \mathbf{Top},$$

and for $f : A \to B$, $\mathbf{S}_M(f) : \mathbf{Top}(M, A) \to \mathbf{Top}(M, B)$ is given by

$$[\mathbf{S}_M(f)](u) = fu, \qquad u \in \mathbf{Top}(M, A).$$

If $\mathbf{K}^*$ is the opposite category, a functor $T : \mathbf{K}^* \to \mathbf{L}$ is called a *contravariant functor* on $\mathbf{K}$; it can be interpreted as a functor on $\mathbf{K}$ that reverses the direction of each morphism when sent to $\mathbf{L}$.

A functor $T : \mathbf{K} \to \mathbf{L}$ transfers commutative diagrams in $\mathbf{K}$ to commutative diagrams in $\mathbf{L}$. Moreover, if u is an isomorphism in $\mathbf{K}$, then $T(u)$ is an isomorphism in $\mathbf{L}$; the converse is not necessarily true.

Let $S, T : \mathbf{K} \to \mathbf{L}$ be two functors. A *natural transformation* $t : S \to T$ of S into T is a function that assigns to each $A \in \mathbf{K}$ a morphism $t^A \in \mathbf{L}(S(A), T(A))$ so that for all A, B in $\mathbf{K}$ and $u \in \mathbf{K}(A, B)$ the diagram

$$\begin{tikzcd} S(A) \arrow[r, "S(u)"] \arrow[d, "t^A"'] & S(B) \arrow[d, "t^B"] \\ T(A) \arrow[r, "T(u)"] & T(B) \end{tikzcd}$$

is commutative. We denote by $\mathrm{Nat}(S, T)$ the set of all natural transformations of S into T. The natural transformation $t : S \to T$ is called a *natural equivalence* if each t^A is an isomorphism in $\mathbf{L}$.

We can form a category of functors from $\mathbf{K}$ to $\mathbf{L}$, denoted by $\mathbf{L}^{\mathbf{K}}$, whose objects are functors and whose morphisms are natural transformations $t : S \to T$. Two functors S, T in this category are isomorphic if there are natural transformations $t : S \to T$ and $s : T \to S$ such that $st = 1_S$ and $ts = 1_T$. It is easily seen that S and T are isomorphic if and only if there is a natural transformation $t : S \to T$ such that $t^A : S(A) \to T(A)$ is an isomorphism in $\mathbf{L}$ for each A in $\mathbf{K}$.

EXAMPLE. Any directed set $\mathcal{D}$ may be regarded as a category whose objects are the elements $\alpha, \beta, \ldots$ of $\mathcal{D}$, the morphisms from α to β are given by $\mathcal{D}(\alpha, \beta) = \{(\beta, \alpha)\}$ if $\alpha \preceq \beta$, and $\mathcal{D}(\alpha, \beta) = \emptyset$ otherwise, and the composition of morphisms is defined by $(\gamma, \beta)(\beta, \alpha) = (\gamma, \alpha)$ whenever $\alpha \preceq \beta \preceq \gamma$. With this terminology, a direct system of abelian groups over $\mathcal{D}$ may be interpreted as a functor $T : \mathcal{D} \to \mathbf{Ab}$, and a morphism of two direct systems T, S of abelian groups as a natural transformation $T \to S$ of functors.

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The bibliography is divided into four parts: I. General Reference Texts; II. Monographs, Lecture Notes, and Surveys; III. Articles; and IV Additional References. The "Articles" part is restricted mostly to the articles dealing directly with the problems discussed in the book; it contains, however, some papers that are not cited in the text but are included to direct the reader's attention to areas closely related to the material presented. In the text, a reference such as Hopf [1927b] is to that author's second article listed for the year 1927 in the "Articles" part.

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IV. Additional References

This part of the bibliography contains additional references on a variety of themes, including some that were not treated in the text.

I. Classical Topics

This section contains references on the following topics:

(a) Results Related to the Banach Theorem
(b) Results Related to the Brouwer and Borsuk Theorems
(c) KKM-Theory. Applications and Related Results
(d) Fixed Points in Linear Topological Spaces
(e) Fixed Points for Kakutani Maps and Related Results

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II. Degree Theory and Fixed Point Index

This section contains references on the following topics:

(a) Results Related to the Leray–Schauder Theory
(b) Lefschetz Theory
(c) Equivariant Degree and Index Theory
(d) Degree and Index for Set-Valued Maps

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III. Special Topics

This section contains references on the following topics:

(a) Nielsen Theory
(b) Poincaré–Birkhoff Theorem and Symplectic Results
(c) Degree for VMO Maps and Sobolev Maps
(d) Degree in Wiener Spaces
(e) Vertical Fixed Point Theory

(a) Nielsen Theory

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IV. Applications

This section contains references on the following topics:

(a) Applications to Nonlinear PDEs and Integral Equations
(b) Applications to Nonlinear ODEs and Differential Inclusions
(c) Applications to Mathematical Economics
(d) Algorithmic Computation of Fixed Points

(a) Applications to Nonlinear PDEs and Integral Equations

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List of Standard Symbols

I. Set-Theoretic and Logical Symbols

$A \Rightarrow B$	A implies B
$A \Leftrightarrow B$	A and B are logically equivalent
$x \in X$	x is a member of the set X
$x \notin X$	x is not a member of the set X
$A \subset B$	A is a subset of B
$\emptyset$	Empty set
$A \cup B$, $\bigcup A_\lambda$	Union
$A \cap B$, $\bigcap A_\lambda$	Intersection
A^c, $\complement A$	Complement
$A - B$	Set-theoretic difference
$A \times B$, $\prod A_i$	Cartesian product
A/R	Quotient set (set of equivalence classes of A with respect to an equivalence relation R)
2^A	Set of all subsets of A
B^A	Set of all maps from A to B
$\{x \mid P(x)\}$	Set of all elements x with the property $P(x)$
$\{a_i\}_{i \in I}$	Family with index set I
$f : X \to Y$	Map f from X to Y
$x \mapsto f(x)$	Map assigning $f(x)$ to x
$f\|A$	Restriction of the map f to A
$g \circ f$, gf	Composite of the maps f and g
1_A, id_A	Identity map $A \to A$
f^n	nth iteration of the map f
(X, A)	Pair consisting of the set X and $A \subset X$
$f : (X, A) \to (Y, B)$	Map $f : X \to Y$ with $f(A) \subset B$
$[n]$	$\{1, 2, \ldots, n\}$

II. Order

(a, b)	Open interval $\{x \mid a < x < b\}$
$[a, b]$	Closed interval $\{x \mid a \le x \le b\}$
$(a, b]$	Half-open interval $\{x \mid a < x \le b\}$

$[a, b)$	Half-open interval $\{x \mid a \le x < b\}$
$\max A$	Maximum of A
$\min A$	Minimum of A
$\sup A$	Supremum, least upper bound of A
$\inf A$	Infimum, greatest lower bound of A

III. Algebra

$a \equiv b \pmod{n}$	a and b are congruent modulo n
$\det A$, $\lvert a_{ij} \rvert$	Determinant of the square matrix A
$\mathrm{tr}\, A$	Trace of the square matrix A
$M \cong N$	Isomorphism of the algebraic systems M and N
M/N	Quotient space of the algebraic system M by N
$\dim M$	Dimension (of a linear space, etc.)
$\mathrm{Im}\, f$	Image of the homomorphism f
$\mathrm{Ker}\, f$	Kernel of the homomorphism f
$\mathrm{Coker}\, f$	Cokernel of the homomorphism f
δ_{ij}	Kronecker delta ($\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \ne j$)
(x, y)	Inner product of x and y
$M \oplus N$	Direct sum of M and N
$M \otimes N$	Tensor product of M and N
$\mathrm{Hom}(M, N)$	Set of homomorphisms $M \to N$

IV. Algebraic Systems

$\boldsymbol{N}$	Set of natural numbers $\{0, 1, 2, \ldots\}$
$\boldsymbol{Z}$	Ring of all rational integers
$\boldsymbol{Z}_m$	$\boldsymbol{Z}/m\boldsymbol{Z}$ (ring of all residue classes modulo m)
$\boldsymbol{Q}$	Field of all rational numbers
$\boldsymbol{R}$	Field of all real numbers
$\boldsymbol{C}$	Field of all complex numbers

V. Topology

$B(x, \varepsilon)$	Open ball in a metric space with center x and radius $\varepsilon > 0$
$K(x, \varepsilon)$	Closed ball with center x and radius ε
(X, ε)	Open "ball" with "center" X and radius $\varepsilon > 0$
$a_n \to a$	Sequence $\{a_n\}$ converges to a
$\lim a_n$	Limit of the sequence $\{a_n\}$
$\overline{X}$, $\mathrm{Cl}\, X$	Closure of the set X
$\mathrm{Int}\, X$	Interior of the set X
∂X, $\mathrm{Fr}\, X$	Boundary of the set X
$\delta(x, y)$, $d(x, y)$	Distance between the points x and y
$f \simeq g$	Homotopy of the mappings f and g
$X \approx Y$	Homeomorphism of the topological spaces X and Y
$\mathrm{diam}\, X$, $\delta(X)$	Diameter of the set X
nbd	Neighborhood
u.s.c.	Upper semicontinuous
l.s.c.	Lower semicontinuous

AR	Absolute retract
ANR	Absolute neighborhood retract
$\mathrm{ES}(Q)$	Class of extensor spaces for the class Q
$\mathrm{NES}(Q)$	Class of neighborhood extensor spaces for the class Q

VI. Functional Analysis

$E, F, \ldots$	Linear topological spaces over the real numbers (sometimes over the complex numbers)
$(E, \\| \\;\\|)$	Normed linear space
$\\|x\\|, \\|x\\|_E$	Norm of the element x in the normed linear space E
$\mathscr{L}(E, F)$	Space of continuous (bounded) linear operators of E into F equipped with the operator norm $\\|A\\| = \sup_{\\|x\\|_E \leq 1} \\|Ax\\|_F$
$\operatorname{conv} A$	Convex hull of the set A
$\operatorname{Conv} A$	Convex closure of the set A
Df	Derivative of the map f
$D_j, \frac{\partial}{\partial x_j}$	Partial derivative
$D^p, \frac{\partial^{\lvert p\rvert}}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}}$	Partial derivative of order $\lvert p\rvert$; $p = (p_1, \ldots, p_n)$, $\lvert p\rvert = p_1 + \ldots + p_n$
E^*	Dual space of the Banach space E
l^2	Hilbert space of sequences $x = (x_1, x_2, \ldots)$ of real numbers with $\\|x\\|^2 = \sum_{i=1}^{\infty} x_i^2 < \infty$

VII. Homology

$[p_0, \ldots, p_s], (p_0, \ldots, p_s), \lvert p_0, \ldots, p_s\rvert$	Simplex with vertices $p_0, \ldots, p_s$
Δ^n	Standard n-simplex
$[\sigma]$	Barycenter of the simplex σ
$\operatorname{St} p$	Star of the vertex p
$\mathrm{Sd}^m \mathscr{K}, \mathscr{K}^{(m)}$	mth barycentric subdivision of the complex $\mathscr{K}$
$\mathrm{Sd}^m : C_*(\mathscr{K}) \to C_*(\mathscr{K}^{(m)})$	Subdivision chain homomorphism
$\lvert\mathscr{K}\rvert$	Support of the simplicial decomposition $\mathscr{K}$
$C_* = (C_n)_{n \geq 0}$	Chain complex
$\partial_n : C_n \to C_{n-1}$	Boundary operator
Z_n	Group of n-cycles
B_n	Group of n-boundaries
H_n	nth homology group
$H_n(X, A)$	Relative homology group
$H_n(X; G)$	Homology group with coefficients in G
$H_n(f) = f_{*n}$	Homomorphism of homology groups induced by f
$C^* = (C^n)_{n \geq 0}$	Cochain complex
$\delta^n : C^n \to C^{n+1}$	Coboundary operator
Z^n	Group of n-cocycles
B^n	Group of n-coboundaries
H^n	nth cohomology group
$H^n(f) = f^{*n}$	Homomorphism of cohomology groups induced by f
$\lambda(f)$	Lefschetz number of the map f
$\chi(f)$	Euler number of the map f
$\chi(K)$	Euler characteristic of the polyhedron K

$d(f)$, $\deg(f)$	(Brouwer) degree of the map f
$\varkappa(f)$	Characteristic of the vector field f
$b_n(X)$	nth Betti number of the space X

VIII. Categories

$\mathbf{K}$	Arbitrary category
$\mathbf{K}(A, B)$	Set of morphisms from A to B in $\mathbf{K}$
$\mathbf{K}^*$	Opposite category
Ens	Category of sets
Ab	Category of abelian groups
GrAb	Category of graded abelian groups
Top	Category of topological spaces
$\mathbf{Top}^2$	Category of pairs of topological spaces
$\mathfrak{C}_E$, $\mathfrak{C}$	Category of compact fields on the normed linear space E
$\mathfrak{L}_E$, $\mathfrak{L}$	Leray–Schauder category of the normed linear space E

IX. Degree and Fixed Point Index

$\mathrm{Fix}(f)$	Fixed point set of the map f
$\deg(f)$	Degree of the map f
$\Lambda(f)$	Generalized Lefschetz number of the map f
$\mathscr{F}(U, X)$	Set of compactly fixed maps $f : U \to X$, where U is open in X
$\mathscr{K}_{\partial U}(\overline{U}, X)$	Set of compact maps $f : \overline{U} \to X$, where $U \subset X$ is open and $\mathrm{Fix}(f\|\partial U) = \emptyset$
$i(f, U)$	Index of the compact map $f \in \mathscr{F}(U, X)$
$I(f, U)$	Index of the map $f \in \mathscr{K}_{\partial U}(\overline{U}, X)$
$J(f, x_0)$	Local index of the isolated fixed point x_0
$i(X, f, U)$	Local index on an admissible class

Index of Names

Index of Terms

GRANAS • DUGUNDJI
Fixed Point Theory

"Granas–Dugundji's book is an encyclopedic survey of the classical fixed point theory of continuous mappings (the work of Poincare, Brouwer, Lefschetz-Hopf, Leray-Schauder) and all its various modern extensions. This is certainly the most learned book ever likely to be published on this subject."

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"The theory of Fixed Points is one of the most powerful tools of modern mathematics. Not only is it used on a daily basis in pure and applied mathematics, but it also serves as a bridge between Analysis and Topology, and provides a very fruitful area of interaction between the two. This book contains a clear, detailed and well-organized presentation of the major results, together with an entertaining set of historical notes and an extensive bibliography describing further developments and applications."

—Haim Brezis, *Universite Pierre et Marie Curie*

"In this monograph, no effort has been spared, even to the smallest detail, be it mathematical, historical or bibliographical. In particular, the necessary background materials are generously provided for non-specialists. In fact, the book could even serve as an introduction to algebraic topology among others. It is certain that the book will be a standard work on Fixed Point Theory for many years to come."

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This monograph gives a carefully worked-out account of the most basic principles and applications of the theory of fixed points. Until now, a treatment of many of the discussed topics has been unavailable in book form. The presentation is self-contained and is accessible to a broad spectrum of readers. The main text is complemented by numerous exercises, detailed comments, and a comprehensive bibliography.

Andrzej Granas studied in Warsaw under Karol Borsuk and then in Moscow, where he earned his Ph.D. in 1958 under Lazar Lusternik. Since 1958, he has held various research and teaching posts in Poland, USA, France, Canada, and elsewhere. During the spring of 1970, he occupied a special chair at the Collège de France. In the early nineties, Dr. Granas founded the journal *Topological Methods in Nonlinear Analysis*, and since 1992, he has served on the editorial board of *Zentralblatt*. He is an Honorary Member of the Gdańsk Scientific Society.

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